

10.7: Taylor and Maclaurin Series

Problem: Assume that a function $f(x)$ has a power series representation about $x = a$:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

and $f(x)$ has derivatives of every order. Find formula for c_n in terms of f .

Solution. We have

$$f(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + C_4(x-a)^4 + \dots$$

Putting $x = a$ in $f(x)$, we get $C_0 = f(a) = \frac{f^{(0)}(a)}{0!}$

$$f'(x) = 0 + C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + 4C_4(x-a)^3 + \dots$$

Substituting $x = a$ we have $C_1 = \frac{f'(a)}{1!}$

Similarly,

$$f''(x) = 2C_2 + 3 \cdot 2C_3(x-a) + 4 \cdot 3 \cdot C_4(x-a)^2 + \dots$$

then $x = a \Rightarrow f''(a) = 2C_2$ and $C_2 = \frac{f''(a)}{2!}$

$$f'''(x) = 3 \cdot 2C_3 + 4 \cdot 3 \cdot 2C_4(x-a) + \dots$$

then $x = a \Rightarrow f'''(a) = 3 \cdot 2C_3$ and $C_3 = \frac{f'''(a)}{3 \cdot 2 \cdot 1} = \frac{f'''(a)}{3!}$

Continuing in this manner, you can see the pattern:

$$f^{(n)}(a) = n! C_n \Rightarrow C_n = \frac{f^{(n)}(a)}{n!}$$

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

The Taylor series for $f(x)$ about $x = a$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n =$$

$$= \underbrace{f(a) + f'(a)(x-a)}_{\text{Linear approximation}} + \frac{f''(a)}{2!} (x-a)^2 + \frac{f'''(a)}{3!} (x-a)^3 + \dots$$

Quadratic approximation

Split the Taylor series as follows:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = \underbrace{\sum_{n=0}^N \frac{f^{(n)}(a)}{n!} (x-a)^n}_{\substack{T_N(x) \\ N\text{-th degree} \\ \text{Taylor polynomial}}} + \underbrace{\sum_{n=N+1}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n}_{\substack{R_N(x) \\ \text{Remainder}}}$$

partial sum

THEOREM 1. If $\lim_{n \rightarrow \infty} R_N(x) = 0$ when $|x-a| < R$ then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n, \quad |x-a| < R.$$

REMARK 2. In all examples that we will be looking at, we assume that $f(x)$ has a power series expansion, i.e.

$\lim_{n \rightarrow \infty} R_N(x) = 0$ for some R . (This means you don't need to show it.)

EXAMPLE 3. Given that function f has power series expansion (i.e. Taylor series) centered at 4. Find this expansion and its radius of convergence if it is given that

$$f^{(n)}(4) = \frac{(-1)^{n-1} \cdot 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-3)}{2^{3n-1} n}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(4)}{n!} (x-4)^n = \sum_{n=0}^{\infty} \frac{(-1)^{n-1} \cdot 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-3)}{2^{3n-1} \cdot n \cdot n!} (x-4)^n$$

Taylor series centered at $x=4$

Ratio Test for R :

$$|a_n| = \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-3) |x-4|^n}{2^{3n-1} n \cdot n!}$$

$$|a_{n+1}| = \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2(n+1)-3) |x-4|^{n+1}}{2^{3(n+1)-1} (n+1) (n+1)!} = \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-3) \cdot (2n-1) |x-4|^{n+1}}{2^{3n+2} (n+1) (n+1)!}$$

$$L = \lim_{n \rightarrow \infty} |a_{n+1}| \cdot \frac{1}{|a_n|} = \lim_{n \rightarrow \infty} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-3) (2n-1) |x-4|^{n+1}}{2^{3n-1} \cdot 2^3 \cdot (n+1)^2 n!} \cdot \frac{2^{3n-1} n \cdot n!}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-3) |x-4|^n}$$

$$= \frac{|x-4|}{8} \lim_{n \rightarrow \infty} \frac{(2n-1)n}{(n+1)^2} = \frac{|x-4|}{8} \cdot 2 = \frac{|x-4|}{4} < 1$$

$$|x-4| < 4$$

$$R = 4$$

EXAMPLE 4. Find Taylor series for $f(x) = e^{3x}$ at $x = 1$. What is the associated radius of convergence?

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} (x-1)^n$$

$$f(x) = e^{3x}$$

$$f'(x) = 3e^{3x}$$

$$f''(x) = 3 \cdot 3e^{3x} = 3^2 e^{3x}$$

⋮

$$f^{(n)}(x) = 3^n e^{3x} \Rightarrow f^{(n)}(1) = 3^n e^3$$

$$f(x) = \sum_{n=0}^{\infty} \frac{3^n e^3}{n!} (x-1)^n$$

Radius of conv. by Ratio Test

$$L = \lim_{n \rightarrow \infty} |a_{n+1}| \cdot \frac{1}{|a_n|} = \lim_{n \rightarrow \infty} \frac{3^{n+1} e^3 |x-1|^{n+1}}{(n+1)! \cdot n!} \cdot \frac{n!}{3^n e^3 |x-1|^n}$$

$$= 3|x-1| \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0 < 1 \text{ for all } x$$

$$R = \infty$$

EXAMPLE 5. Find Taylor series for $f(x) = \ln x$ at $x = 1$. What is the associated radius of convergence?

