3 FUNCTIONS

3.1 Definition and Basic Properties

DEFINITION 1. *Let A and B be nonempty sets. A* **function** *f from the set A to the set B is a correspondence that assigns to each element a in the set A one and only one element b in the set B, which is denoted by $f(a)$.*

We call A the **domain** *of f and B the* **codomain** *of f.*

If $a \in A$ and $b \in B$ are such that $b = f(a)$, then b is called the **value** of f at a, or the **image** of a under f. We may also say that f **maps** a to b.

Using diagram

DEFINITION 2. *Two functions f and g are* **equal** *if they have the same domain and the same codomain and if $f(a) = g(a)$ for all a in domain.*

DEFINITION 3. *The* **graph** *of $f : A \to B$ is the set*

$$G_f = \{(a, b) \in A \times B | b = f(a)\}.$$

REMARK 4. We can determine a function from its domain, codomain, and graph.

EXAMPLE 5. *Let $A = \{2, 4, 6, 10\}$ and $B = \{0, 1, -1, 8\}$. Write out three functions with domain A and codomain B, and find its graph.*

Some common functions

- **Identity** function $i_A : A \to A$ maps every element to itself:

- **Polynomial** of degree n with real coefficients $a_0, a_1, \ldots, a_n$ is a function from $\mathbb{R}$ to $\mathbb{R}$

  Particular cases of polynomial:

  Quadratic function $f : \mathbb{R} \to \mathbb{R}$ is defined by

  Linear function $f : \mathbb{R} \to \mathbb{R}$ is defined by

  Constant function $f : \mathbb{R} \to \mathbb{R}$ is defined by

  The zero polynomial $f : \mathbb{R} \to \mathbb{R}$ is defined by

DEFINITION 6. *Let A and B be nonempty sets. We define*

$$F(A,B) =$$

the set of all functions from A to B.

If $A = B$, we simply write $F(A)$.

EXAMPLE 7. *Let $f \in F(\mathbb{R})$ be defined by $f(x) = x^3 + 3$. Find its graph and illustrate it.*

Image of a Function

DEFINITION 8. *Let $f \in F(A,B)$. The* **image** *of f is*

$$\mathrm{Im}(f) = \{y \in B | y = f(x) \quad for \quad some \quad x \in A\}.$$

EXAMPLE 9. *Let $f \in F(\mathbb{R})$ and $g \in F(\mathbb{R}, [0,\infty))$ defined by $f(x) = g(x) = x^4$. Find* $\mathrm{Im}(f)$, $\mathrm{Im}(g)$, *G_f and G_g. Determine whether $f = g$.*

EXAMPLE 10. *Let* $f, g : F(\mathbb{R})$ *be defined by* $f(x) = \cos x$ *and* $g(x) = |\cos x|$. *Find* $\operatorname{Im}(f)$ *and* $\operatorname{Im}(g)$.

EXAMPLE 11. *Let* $S = \{y \in \mathbb{R} | y \geq 0\}$. *Prove that if* $f : \mathbb{R} \to \mathbb{R}$ *is defined by* $f(x) = x^4$ *then* $\operatorname{Im}(f) = S$.

Section 3.2 Surjective and Injective Functions

Surjective functions ("onto")

DEFINITION 12. *Let $f : A \to B$ be a function. Then f is* **surjective** *(or a surjection) if the image of f coincides with its codomain, i.e.*

$$\mathrm{Im} f = B.$$

Note: surjection is also called "onto".

`Proving surjection:`

We know that for all $f : A \to B$: ________________________

Thus, *to show that $f : A \to B$ is a surjection it is sufficient to prove that* ________________

In other words,
to prove that $f : A \to B$ is a surjective function it is sufficient to show that ________________

Question: How to disprove surjectivity?

EXAMPLE 13. *Determine which of the following functions are surjective.*

(a) *Identity function*

(b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^4$.

(c) $g : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$, $g(x) = x^4$.

(d) $f : \mathbb{Z} \to \mathbb{Z},\ f(n) = \begin{cases} n-2 & \text{if} \quad n \in \mathbb{E}, \\ 2n-1 & \text{if} \quad n \in \mathbb{O}. \end{cases}$

(e) $f : \mathbb{Z} \to \mathbb{Z},\ f(n) = \begin{cases} n+1 & \text{if} \quad n \in \mathbb{E}, \\ n-3 & \text{if} \quad n \in \mathbb{O}. \end{cases}$

Injective functions ("one to one")

DEFINITION 14. *Let $f : A \to B$ be a function. Then f is* **injective** *(or an injection) if whenever $a_1, a_2 \in A$ and $a_1 \neq a_2$, we have $f(a_1) \neq f(a_2)$.*

Recall that surjection is also called "onto". Using diagram:

EXAMPLE 15. *Given $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$.*

(a) *Write out graph of an injective function with domain A and codomain B. Justify your answer.*

(b) *Write out graph a non injective function with domain A and codomain B. Justify your answer.*

`Proving injection:`
Let $P(a_1, a_2) : a_1 \neq a_2$ and $Q(a_1, a_2) : f(a_1) \neq f(a_2)$.

