10.5: Power Series

DEFINITION 1. *A power series about $x = a$ (or centered at $x = a$), or just* **power series**, *is any series that can be written in the form*

center

general term

$a_n = c_n(x-a)^n$

$$\sum_{n=0}^{\infty} c_n(x-a)^n,$$

coefficient

where a and c_n are numbers. The c_n's are called the **coefficients** *of the power series.*

For example $\sum_{n=0}^{\infty} x^n$ then center $a=0$

coeff. $c_n = 1$

general term $a_n = x^n$

THEOREM 2. *For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only 3 possibilities:*

1. *The series converges only for $x = a$.* $R=0$
2. *The series converges for all x.* $R=\infty$
3. *There is $R > 0$ such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$. We call such R the* radius of convergence.

strict inequality

REMARK 3. In case 1 of the theorem we say that $R = 0$ and in case 2 we say that $R = \infty$

$|x-a| < R \Rightarrow -R < x-a < R$ $+a$ $+a$ $+a$

$a - R < x < a + R$

DEFINITION 4. *An* **interval of convergence** *is the interval of all x's for which the power series converges.*

Points $x = a \pm R$ are called end points and should be investigated separately

$x = -3$ center

EXAMPLE 5. *Assume that it is known that the series* $\sum_{n=0}^{\infty} c_n(x+3)^n$ *converges when* $x = -10$ *and diverges when* $x = 6$. *What can be said about the convergence or divergence of the following series:*

$\sum_{n=0}^{\infty} c_n 2^n$

$x+3=2 \Rightarrow x=-1$

converges
(see diagram below)

$\sum_{n=0}^{\infty} c_n(-11)^n$

$x+3=-11 \Rightarrow x=-14$

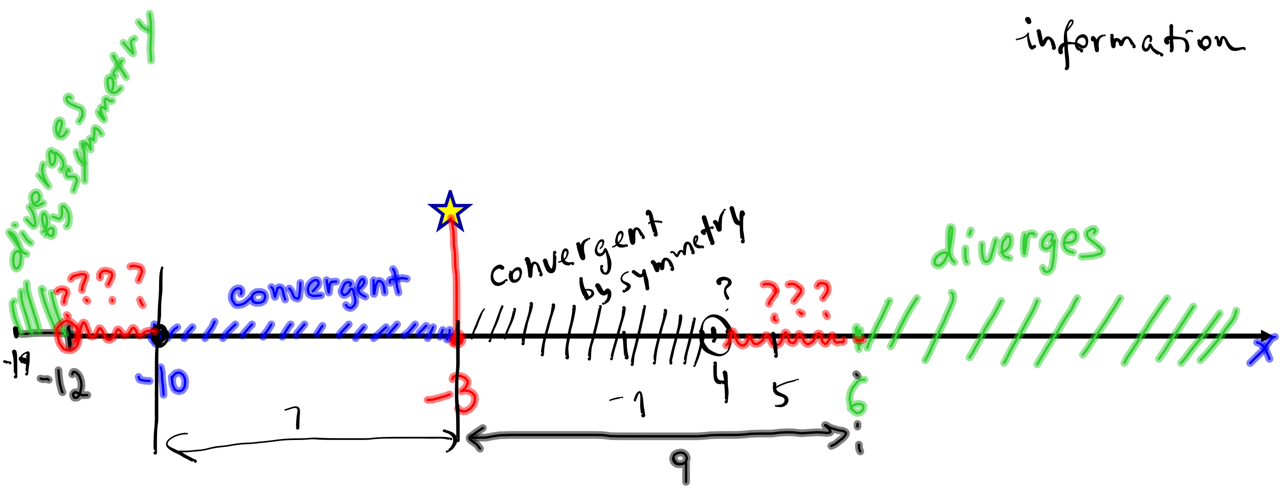
diverges

$\sum_{n=0}^{\infty} c_n 8^n$

$x+3=8$

$x=5$

not enough information



Ex 0.

$$\sum_{n=0}^{\infty} \underbrace{x^n}_{a_n} \swarrow \text{ geometric series with } r = x$$

$\Rightarrow$ converges $|r| < 1$, or $|x| < 1$

interval of conv. $\boxed{-1 < x < 1}$ $\boxed{R = 1}$

Apply Ratio Test

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x^{n+1}|}{|x^n|} = \lim_{n\to\infty} \frac{|x|^{n+1}}{|x|^n} = \lim_{n\to\infty} |x| = |x| < 1$$

$\Downarrow$

$R = 1$.

