

10.3 deals with positive series only
 partial sums $\{S_n\}$ positive & increasing
bounded from below

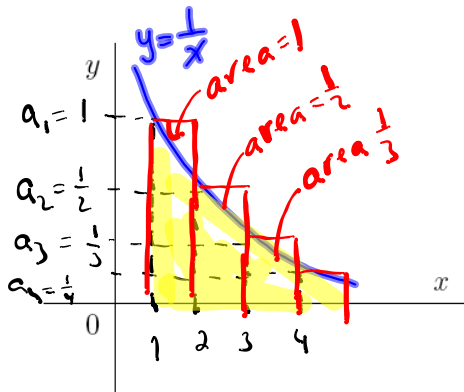
10.3: The Integral and Comparison Tests; Estimating Sums

QUESTION: For what values of p the p -series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent?

If $p = 1$ then $\sum_{n=1}^{\infty} \frac{1}{n}$ is called **harmonic series**.

$$f(x) = \frac{1}{x}, \quad x \geq 1$$

$$f(n) = \frac{1}{n}$$



$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} + \dots$$

$$\int_1^{\infty} f(x) dx = \int_1^{\infty} \frac{dx}{x} < 1 + \frac{1}{2} + \frac{1}{3} + \dots = \sum_{n=1}^{\infty} \frac{1}{n}$$

divergent

$\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent

THE INTEGRAL TEST Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x) dx$ is convergent. In other words:

If $\int_1^{\infty} f(x) dx$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges

If $\int_1^{\infty} f(x) dx$ diverges $\Rightarrow \sum_{n=1}^{\infty} a_n$ diverges

Note that one can consider interval $[a, \infty)$ instead of $[1, \infty)$ if $\sum_{n=a}^{\infty} a_n$.

FACT: The p -series, $\sum_{n=1}^{\infty} \frac{1}{n^p}$, converges if $p > 1$ and diverges if $p \leq 1$.

EXAMPLE 1. Determine if the following series is convergent or divergent:

(a) $\sum_{n=1}^{\infty} \frac{1000}{n\sqrt{n}}$

$$f(x) = \frac{1000}{x\sqrt{x}} = \frac{1000}{x^{3/2}}, \quad x \geq 1$$

On $[1, \infty)$ $f(x)$ is positive ✓
 continuous ✓

$$f'(x) = \left(\frac{1000}{x^{3/2}} \right) = 1000 \left(-\frac{3}{2} \right) x^{-5/2} < 0$$

decreasing ✓

We can apply Int. Test

$$\int_1^{\infty} f(x) dx = \int_1^{\infty} \frac{1000}{x^{3/2}} dx$$

$x^{-3/2+1}$	Anti-derivat
x	
$1000 \frac{\quad}{-3/2 + 1}$	$= -\frac{2000}{\sqrt{x}}$

$$= \lim_{t \rightarrow \infty} \int_1^t \frac{1000 dx}{x^{3/2}} = -2000 \lim_{t \rightarrow \infty} \frac{1}{\sqrt{x}} \Big|_1^t = -2000 \lim_{t \rightarrow \infty} \frac{1}{\sqrt{t}} - 1$$

$= 2000 \Rightarrow$ the integral converges

\Downarrow
 the series converges

$$(b) \sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

$$f(x) = \frac{1}{x \ln x} \quad x \geq 2$$

On $[2, \infty)$

$f(x)$ is

positive ✓

continuous ✓

decreasing ✓

(x is increasing } $\Rightarrow \frac{1}{x} \cdot \frac{1}{\ln x}$
 $\ln x$ is increasing } is decr.

Apply Int. Test

$$\int_2^{\infty} f(x) dx = \int_2^{\infty} \frac{dx}{x \ln x} = \lim_{t \rightarrow \infty} \int_2^t \frac{dx}{x \ln x}$$

$$= \lim_{t \rightarrow \infty} \ln(\ln(x)) \Big|_2^t$$

$$= \lim_{t \rightarrow \infty} \underbrace{\ln(\ln t)}_{\infty} - \ln(\ln 2)$$

$$\int \frac{dx}{x \ln x} = \int \frac{du}{u} = \ln|u| = \ln(\ln x)$$

$$u = \ln x \Rightarrow du = \frac{dx}{x}$$

The integral diverges \Rightarrow the given series diverges

A disadvantage of the Integral Test: it does force us to do improper integrals which are in some cases may be impossible to determine the convergence of. For example, consider

$$\sum_{n=0}^{\infty} \frac{1}{4^n + n^4}$$

positive

Try to apply Int. Test

$$f(x) = \frac{1}{4^x + x^4}$$

What is anti-derivative of $f(x) = ?$



Partial sum

$$S_n = \sum_{k=0}^n \frac{1}{4^k + k^4}$$

$\{S_n\}$ is positive (= bounded from below by 0)

