10.9: Applications of Taylor Polynomials

Recall that the Nth degree Taylor Polynomial is defined by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \underbrace{\sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n}_{\substack{T_N(x) \\ N-\text{th degree} \\ \textbf{Taylor polynomial}}} + \underbrace{\sum_{n=N+1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}_{\substack{R_N(x) \\ \textbf{Remainder}}}$$

partial sum

EXAMPLE 1. *For* $f(x) = \cos x$ *find* $T_N(x)$ *for* $N = 0, 1, 2, \ldots, 8$ at $x=0$.

Find n-th degree Taylor polynomials $T_0, T_1, \ldots, T_8$

We know

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$\cos x = 1 \cdot x^0 + 0 \cdot x - \frac{x^2}{2} + 0 \cdot x^3 + \frac{x^4}{24} + 0 \cdot x^5 - \frac{x^6}{720} + 0 \cdot x^7 + \frac{x^8}{40320} + \cdots$$

$T_0(x)$

$T_1(x)$ linear approximation

$T_2(x)$ Quadratic approximation

$T_3(x)$

$T_4(x)$

$T_5(x)$

$$T_0(x) = 1$$

$$T_1(x) = 1$$

$$T_2(x) = 1 - \frac{x^2}{2}$$

$$T_3(x) = 1 - \frac{x^2}{2}$$

$$T_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

$$T_5(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

$$T_6(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$$

$$T_7(x) = T_6(x)$$

$$T_8(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320}$$

REMARK 2. As the degree of the Taylor polynomial increases, it starts to look more and more like the function itself (and thus, it approximates the function better).

Example 1'. Find Taylor polynomials T_N, $N = 0, 1, 2, 3, 4$

for e^x at $x = 0$.

$$e^x = \underbrace{1}_{T_0(x)} + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$T_0(x) = 1$$

$$T_1(x) = 1 + x$$

$$T_2(x) = 1 + x + \frac{x^2}{2}$$

$$T_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

$$T_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

REMARK 3. The first degree Taylor polynomial

$$T_1(x) = f(a) + f'(a)(x - a)$$

is the same as linear approximation of f at $x = a$.

In general, $f(x)$ is the sum of its Taylor series if $T_N(x) \to f(x)$ as $n \to \infty$. So, $T_N(x)$ can be used as an approximation:

$$f(x) \approx T_N(x).$$

How to estimate the Remainder $|R_N(x)| = |f(x) - T_N(x)|$? on interval I

- Use graph of $R_N(x)$. $\max_{I} |R_N(x)|$
- If the series happens to be an alternating series, you can use the Alternating Series Theorem.
- In all cases you can use **Taylor's Inequality**:

$$|R_N(x)| \le \frac{M}{(N+1)!} \underbrace{|x-a|}_{\text{replace by } \max|x-a|}{}^{N+1}$$

where $|f^{(N+1)}(x)| \le M$ for all x in an interval containing a.

$M = \max |f^{(N+1)}(x)|$

M is absolute max of $|f^{(N+1)}(x)|$ on the given interval

EXAMPLE 4. *Let* $f(x) = e^{x^2}$.

(a) *Approximate* $f(x)$ *by a Taylor polynomial of degree 3 at* $a = 0$.

We know $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots \frac{x^n}{n!} + \dots$

and then $e^{x^2} = 1 + x^2 + \frac{(x^2)^2}{2} + \frac{(x^2)^3}{6} + \frac{(x^2)^4}{24} + \dots + \frac{(x^2)^n}{n!} + \dots$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24} + \dots + \frac{x^{2n}}{n!} + \dots$$

$$e^{x^2} = \underbrace{1 + 0\cdot x + x^2 + 0\cdot x^3}_{T_3(x)} + \frac{x^4}{2} + \dots$$