EXAMPLE 6. *Given* $\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}(x+3)^n$.

$$|a_n| = \left|\frac{(-1)^n n}{4^n}(x+3)^n\right| = \frac{n}{4^n}|x+3|^n$$

$$|a_{n+1}| = \frac{n+1}{4^{n+1}}|x+3|^{n+1}$$

(a) *Find the radius of convergence.*

Apply Ratio Test

$$L = \lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)|x+3|^{\cancel{n+1}}}{4^{\cancel{n+1}}} \cdot \frac{\cancel{4^n}}{n\,\cancel{|x+3|^n}}$$

$$= \frac{|x+3|}{4} \underbrace{\lim_{n\to\infty} \frac{n+1}{n}}_{1} = \frac{|x+3|}{4} < 1 \Rightarrow |x+3| < 4$$

$$\Downarrow$$

$$\boxed{R = 4}$$

(b) *Find the interval of convergence.*

$\underbrace{L<1}$ and $L=1$ (end points)

By (a): $|x+3| < 4$

$$-4 < x+3 < 4$$

$$^{-3}\quad\ ^{-3}\quad\ ^{-3}$$

$$-7 < x < 1$$

By (a) $L=1 \Rightarrow |x+3| = 4$

$$x+3 = \pm 4$$

Plug in $x+3=4$ (i.e. $x=1$)

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} \cdot 4^n = \sum_{n=1} (-1)^n n$$

Diverges by Divergent Test!

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n n \quad \text{DNE}$$

Plugin second end point where $x+3=-4$ (i.e. $x=-7$)

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}(-4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n n (-1)^n 4^n}{4^n}$$

$$= \sum_{n=1}^{\infty} (-1)^{2n} n = \sum_{n=1}^{\infty} n \quad \text{diverges}$$

by DT $\lim_{n\to\infty} n = \infty$.

Conclusion No convergence at end points and then $(-7, 1)$ is interval of convergence.

EXAMPLE 7. *Given* $\sum_{n=1}^{\infty} \frac{2^n}{n}(3x-6)^n = \sum_{n=1}^{\infty} \frac{2^n}{n}\left(3(x-2)\right)^n = \sum_{n=1}^{\infty} \frac{2^n 3^n}{n}(x-2)^n$

$$= \sum_{n=1}^{\infty} \frac{6^n}{n}(x-2)^n$$

(a) *Find the radius of convergence.*

$$|a_n| = \frac{6^n}{n}|x-2|^n$$

$$|a_{n+1}| = \frac{6^{n+1}}{n+1}|x-2|^{n+1}$$

Apply Ratio Test

$$L = \lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|} = \lim_{n\to\infty} \frac{6^{\cancel{n+1}}\,|x-2|^{\cancel{n+1}}\; n}{(n+1)\;\cancel{6^n}\;\cancel{|x-2|^n}}$$

$$= 6|x-2| \underbrace{\lim_{n\to\infty} \frac{n}{n+1}}_{1} = 6|x-2| < 1$$

$$|x-2| < \frac{1}{6}$$

$$\boxed{R = \frac{1}{6}}$$

(b) *Find the interval of convergence.*

$L < 1$ & inspect $L = 1$ (end points)

$|x-2| < \frac{1}{6}$

$-\frac{1}{6} < x-2 < \frac{1}{6}$

$-\frac{1}{6} + 2 < x-2+2 < \frac{1}{6} + 2$

$\frac{11}{6} < x < \frac{13}{6}$

$|x-2| = \frac{1}{6}$

$x-2 = -\frac{1}{6} \Rightarrow x = \frac{11}{6}$ OR $x-2 = \frac{1}{6} \Rightarrow x = \frac{13}{6}$

Plug in

$\sum_{n=1}^{\infty} \frac{6^n}{n}\left(-\frac{1}{6}\right)^n$

$\| $

$\sum_{n=1}^{\infty} \frac{\cancel{6^n}}{n} \cdot \frac{(-1)^n}{\cancel{6^n}}$

$\|$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

converges by AST

$\left(\lim_{n\to\infty} \frac{1}{n} = 0 \text{ and } \left\{\frac{1}{n}\right\} \downarrow\right)$

$\sum_{n=1}^{\infty} \frac{6^n}{n}\left(\frac{1}{6}\right)^n$

$\|$

$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}$

divergent as p-series $p = 1$ (or use Integral Test)

Conclusion: The interval of convergence is $\frac{11}{6} \leq x < \frac{13}{6}$

OR $\left[\frac{11}{6}, \frac{13}{6}\right)$

EXAMPLE 8. *Given* $\sum_{n=1}^{\infty} \frac{(-1)^n}{(3n+1)!}(x+8)^n$.