$\{S_n\}$ increasing sequence, because $S_{n+1} = S_n + a_{n+1} > S_n$

$$S_n = \sum_{k=0}^n \frac{1}{4^k + k^4} < \sum_{k=0}^n \frac{1}{4^k} < \sum_{k=0}^{\infty} \frac{1}{4^k} = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}$$

geometric
 $r = \frac{1}{4}$
 $a = 1$
 $S = \frac{a}{1-r}$

$S_n \leq \frac{4}{3} \Rightarrow S_n$ is bdd above

Conclusion $\{S_n\}$ is monotonic and bounded \Rightarrow

$\Rightarrow \{S_n\}$ is convergent

\Rightarrow the given series converges

THE COMPARISON TEST Suppose that $\sum a_n$ and $\sum b_n$ are series with **positive** terms and $a_n \leq b_n$ for all n .

- If $\sum b_n$ is convergent then $\sum a_n$ is also convergent.
- If $\sum a_n$ is divergent then $\sum b_n$ is also divergent.

EXAMPLE 2. Determine if the following series is convergent or divergent:

$$(a) \sum_{n=1}^{\infty} \frac{n^4 + 4}{n^6 + 6} \sim \sum \frac{n^4}{n^6} = \sum \frac{1}{n^2} \text{ Conv.}$$

$$\frac{n^4 + 4}{n^6 + 6} \leq \frac{n^4 + 4}{n^6} = \frac{1}{n^2} + \frac{4}{n^6}$$

$$\left. \begin{array}{l} \sum \frac{1}{n^2} \\ \sum \frac{4}{n^6} \end{array} \right\} \begin{array}{l} \text{conv. as} \\ \text{p-series} \\ p > 1 \end{array} \Rightarrow \sum \frac{1}{n^2} + \frac{4}{n^6} \text{ converges}$$

↓ by Comp. Test
the given series
converges

$$(b) \sum_{n=1}^{\infty} \frac{n^{2000}}{n^{2001} - \sin^{2000} n} \sim \sum \frac{n^{2000}}{n^{2001}} = \sum \frac{1}{n} \text{ divergent}$$

$$\frac{n^{2000}}{n^{2001} - \sin^{2000} n} \geq \frac{n^{2000}}{n^{2001}} = \frac{1}{n}$$

The series $\sum \frac{1}{n}$ is divergent ($p=1$)

\Rightarrow By Comp. Test the given series is divergent

$$(c) \sum_{n=1}^{\infty} \frac{\cos^4 n}{n^2 \sqrt{n}} \sim \sum \frac{1}{n^2 \sqrt{n}} = \underbrace{\sum \frac{1}{n^{5/2}} \text{ converges } p = 5/2}$$

$$\frac{\cos^4 n}{n^2 \sqrt{n}} \leq \frac{1}{n^2 \sqrt{n}} = \frac{1}{n^{5/2}}$$

By Comp. Test
the given series
converges

$$(d) \sum_{n=1}^{\infty} \frac{5^n + 1}{4^n} = \sum_{n=1}^{\infty} \frac{5^n}{4^n} + \frac{1}{4^n} = \sum \left(\frac{5}{4}\right)^n + \frac{1}{4^n}$$

$\sum \left(\frac{5}{4}\right)^n$ $\sum \frac{1}{4^n}$
 geometric series
 $r = \frac{5}{4} > 1$ $r = \frac{1}{4}$
 div. conv.

$$\frac{5^n + 1}{4^n} = \left(\frac{5}{4}\right)^n + \frac{1}{4^n} > \left(\frac{5}{4}\right)^n$$

$\sum_{n=1}^{\infty} \left(\frac{5}{4}\right)^n$ diverges

} → By Comp. Test
 the given
 series
 diverges

$$(e) \sum_{n=0}^{\infty} \frac{1}{4^n + n^4}$$

$$\left. \begin{array}{l} \frac{1}{4^n + n^4} \leq \frac{1}{4^n} \\ \sum_{n=0}^{\infty} \frac{1}{4^n} \text{ converges} \end{array} \right\} \Rightarrow \text{by Comp. Theorem} \quad \sum_{h=0}^{\infty} \frac{1}{4^h + n^4} \text{ converges}$$

2nd way

$$\frac{1}{4^n + n^4} < \frac{1}{n^4}$$

$\sum \frac{1}{h^4}$ conv. p-series $p=4 > 1$

WARNING: Distinguish between $\sum_{n=1}^{\infty} n^p$ and $\sum_{n=1}^{\infty} p^n$.