$T_3(x) = 1 + x^2$

Note that $T_2(x) = 1 + x^2$

and also $e^{x^2} \approx 1 + x^2$

(b) *How accurate is this approximation when* $0 \le x \le 0.1$

Method 1

Find $\max\limits_{0 \le x \le 0.1} |R_3(x)|$ absolute extremum problem

$$R_3(x) = f(x) - T_3(x) = e^{x^2} - (1 + x^2)$$

$$R_3(x) = e^{x^2} - 1 - x^2$$

To find absolute extremum find critical numbers on $[0, 0.1]$ and plug in + end points

$$R_3'(x) = 2x e^{x^2} - 2x = 2x(e^{x^2} - 1) = 0$$

$x = 0$ OR $e^{x^2} - 1 = 0$, $e^{x^2} = 1$, $x = 0$

Note that $x = 0$ is in $[0, 0.1]$

crit. number, end point: $R_3(0) = e^{0^2} - 1 - 0^2 = \boxed{0}$

another end point: $R_3(0.1) = e^{0.1^2} - 1 - 0.1^2 \approx \boxed{5 \cdot 10^{-9}}$

$$\max_{[0, 0.1]} |R_3(x)| \approx 5 \cdot 10^{-9}$$

$\Downarrow$

$|R_3(x)| \le 5 \cdot 10^{-9}$ for all $0 \le x \le 0.1$

Method 2 Use Taylor Inequality

given interval

(b) *How accurate is this approximation when* $0 \le x \le 0.1$

Method 2: Use Taylor Inequality: $|R_N(x)| \le \frac{M}{(N+1)!}|x-a|^{N+1}$

where $M = \max|f^{(N+1)}(x)|$

In our case we have $a=0$, $f(x)=e^{x^2}$, $N=3$

$$|R_3(x)| \le \frac{M}{(3+1)!}|x-0|^{3+1} = \frac{M}{4!}|x|^4 = \frac{M}{24}\cdot\max_{0\le x\le 0.1}|x|^4 = \frac{M}{24}\cdot 0.1^4$$

where $M = \max_{0\le x\le 0.1}|f^{(4)}(x)|$

It remains to find M (solve absolute extremum problem for that. Review Chapter 5 if necessary).

$f(x) = e^{x^2}$

$f'(x) = 2xe^{x^2}$

$f''(x) = 2[e^{x^2} + 2x^2e^{x^2}]$

$f'''(x) = 2[2xe^{x^2} + 4xe^{x^2} + 4x^3e^{x^2}] = 2[6xe^{x^2} + 4x^3e^{x^2}]$

$f^{(4)}(x) = 2[6e^{x^2} + 12x^2e^{x^2} + 12x^2e^{x^2} + 8x^4e^{x^2}] =$

$= 2e^{x^2}[6 + 24x^2 + 8x^4] = h(x)$

Find abs. max and min of $h(x)$ on $[0, 0.1]$

$h'(x) = 4xe^{x^2}[6+24x^2+8x^4] + 2e^{x^2}[48x + 32x^3] > 0$ on $[0, 0.1]$ $\Rightarrow$

$\Rightarrow$ $h(x)$ is monotonically increasing on $[0, 0.1]$

$\Rightarrow$ absolute extemum is attained at end points

$h(0) = 2\cdot e^0\cdot 6 = 12$

$h(0.1) = 2e^{0.01}(6+24\cdot 0.1^2 + 8\cdot 0.1^4) \approx 12.607$

$M = \max\{|h(0)|, |h(0.1)|\} \approx 12.607$

Finally,

$$|R_3(x)| \le \frac{M\cdot(0.1)^4}{24} \approx \frac{12.607\cdot 10^{-4}}{24} \approx \boxed{5.3\cdot 10}$$

EXAMPLE 5. *Find $T_2(x)$ for $f(x) = \cos x$ at $x = \pi/4$. How accurate this approximation when $\pi/6 \le x \le 2\pi/3$.*

Taylor series $\quad \underbrace{f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2}_{T_2(x) = \text{2nd degree Taylor polynomial}} + \frac{f'''(a)}{3!}(x-a)^3 + \dots$