$f(x) = \ln x$	$x=1$	$f(1) = 0 = 0!$	
$f'(x) = \frac{1}{x}$		$f'(1) = 1 = 0!$	$f^{(n)}(1) = (-1)^{n+1} (n-1)!$
$f''(x) = -\frac{1}{x^2}$		$f''(1) = -1 = -1!$	
$f'''(x) = \frac{2}{x^3}$		$f'''(1) = 2 = 2!$	
$f^{(4)}(x) = -\frac{2 \cdot 3}{x^4}$		$f^{(4)}(1) = -2 \cdot 3 = -3!$	
$f^{(5)}(x) = \frac{2 \cdot 3 \cdot 4}{x^5}$		$f^{(5)}(1) = 2 \cdot 3 \cdot 4 = 4!$	
		$f^{(6)}(1) = -2 \cdot 3 \cdot 4 \cdot 5 = -5!$	
		<u>Note</u> $f^{(n)}(x) = \frac{(-1)^{n+1} (n-1)!}{x^n}$	

$$f(x) = \sum_{n=0}^{\infty} C_n (x-1)^n$$

$$C_n = \frac{f^{(n)}(1)}{n!} = \begin{cases} 0, & n=0 \\ \frac{(-1)^{n+1} (n-1)!}{\underbrace{n!}_{(n-1)!n}} = \frac{(-1)^{n+1}}{n}, & n \geq 1 \end{cases}$$

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n$$

Ratio Test:

$$L = \lim_{n \rightarrow \infty} |a_{n+1}| \cdot \frac{1}{|a_n|} = \lim_{n \rightarrow \infty} \frac{|x-1|^{n+1}}{n+1} \cdot \frac{n}{|x-1|^n}$$

$$L = |x-1| \lim_{n \rightarrow \infty} \frac{n}{n+1} = |x-1| < 1$$

$$\Downarrow$$

$R = 1$

EXAMPLE 6. Find Taylor series for $\ln(1+x)$ centered at $x=0$. What is the associated radius of convergence?

By Example 5 we have

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n, \quad |x-1| < 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (1+x-1)^n, \quad |1+x-1| < 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \underbrace{\frac{(-1)^{n+1}}{n}}_{=c_n} x^n, \quad |x| < 1 \Rightarrow \boxed{R=1}$$

Taylor series centered at zero = **Maclaurin series**

Question $f(x) = \ln(1+x)$

What is $f^{(100)}(0)$?

$$c_n = \frac{f^{(n)}(0)}{n!} \Rightarrow c_{100} = \frac{f^{(100)}(0)}{100!}$$

$$f^{(100)}(0) = 100! c_{100} = \frac{100! (-1)^{100+1}}{100}$$

$$f^{(100)}(0) = -99!$$

The Maclaurin series is the Taylor series about $x = 0$ (i.e. $a=0$):

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

EXAMPLE 7. Find the Maclaurin series for $f(x)$:

(a) $f(x) = e^x$ $f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = e^0 = 1$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad R = \infty \quad (\text{check it!})$$

(b) $f(x) = e^{-x}$

using (a): $e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$

(c) $f(x) = x^5 e^{-2x^2}$ Use (a):

$$f(x) = x^5 \cdot \sum_{n=0}^{\infty} \frac{(-2x^2)^n}{n!} = \sum_{n=0}^{\infty} x^5 \frac{(-2)^n x^{2n}}{n!}$$

$$x^5 e^{-2x^2} = \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} x^{2n+5}$$

EXAMPLE 8. Find the Maclaurin series for $f(x)$:

(a) $f(x) = \cos x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

	$x=0$
$f(x) = \cos x$	1
$f'(x) = -\sin x$	0
$f''(x) = -\cos x$	-1
$f'''(x) = \sin x$	0
$f^{(4)}(x) = \cos x$	1

cycle

odd derivatives

$$f^{(n)}(0) = \begin{cases} 0, & n = 2k+1 \text{ (or } n \text{ is odd)} \\ (-1)^k, & n = 2k \text{ (or } n \text{ is even)} \end{cases}$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

Use Ratio Test to find radius of convergence

$$L = \lim_{k \rightarrow \infty} |a_{k+1}| \cdot \frac{1}{|a_k|} = \lim_{k \rightarrow \infty} \frac{|x|^{2(k+1)}}{(2(k+1))!} \cdot \frac{(2k)!}{|x|^{2k}}$$

$$= \lim_{k \rightarrow \infty} \frac{|x|^{2k+2} \cdot (2k)!}{(2k)! \cdot (2k+1)(2k+2) |x|^{2k}} = |x|^2 \lim_{k \rightarrow \infty} \frac{1}{(2k+1)(2k+2)} = 0 < 1$$

for all x

$$R = \infty$$

(b) $f(x) = \sin x$

Way 1 Use the same method as in item (a)