↓
p-series

↓ geom. series
with $r=p$

$$\sum \frac{1}{n^{-p}}$$

In some cases inequalities are useless. For example, for the series

$$\sum_{n=0}^{\infty} \frac{1}{4^n - n}$$

we have

$$\frac{1}{4^n - n} > \frac{1}{4^n}$$

But $\sum \frac{1}{4^n}$ convergent

geometric

$$-1 < r = \frac{1}{4} < 1$$

} \Rightarrow no conclusion
can be
made
using
Comparison
Test.

\Downarrow
use another test

THE LIMIT COMPARISON TEST Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

If

$$\lim_{n \rightarrow \infty} a_n \cdot \frac{1}{b_n} = \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

$c = 0$ or $c = \infty \Rightarrow$ TEST FAILS \Rightarrow use other test

EXAMPLE 3. Determine if the following series is convergent or divergent:

(a) $\sum_{n=1}^{\infty} \frac{1}{4^n - n} \sim \sum \frac{1}{4^n}$ convergent as geom. series $r = \frac{1}{4} < 1$

$$c = \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{4^n}{4^n - n} \stackrel{\frac{\infty}{\infty}}{=} \lim_{n \rightarrow \infty} \frac{4^n \ln 4 / 4^n}{(4^n \ln 4 - 1) / 4^n} =$$

L'Hospital's Rule

$$= \lim_{n \rightarrow \infty} \frac{\ln 4}{\ln 4 - \frac{1}{4^n}} = \frac{\ln 4}{\ln 4 - 0} = 1 > 0$$

The given series converges by LCT.

(b) $\sum_{n=2}^{\infty} \frac{n^2 + n}{\sqrt{n^5 + n^3}} \sim \sum \frac{n^2}{\sqrt{n^5}} = \sum \frac{n^2}{n^{5/2}} = \sum \frac{1}{n^{1/2}}$ divergent p-series with $p = \frac{1}{2} < 1$

$$c = \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} a_n \cdot \left(\frac{1}{b_n} \right) = \lim_{n \rightarrow \infty} \frac{(n^2 + n) \sqrt{n}}{\sqrt{n^5 + n^3}} =$$

$$= \lim_{n \rightarrow \infty} \sqrt{\frac{(n^2 + n)^2 n}{n^5 + n^3}} = 1 > 0$$

By LCT the given series diverges

Illustration: Why c in the Limit Comparison Test must be positive and finite:

$$c > 0 \Rightarrow \text{LCT}$$

$$c = 0$$

$$c = \infty$$

c DNE (see Ex. 5A)

$c < 0$ (impossible & / c we apply LCT only to positive series)

} LCT fails

For example, $\sum \underbrace{\frac{1}{n}}_{a_n} \text{ div.}$ and $\sum \underbrace{\frac{1}{n^2}}_{b_n} \text{ conv.}$

$$c = \lim_{n \rightarrow \infty} a_n \cdot \frac{1}{b_n} = \lim_{n \rightarrow \infty} \frac{n^2}{n} = \lim_{n \rightarrow \infty} n = \infty \Rightarrow \text{LCT fails}$$

For example, $\sum \frac{1}{n^2} \text{ conv.}$ $\sum \frac{1}{n^3} \text{ conv.}$

$$c = \infty$$

$$\sum \frac{1}{n^4} \text{ conv.}$$

$$\sum \frac{1}{n^2} \text{ conv.}$$

$$\Rightarrow c = 0$$

If $\sum_{n=1}^{\infty} a_n$ converges \Rightarrow

$$S = \sum_{n=1}^{\infty} a_n = \underbrace{a_1 + a_2 + a_3 + \dots + a_{n-1} + a_n}_{\text{partial sum } S_n} + \underbrace{a_{n+1} + a_{n+2} + a_{n+3} + \dots}_{R_n \text{ remainder}}$$

$$S = S_n + R_n$$

REMAINDER ESTIMATE FOR THE INTEGRAL TEST

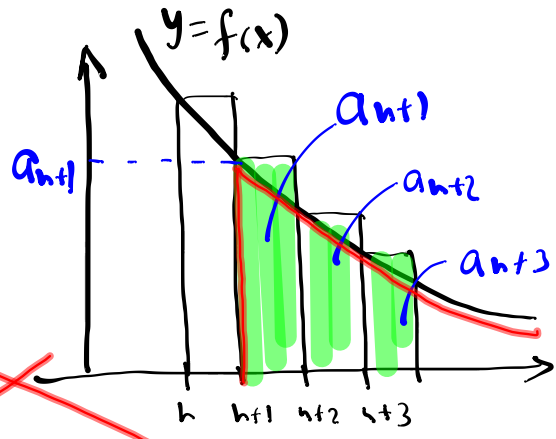
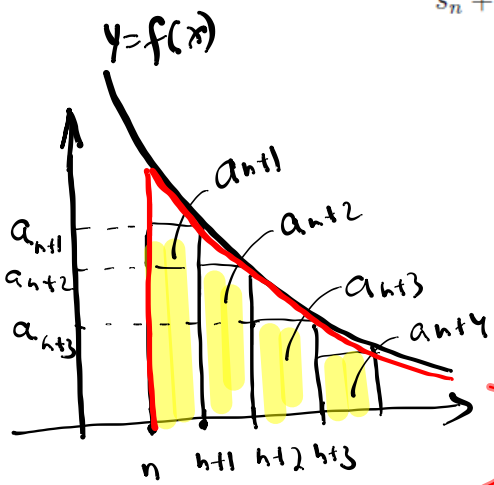
If $\sum a_n$ converges by the Integral Test and $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x) dx \leq R_n \leq \int_n^{\infty} f(x) dx$$