Then by definition f is injective if ____________________.

Using contrapositive, we have ____.
In other words, to prove injection show that:

Question: How to disprove injectivity?

EXAMPLE 16. *Prove or disprove injectivity of the following functions.*

(a) $f : \mathbb{R} \to \mathbb{R},\ f(x) = \sqrt[5]{x}.$

(b) $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^4.$

(c) $f : \mathbb{Z} \to \mathbb{Z},\ f(n) = \begin{cases} n/2 & \text{if} \quad n \in \mathbb{E}, \\ 2n & \text{if} \quad n \in \mathbb{O}. \end{cases}$

`Discussion Exercise.`

- `Must a strictly increasing or decreasing function be injective?`

- `Must an injective function be strictly increasing or decreasing?`

EXAMPLE 17. *Prove or disprove injectivity of the following functions. In each case, $f \in F(\mathbb{R})$.*

(a) $f(x) = 3x^5 + 5x^3 + 2x + 2016.$

(b) $f(x) = x^{12} + x^8 - x^4 + 11.$

Bijective functions

DEFINITION 18. *A function that is both surjective and injective is called* **bijective** *(or bijection.)*

EXAMPLE 19. *Determine which of the following functions are bijective.*

(a) $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^3$.

(b) $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$.

EXAMPLE 20. *Prove that* $f \in F(\mathbb{R} - \{2\}, \mathbb{R} - \{3\})$ *defined by* $f(x) = \dfrac{3x}{x-2}$ *is bijective.*

3.3 Composition and Invertible Functions

Composition of Functions

DEFINITION 21. *Let A, B, and C be nonempty sets, and let $f \in F(A,B)$, $g \in F(B,C)$. We define a function*

$$g \circ f \in F(A,C),$$

called the **composition** *of f and g, by*

$$gf(a) =$$

`Alternative notation for composition of` f `and` g`:` gf

EXAMPLE 22. *Let $A = \{1,2,3,4\}$, $B = \{a,b,c,d\}$, $C = \{r,s,t,u,v\}$ and define the functions $f \in F(A,B)$, $g \in F(B,C)$ by their graphs:*

$$G_f = \{(1,b),(2,d),(3,a),(4,a)\}, \qquad G_g = \{(a,u),(b,r),(c,r),(d,s)\}.$$

Find $G_{g\circ f}$. What is about $f \circ g$?

EXAMPLE 23. *Let $f, g \in F(\mathbf{R})$ be defined by $f(x) = e^x$ and $g(x) = x \sin x$. Find $f \circ g$ and $g \circ f$.*

PROPOSITION 24. *Let $f \in F(A,B)$, $g \in F(B,C)$, and $h \in F(C,D)$. Then*

$$(h \circ g) \circ f = h \circ (g \circ f),$$

i.e. composition of functions is associative.

Proof.

EXAMPLE 25. *Let $f \in F(\mathbf{R} - \{0,1\})$ be defined by $f(x) = 1 - \frac{1}{x}$. Determine $f \circ f \circ f$.*

EXAMPLE 26. *Let $f, g \in F(\mathbf{Z})$ be defined by*

$$f(n) = \begin{cases} n+4, & \text{if } n \in \mathbf{E} \\ 2n-3, & \text{if } n \in \mathbf{O} \end{cases} \qquad g(n) = \begin{cases} 2n-4, & \text{if } n \in \mathbf{E} \\ (n-1)/2, & \text{if } n \in \mathbf{O} \end{cases}$$

Find gf and fg.

PROPOSITION 27. *Let $f \in F(A, B)$ and $g \in F(B, C)$. Then*

i. *If f and g are surjections, then gf is also a surjection.*

`Proof.`

ii. *If f and g are injections, then gf is also an injection.*

`Proof.`

COROLLARY 28. *If f and g are bijections, then gf is also a bijection.*

PROPOSITION 29. *Let $f \in F(A,B)$. Then $fi_A = f$ and $i_B f = f$.*

Inverse Functions

DEFINITION 30. *Let $f \in F(A,B)$. Then f is* **invertible** *if there is a function $f^{-1} \in F(B,A)$ such that*

$$f^{-1}f = i_A \quad \text{and} \quad ff^{-1} = i_B.$$

If f^{-1} exists then it is called the **inverse** *function of f.*

`Question:` *`What is the inverse of`* f^{-1}*?*

REMARK 31. f is invertible if and only if f^{-1} is invertible.

