2. Sets (Part II: Proofs Involving Sets)

Fundamental properties of sets

THEOREM 1. *The following statements are true for all sets A, B, and C.*

1. $A \cup B = B \cup A$ (`commutative`)
2. $A \cap B = B \cap A$ (`commutative`)
3. $(A \cup B) \cup C = A \cup (B \cup C)$ (`associative`)
4. $(A \cap B) \cap C = A \cap (B \cap C)$ (`associative`)
5. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (`distributive`)
6. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (`distributive`)

   `DeMorgan's Laws:` *If A and B are the sets contained in some universal set U then*
7. $\overline{A \cup B} = \bar{A} \cap \bar{B}$.
8. $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

Proving set properties

Use the following tautologies:

- $x \in A \cap B \Leftrightarrow (x \in A \wedge x \in B)$
- $x \in A \cup B \Leftrightarrow$
- $x \in A - B \Leftrightarrow$
- $A = B \Leftrightarrow (x \in A \Leftrightarrow x \in B)$
- $A \subseteq B \Leftrightarrow (x \in A \Rightarrow x \in B)$
- $x \in \bar{A} \Leftrightarrow x \notin A$

Methods:

- To prove $A \subseteq B$ it is sufficient to prove $x \in A \Rightarrow x \in B$.
- To prove $A = B$ it is sufficient to prove $x \in A \Leftrightarrow x \in B$.
- To prove $A = B$ it is sufficient to prove $A \subseteq B$ and $B \subseteq A$.
- To show that $A = \emptyset$ it is sufficient to show that $x \in A$ implies a false statement.

THEOREM 2. *The following statements are true for all sets A and B.*

1. $A \subseteq A \cup B$.
2. $A \cap B \subseteq A$.
3. *The empty set is a subset of every set. (Namely, for every set A, $\emptyset \subseteq A$. If $A \neq \emptyset$, then $\emptyset \subset A$.).*
4. $A \cup \emptyset = A$.
5. $A \cap \emptyset = \emptyset$.

EXAMPLE 3. *Let A and B be sets. Show that $(A - B) \cap B = \emptyset$.*

PROPOSITION 4. *For every two sets A and B,*

$$A - B = A \cap \bar{B}$$

EXAMPLE 5. *Let A, B and C be sets. Prove that*

$$A - (B \cup C) = (A - B) \cup (A - C)$$

EXAMPLE 6. *For the sets A, B and C prove that*

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

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PROPOSITION 7. *Let $A, B,$ and C be sets, and suppose $A \subseteq B$ and $B \subseteq C$. Then $A \subseteq C$.*

EXAMPLE 8. *Let* A, B, C *and* D *be sets. If* $A \subseteq C$ *and* $B \subseteq D$*, then* $A \times B \subseteq C \times D$*.*

EXAMPLE 9. *Prove the following statement. Let* A *and* B *be subsets of a universal set* U*. Then* $A \subseteq B \Leftrightarrow A \cup B = B$*.*

EXAMPLE 10. *Let* A *and* B *be subsets of a universal set* U*. Prove that*

$$A = A - B \Leftrightarrow A \cap B = \emptyset.$$