$\int_n^{\infty} f(x) dx = s - s_n$

which implies

$$s_n + \int_{n+1}^{\infty} f(x) dx \leq s \leq s_n + \int_n^{\infty} f(x) dx$$



$$\int_{h+1}^{\infty} f(x) dx \leq R_n = a_{n+1} + a_{n+2} + a_{n+3} + \dots \leq \int_n^{\infty} f(x) dx$$

EXAMPLE 4. Given $\sum_{n=1}^{\infty} \frac{1}{n^3}$ convergent p-series, $p=3 > 1$

(a) Approximate the sum of the series by using the sum of the first 10 terms (Use Calculator)

$$S_{10} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \dots + \frac{1}{10^3} \approx 1.1975$$

(b) Estimate the error $= R_{10}$

$$\int_{11}^{\infty} f(x) dx \leq R_{10} \leq \int_{10}^{\infty} f(x) dx, \text{ where } f(x) = \frac{1}{x^3}$$

$$\int_{11}^{\infty} \frac{dx}{x^3} \leq R_{10} \leq \int_{10}^{\infty} \frac{dx}{x^3}$$

$$\frac{1}{2 \cdot 11^2} \leq R_{10} \leq \frac{1}{2 \cdot 10^2}$$

$$\boxed{\frac{1}{242} \leq R_{10} \leq \frac{1}{200}} \text{ Final answer}$$

↓

$$0.00413 \lesssim R_{10} \lesssim 0.005$$

↓

$$1.1975 + 0.00413 \lesssim S = S_{10} + R_{10} \leq 1.1975 + 0.005$$

↓

$$\boxed{1.2016 \lesssim S \leq 1.2025} \text{ sum estimate with the error } R_{10}$$

(c) How many terms are required to ensure that the sum is accurate to within 0.0005?

Find n such that

$$R_n \leq \int_n^{\infty} \frac{dx}{x^3} = \frac{1}{2n^2} \leq 0.0005$$

$$\frac{1}{2n^2} \leq 5 \cdot 10^{-4}$$

$$1 \leq 5 \cdot 10^{-4} n^2 \cdot 2$$

$$1 \leq 10 \cdot 10^{-4} n^2$$

$$1 \leq 10^{-3} n^2$$

$$10^3 \leq n^2$$

$$n \geq \sqrt{1000} \approx 31.6$$

Conclusion

We need $\boxed{32}$ terms to ensure that the sum is accurate to within 0.0005.

EXAMPLE 5. Given

$$\sum_{n=1}^{\infty} \underbrace{\frac{1 + \sin n}{2n^3}}_{a_n} \sim \sum \frac{2}{2n^3} = \sum \frac{1}{n^3} \text{ Conv.}$$

(a) Prove the convergence.

$$c = \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{1 + \sin n}{2n^3} \cdot n^3 = \lim_{n \rightarrow \infty} \frac{1 + \sin n}{2} = \text{DNE}$$

oscillating

\Rightarrow LCT fails \Rightarrow use another test

Use Comparison Test

$$0 \leq \frac{1 + \sin n}{2n^3} \leq \frac{2}{2n^3} = \frac{1}{n^3}$$

$\sum \frac{1}{n^3}$ converges $\overset{CT}{\Rightarrow}$ the given series converges.

(b) By comparison the series to a p-series, estimate the error in using s_{100} to approximate the sum of the series,

By Example 4 for the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ we have

$$R_n \leq \int_n^{\infty} \frac{dx}{x^3} = \frac{1}{2n^2}$$

$$\Downarrow$$

$$R_{100} \leq \frac{1}{2 \cdot 100^2} = 0.5 \cdot 10^{-4}$$

Thus the error r_{100} for the given series $\sum_{n=1}^{\infty} \frac{1 + \sin n}{n^3}$

will be $r_{100} \leq R_{100} \leq \boxed{0.5 \cdot 10^{-4}}$