10.6: Representation of Functions as Power Series

Problem: Find the sum of the following series: $\sum_{n=0}^{\infty} x^n$

geometric series

$a = 1$, $r = x$

converges when $|r| < 1$,

or $|x| < 1$

$\frac{1}{1-x}$, $|x| < 1$

power series

centered at 0, $C_n = 1$

with $R = 1$

interval of conv. $-1 < x < 1$, or $(-1, 1)$

EXAMPLE 1. *Find a power series representation for $f(x)$ and determine the interval of convergence.*

(a) $f(x) = \frac{1}{5-x}$ $= \frac{1}{5} \cdot \frac{1}{1-\frac{x}{5}} = \frac{1}{5} \sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n = \sum_{n=0}^{\infty} \frac{1}{5} \frac{x^n}{5^n}$

Sum of geom. series

$$\boxed{\frac{1}{5-x} = \sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}}}$$

$\left|\frac{x}{5}\right| < 1$

$|x| < 5$ $\Rightarrow$ $\boxed{(-5, 5)}$

(b) $f(x) = \frac{x}{5-x}$ $= x \frac{1}{5-x} = x \sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}}$ $\boxed{(-5, 5)}$

$$\boxed{\frac{x}{5-x} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{5^{n+1}}}$$

(c) $f(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$, where $|-x^2| < 1$

$\Downarrow$

$|x^2| < 1$

$|x| < 1$

$\boxed{(-1, 1)}$

$f(x) = \sum_{n=0}^{\infty} (-1)^n (x^2)^n$

$$\boxed{f(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n}}$$

(d) $f(x) = \frac{x^2}{16x^4 - 9}$ $= x^2 \cdot \frac{1}{-(9-16x^4)} = -\frac{x^2}{9} \cdot \frac{1}{\underbrace{1-\frac{16x^4}{9}}_{\text{sum of geom. series}}}$

$$f(x) = -\frac{x^2}{9} \cdot \sum_{n=0}^{\infty} \left(\frac{16x^4}{9}\right)^n, \quad \left|\frac{16x^4}{9}\right| < 1$$

$$f(x) = \sum_{n=0}^{\infty} -\frac{x^2}{9} \cdot \frac{16^n}{9^n} x^{4n}$$

$$x^4 < \frac{9}{16}$$

$$x^2 < \frac{3}{4}$$

$$|x| < \frac{\sqrt{3}}{2}$$

$$\boxed{\left(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)}$$

$$\boxed{f(x) = \sum_{n=0}^{\infty} -\frac{16^n}{9^{n+1}} x^{4n+2}}$$

Differentiation and Integration of power series

THEOREM 2. *If the power series* $\sum_{n=0}^{\infty} c_n(x-a)^n$ *has radius of convergence* $R > 0$*, then the function* f *defined by*

sum of power series $\leftarrow$ $$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ *and*

- $f'(x) = \sum_{n=0}^{\infty} nc_n(x-a)^{n-1}$ TERMWISE DIFFERENTIATION
- $\int f(x)\,\mathrm{d}x = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$ TERMWISE INTEGRATION

The radii of convergence of the power series for $f'(x)$ *and* $\int f(x)\,\mathrm{d}x$ *are both* R.

$$f'(x) = \frac{d}{dx}\sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} c_n \frac{d}{dx}(x-a)^n$$

$$= \sum_{n=0}^{\infty} c_n \cdot n\,(x-a)^{n-1}$$

EXAMPLE 3. *Find a power series representation for $f(x)$ and determine the radius of convergence.*

(a) $f(x) = \frac{1}{(5-x)^2}$

Integrate w.r.t. x:

$$\int f(x)\,dx = \int \frac{dx}{(5-x)^2} \overset{\text{u-subst.}}{=} \underbrace{\frac{1}{5-x}}_{} + C$$

By Ex. 1a

$$= \sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}} + C \text{, where } |x|<5 \quad \boxed{R=5}$$

Return to $f(x)$:

$$\frac{d}{dx}\int f(x)\,dx = \frac{d}{dx}\sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}} + C$$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{5^{n+1}} \frac{d}{dx}(x^n) = \boxed{\sum_{n=\cancel{0}\,1}^{\infty} \frac{n\,x^{n-1}}{5^{n+1}}, \quad R=5}$$

(b) $f(x) = \ln(2+x)$

$$f'(x) = \frac{1}{2+x} = \frac{1}{2}\cdot\frac{1}{1+\frac{x}{2}} = \frac{1}{2}\cdot\frac{1}{1-\left(-\frac{x}{2}\right)}$$