Way 2 Use termwise integration or differentiation

$$\begin{aligned} \sin x &= \int \cos x \, dx \stackrel{(a)}{=} \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \, dx \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \int x^{2n} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)! \cdot 2n+1} x^{2n+1} + C \end{aligned}$$

To determine C plugin the center of the series: $x=0$

$$\sin 0 = 0 + C \Rightarrow C = 0$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad R = \infty.$$

Remark Decomposition of $\sin x$ contains only odd polynomials: x, x^3, x^5, \dots

Decomp. of $\cos x$ contains only even polynomials: $1, x^2, x^4, \dots$

This happen because $\sin x$ is odd function
 $\sin(-x) = -\sin x$

and $\cos x$ is even function
 $\cos(-x) = \cos x$

Known Mclaurin series and their intervals of convergence you must have memorized:

geom. s. r.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \quad (1, 1)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad (-\infty, \infty)$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad (-\infty, \infty)$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad (-\infty, \infty)$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \quad [-1, 1]$$

intervals of convergence

↓
 very useful for limit calculations
 and for Alternating Series Remainder
 estimate

Use the previous table

EXAMPLE 9. Find the sum of the series:

$$(a) \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n} (2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{\pi}{3}\right)^{2n} = \cos \frac{\pi}{3} = \frac{1}{2}$$

$$(b) \sum_{n=0}^{\infty} \frac{2012^n}{n!} = e^{2012}$$

$$(c) \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{2n+1}$$

$$= \arctan(x^2)$$

EXAMPLE 10. (a) Determine Maclaurin Series for $\int \frac{\sin x}{x} dx$

Use the table:

$$\frac{\sin x}{x} = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n}$$

We don't know anti-derivative

Use termwise integration:

$$\int \frac{\sin x}{x} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int x^{2n} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{x^{2n+1}}{2n+1}$$

(b) Evaluate $\int_0^1 \frac{\sin x}{x} dx$ correct to within an error of 0.001 = $\frac{1}{1000}$

In other words, approximate $\int_0^1 \frac{\sin x}{x} dx \approx S_n$

such that $|R_n| \leq 0.001$.

find n such that $|R_n| \leq 0.001$.

$$\int_0^1 \frac{\sin x}{x} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! (2n+1)} x^{2n+1} \Big|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! (2n+1)}$$

Alternating series

$$= 1 - \frac{1}{3! \cdot 3} + \frac{1}{5! \cdot 5} - \frac{1}{7! \cdot 7} + \dots$$

$$\int_0^1 \frac{\sin x}{x} dx \approx S_0 = 1 \Rightarrow |R_0| \leq \frac{1}{3! \cdot 3} = \frac{1}{6 \cdot 3} = \frac{1}{18} > 0.001 = \frac{1}{1000}$$

$$\int_0^1 \frac{\sin x}{x} dx \approx S_1 = 1 - \frac{1}{3! \cdot 3} \Rightarrow |R_1| = \frac{1}{5! \cdot 5} = \frac{1}{120 \cdot 5} = \frac{1}{600} > \frac{1}{1000}$$

$$\int_0^1 \frac{\sin x}{x} dx \approx S_2 = \boxed{1 - \frac{1}{3! \cdot 3} + \frac{1}{5! \cdot 5} \approx 0.9461}$$

$$\Rightarrow |R_2| = \frac{1}{7! \cdot 7} = \frac{1}{35280} < \frac{1}{1000} \text{ 😊😊😊}$$

EXAMPLE 11. Use series to find the limit: $\lim_{x \rightarrow 0} \frac{1 - \cos x}{1 + x - e^x}$

We know Maclaurin series for $\cos x$ and e^x :

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{1 + x - e^x} = \frac{\cancel{1} - \left(\cancel{1} - \frac{x^2}{2} + \frac{x^4}{4!} + \dots \right)}{\cancel{1} + \cancel{x} - \left(\cancel{1} + \cancel{x} + \frac{x^2}{2} + \frac{x^3}{3!} + \dots \right)}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{x^2}{2} - \frac{x^4}{4!} + \dots}{\frac{x^2}{2} + \frac{x^3}{3!} + \dots} = \lim_{x \rightarrow 0} \frac{\cancel{x^2} \left(\frac{1}{2} - \frac{x^2}{4!} + \dots \right)}{\cancel{x^2} \left(\frac{1}{2} + \frac{x}{3!} + \dots \right)}$$

$$= - \frac{\frac{1}{2} - 0 + \dots}{\frac{1}{2} + 0 + 0 + \dots} = \boxed{-1}$$