(a) *Find the radius of convergence.*

$$|a_n| = \left|\frac{(-1)^n (x+8)^n}{(3n+1)!}\right| = \frac{|x+8|^n}{(3n+1)!}$$

$$|a_{n+1}| = \frac{|x+8|^{n+1}}{(3(n+1)+1)!} = \frac{|x+8|^{n+1}}{(3n+4)!}$$

Apply Ratio Test

$$L = \lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|} = \lim_{n\to\infty} \frac{|x+8|^{\cancel{n+1}}\,(3n+1)!}{(3n+4)!\;\cancel{|x+8|^n}} =$$

$$= |x+8| \cdot \lim_{n\to\infty} \frac{\cancel{(3n+1)!}}{\cancel{(3n+1)!}\,(3n+2)(3n+3)(3n+4)} = 0 < 1$$

for all x

$\Downarrow$

series converges absolutely for all x.

$$R = \infty$$

(b) *Find the interval of convergence.* $(-\infty, \infty)$

EXAMPLE 9. *Given* $\sum_{n=1}^{\infty} \frac{(2n)!}{9^{n-1}}(x+8)^n$.

$$|a_n| = \frac{(2n)!\,|x+8|^n}{9^{n-1}}$$

(a) *Find the radius of convergence.*

$$|a_{n+1}| = \frac{(2(n+1))!\,|x+8|^{n+1}}{9^{n+1-1}} = \frac{(2n+2)!\,|x+8|^{n+1}}{9^n}$$

Apply Ratio Test

$$L = \lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|} = \lim_{n\to\infty} \frac{(2n+2)!\,|x+8|^{\cancel{n+1}}}{9^{\cancel{n}}} \cdot \frac{\cancel{9^{n-1}}}{(2n)!\,\cancel{|x+8|^n}}$$

$$= \frac{|x+8|}{9} \lim_{n\to\infty} \frac{\cancel{(2n)!}(2n+1)(2n+2)}{\cancel{(2n)!}} = \begin{cases} \infty > 1, & x \neq -8 \\ 0 < 1, & x = -8 \end{cases}$$

The series converges at $x=-8$ only

$\Rightarrow$ $\boxed{R = 0}$

(b) *Find the interval of convergence.* = singleton $\{-8\}$

EXAMPLE 10. *Given* $\sum_{n=1}^{\infty} \frac{x^{5n+5}}{3^{n+1}(n+1)}$. power series

(a) *Find the radius of convergence.*

$$|a_n| = \frac{|x|^{5n+5}}{3^{n+1}(n+1)}$$

$$|a_{n+1}| = \frac{|x|^{5(n+1)+5}}{3^{(n+1)+1}((n+1)+1)} = \frac{|x|^{5n+10}}{3^{n+2}(n+2)}$$

Apply Ratio Test

$$L = \lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|} = \lim_{n\to\infty} \frac{|x|^{\cancel{5n+10}\ 5}}{3^{\cancel{n+2}}(n+2)} \cdot \frac{\cancel{3^{n+1}}(n+1)}{\cancel{|x|^{5n+5}}}$$

$$= \frac{|x|^5}{3} \lim_{n\to\infty} \frac{n+1}{n+2} = \underbrace{\frac{|x|^5}{3} < 1}$$

$$|x|^5 < 3$$

$$|x| < \sqrt[5]{3} \implies \boxed{R = \sqrt[5]{3}}$$

(b) *Find the interval of convergence.*

$L < 1$ & $L = 1 \Rightarrow |x| = \sqrt[5]{3} \Rightarrow x = \pm\sqrt[5]{3}$

$|x| < \sqrt[5]{3}$

$-\sqrt[5]{3} < x < \sqrt[5]{3}$

Plug in to $\sum_{n=1}^{\infty} \frac{(x^5)^{n+1}}{3^{n+1}(n+1)}$

$x = -\sqrt[5]{3}$

$\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{3^{n+1}(n+1)}$

$\shortparallel$

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cancel{3^{n+1}}}{\cancel{3^{n+1}}(n+1)}$

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$

converges by AST:

$\lim \frac{1}{n+1} = 0$

$\left\{\frac{1}{n+1}\right\} \downarrow$

$x = \sqrt[5]{3}$

$\sum_{n=1}^{\infty} \frac{\cancel{3^{n+1}}}{\cancel{3^{n+1}}(n+1)}$

$\shortparallel$

$\sum_{n=1}^{\infty} \frac{1}{n+1}$

diverges by integral Test.

$\int_1^{\infty} \frac{dx}{x+1} = \infty$

Conclusion The interval of convergence is

$-\sqrt[5]{3} \leq x < \sqrt[5]{3}$

or $[-\sqrt[5]{3}, \sqrt[5]{3})$.