Sum of geom. series

$$f'(x) = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n$$

where $\left|-\frac{x}{2}\right| < 1$

$|x| < 2$

$\boxed{R=2}$

$$f'(x) = \sum_{n=0}^{\infty}\frac{1}{2}\cdot\frac{(-1)^n x^n}{2^n}$$

$$f'(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^{n+1}}$$

Integrate

$$\underbrace{\int f'(x)\,dx}_{} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}\int x^n\,dx$$

$$\ln(2+x) = f(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{2^{n+1}(n+1)} + C, \quad R=2$$

To determine C we can plugin a point from the interval of convergence

$$\ln(2+x) = \underset{n=0}{\square}\,x + \underset{n=1}{\square}\,x^2 + \square\,x^3 + \ldots + C$$

$x=0 \Rightarrow \ln 2 = 0 + C \Rightarrow \boxed{C = \ln 2}$

$$\boxed{\ln(2+x) = \ln 2 + \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{2^{n+1}(n+1)}, \quad R=2}$$

(c) $f(x) = \arctan(2x)$

$$f'(x) = \frac{d}{dx}(\arctan(2x)) \overset{\text{Chain Rule}}{=} \frac{2}{1+(2x)^2} = 2\cdot\frac{1}{1+4x^2}$$

$$f'(x) = 2\cdot\frac{1}{1-(-4x^2)} = 2\sum_{n=0}^{\infty}(-4x^2)^n \quad \text{where } |-4x^2|<1$$

$$x^2 < \frac{1}{4}$$

$$|x| < \frac{1}{2}$$

$$\boxed{R=\frac{1}{2}}$$

$$f'(x) = \sum_{n=0}^{\infty} 2\cdot(-1)^n(2^2)^n(x^2)^n$$

$$f'(x) = \sum_{n=0}^{\infty}(-1)^n 2^{2n+1} x^{2n}, \quad R=\frac{1}{2}$$

Integrate: $\displaystyle\int f'(x)\,dx = \sum_{n=0}^{\infty}(-1)^n\cdot 2^{2n+1}\int x^{2n}\,dx$

$$\arctan(2x) = f(x) = \sum_{n=0}^{\infty}(-1)^n 2^{2n+1}\frac{x^{2n+1}}{2n+1} + C, \quad R=\frac{1}{2}$$

To find C plug in a point from interval of convergence (namely, the center of the series, $x=0$)

$$\arctan(2\cdot 0) = 0 + C \Rightarrow C=0$$

$$\boxed{\arctan 2x = \sum_{n=0}^{\infty}(-1)^n 2^{2n+1}\frac{x^{2n+1}}{2n+1}, \quad R=\frac{1}{2}}$$

EXAMPLE 4. *Use a power series decomposition to approximate the integral*

Means that we need $|R_n| \sim 10^{-7}$ or less

$$\int_0^{0.2} \underbrace{\frac{1}{1+x^4}}_{\text{sum of geom. series}} \mathrm{d}x \qquad r = -x^4$$

$\underbrace{\textit{to six decimal places.}}$

$|x|<1$

$$\int_0^{0.2} \frac{dx}{1+x^4} = \int_0^{1/5} \sum_{n=0}^{\infty} (-x^4)^n dx = \int_0^{1/5} \sum_{n=0}^{\infty} (-1)^n x^{4n} dx = \sum_{n=0}^{\infty} (-1)^n \int_0^{1/5} x^{4n} dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \left. \frac{x^{4n+1}}{4n+1} \right|_0^{1/5} = \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{4n+1} \cdot \frac{1}{5^{4n+1}}}_{\text{Alternating series}}$$

$$\int_0^{1/5} \frac{dx}{1+x^4} = \underset{n=0}{\frac{1}{5}} - \underset{n=1}{\frac{1}{5 \cdot 5^5}} + \underset{n=2}{\frac{1}{9 \cdot 5^9}} - \frac{1}{13 \cdot 5^{13}} + \dots$$

$S \approx S_0 \pm R_0$

$S \approx 0.2 \pm 6.4 \cdot 10^{-5}$

$R_0 = \frac{1}{5 \cdot 5^2} \approx 6.4 \cdot 10^{-5}$

$S \approx S_1 \pm R_1$

$S_1 = \frac{1}{5} - \frac{1}{5 \cdot 5^5} \approx \boxed{0.199936}$

$|R_1| \le \frac{1}{9 \cdot 5^9} \approx 5.7 \cdot 10^{-8}$