EXAMPLE 32. *Discuss the inverse of a function defined by $f(x) = \sin x$.*

PROPOSITION 33. *The inverse function is unique.*

`Proof.`

EXAMPLE 34. *The function* $f : \mathbb{R} - \{2\} \to \mathbb{R} - \{3\}$ *defined by* $f(x) = \dfrac{3x}{x-2}$ *is known to be bijective (*see Example 20, Section 3.2*). Determine the inverse function.*

REMARK 35. Finding the inverse of a bijective function is not always possible by algebraic manipulations. For example,

if $f(x) = e^x$ then $f^{-1}(x) =$ _________

The function $f(x) = 3x^5 + 5x^3 + 2x + 2016$ is known to be bijective, but there is no way to find expression for its inverse.

THEOREM 36. *A function $f \in F(A,B)$ is invertible if and only if f is bijective.*

COROLLARY 37. *If a function $f \in F(A,B)$ is bijective, so is f^{-1}.*

Functions and Sets

Image of a Set

DEFINITION 38. *Let $f : A \to B$ be a function. If $X \subseteq A$, we define $f(X)$, the* **image** *of X under f, by*

$$f(X) = \{y \in B | y = f(x) \quad for \quad some \quad x \in X\}.$$

EXAMPLE 39. *Let $f \in F(A, B)$. Complete:*

(a) *If $X \subseteq A$ then $f(X) \subseteq$ ____ $\subseteq$ ____*

(b) $f(A) =$ ____

EXAMPLE 40. *$f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = \cos x$. Find $f([-\pi/2, \pi/2])$.*

Inverse Image

DEFINITION 41. *Let $f : A \to B$ be a function and let W be a subset of its codomain (i.e. $W \subseteq B$). Then the* **inverse image** *of W (written $f^{-1}(W)$) is the set*

$$f^{-1}(W) = \{a \in A | f(a) \in W\}.$$

EXAMPLE 42. *Let $f \in F(\mathbb{R})$ be defined by $f(x) = x^4$. Find the following sets:*

(a) $f([1,2]) =$

(b) $f([-2,-1]) =$

(c) $f^{-1}([1,16]) =$

(d) $f^{-1}([-16,-1]) =$

(e) $f^{-1}([-1,1]) =$

(f) $f((-\infty,\infty)) =$

(g) $f^{-1}(\mathbb{R}) =$

(h) $f^{-1}(\mathbb{R}^+) =$

EXAMPLE 43. *Let $A = \{a,b,c,d,e,f\}$ and $B = \{7,9,11,12,13\}$ and let the function $g : A \to B$ be given by*

$$g(a) = 11, g(b) = 9, g(c) = 9, g(d) = 11, g(e) = 9, g(f) = 7.$$

Find

$g^{-1}(\{7,9\}) =$

$g^{-1}(\{12,13\}) =$

$g^{-1}(\{11,12\}) =$

Summary

Let $f : A \to B$. Then above definitions imply the following tautologies

- $(y \in \mathrm{Im}(f)) \Leftrightarrow (\exists x \in A \ni f(x) = y)$.
- $(y \in f(X)) \Leftrightarrow (\exists x \in X \ni f(x) = y)$.
- $(x \in f^{-1}(W)) \Leftrightarrow (f(x) \in W)$.

Also note that

- If $W \subseteq \mathrm{Im}(f)$ then $(S = f^{-1}(W)) \Rightarrow (f(S) = W)$.

EXAMPLE 44. *$f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 5x - 4$. Find $f([0,1])$. Justify your answer.*

EXAMPLE 45. *Let* $f:\mathbb{R}\to\mathbb{R}$ *be defined by* $f(x)=3x+4$ *and let* $W=\{x\in\mathbb{R}|x>0\}$. *Find* $f^{-1}(W)$.

EXAMPLE 46. *Prove that* $f(\mathbb{E})=\mathbb{O}$, *where* $f\in F(\mathbb{Z})$ *is defined by* $f(n)=\begin{cases} n-1, & if \ \ n\in\mathbb{E}, \\ n+1, & if \ \ n\in\mathbb{O}. \end{cases}$

EXAMPLE 47. *Let* $f \in F(\mathbb{Z})$ *be defined by* $f(n) = \begin{cases} n/2 & if \;\; n \in \mathbb{E}, \\ n+1 & if \;\; n \in \mathbb{O}. \end{cases}$ *Compute the following.*

(a) $f^{-1}(\{6,7\}) =$

(b) $f^{-1}(\mathbb{O})$

PROPOSITION 48. *Let* $f \in F(A,B)$. *If* $X \subseteq Y \subseteq A$ *then* $f(X) \subseteq f(Y)$.

Proof.

PROPOSITION 49. *Let $f \in F(A,B)$. If W and V are subsets of B then*

(a) $f^{-1}(W \cup V) = f^{-1}(W) \cup f^{-1}(V)$.

(b) $f^{-1}(W \cap V) = f^{-1}(W) \cap f^{-1}(V)$.

PROPOSITION 50. *Let $f \in F(A,B)$. If $X \subseteq A$ and $Y \subseteq A$ then*

(a) $f(X \cup Y) = f(X) \cup f(Y)$.

(b) $f(X \cap Y) \subseteq f(X) \cap f(Y)$.

Proof