In our case

$a = \frac{\pi}{4}, \quad f(x) = \cos x \Rightarrow f'(x) = -\sin x \Rightarrow f''(x) = -\cos x$

$f(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}, \quad f'(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}, \quad f''(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$

$$T_2(x) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2$$

Estimate $R_2(x)$ for $\frac{\pi}{6} \le x \le \frac{2\pi}{3}$. Apply Taylor inequality for $N = 2$:

$$|R_2(x)| \le \frac{M}{(2+1)!}|x-a|^{2+1} = \frac{M}{6}\left|x - \frac{\pi}{4}\right|^3 \le \frac{M}{6}\cdot\left(\frac{5\pi}{12}\right)^3$$

$\frac{\pi}{6} \le x \le \frac{2\pi}{3}$

$\frac{\pi}{6} - \frac{\pi}{4} \le x - \frac{\pi}{4} \le \frac{2\pi}{3} - \frac{\pi}{4}$

$-\frac{\pi}{12} \le x - \frac{\pi}{4} \le \frac{5\pi}{12}$

$\left|x - \frac{\pi}{4}\right| \le \frac{5\pi}{12}$

where $\quad M = \max_{\frac{\pi}{6} \le x \le \frac{2\pi}{3}} |f^{(3)}(x)|$

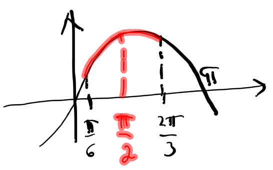
Determine M:

$f^{(3)}(x) = (f''(x))' = (-\cos x)' = \sin x = h(x)$

Find abs. extremum of $h(x) = \sin x$ on $[\frac{\pi}{6}, \frac{2\pi}{3}]$

use graph

critical points + end points



$\max_{\frac{\pi}{6} \le x \le \frac{2\pi}{3}} h(x) = h(\frac{\pi}{2}) = \sin\frac{\pi}{2} = 1, \quad h(x) = |h(x)|$

$\Rightarrow M = \max_{\frac{\pi}{6} \le x \le \frac{2\pi}{3}} |h(x)| = 1$

Finally,

$|R_2(x)| \le \frac{1}{6}\left(\frac{5\pi}{12}\right)^3 \approx \boxed{0.373822}$

EXAMPLE 6. *How many terms of the Maclaurin series for $f(x) = \ln(x+1)$ do you need to use to estimate $\ln 1.2$ to within 0.001.*

We already know (see Sections 10.7, 10.5) that

$$f(x) = \ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

$$\ln 1.2 = f(0.2) = f\left(\frac{1}{5}\right) = \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{1}{5^{n+1}(n+1)}}_{\text{Alternating series}}$$

Find n such that $|R_n| \lesssim 0.001 = \frac{1}{1000}$

Note that $|R_n| \leq b_{n+1} = \frac{1}{5^{n+2}(n+2)} \leq \frac{1}{1000}$

However it is difficult to get general solution for the last inequality.

Thus,

$$\ln 1.2 = \underset{n=0}{\frac{1}{50}} - \underset{n=1}{\frac{1}{3\cdot 125}} + \underset{n=2}{\frac{1}{5^4\cdot 4}} - \underset{n=3}{\frac{1}{5^5 \cdot 5}} + \dots$$

$n=0$ $\quad |R_0| \leq b_1 = \frac{1}{50} > 0.001$

$n=1$ $\quad |R_1| \leq b_2 = \frac{1}{3\cdot 125} > 0.001 = \frac{1}{1000}$

$n=2$ $\quad |R_2| \leq b_3 = \frac{1}{5^4 \cdot 4} = \frac{1}{625\cdot 4} < \frac{1}{1000} = 0.001$

We need 3 terms of Maclaurin series for the desired accuracy:

$$\ln(1.2) \approx \frac{1}{5} - \frac{1}{50} + \frac{1}{3\cdot 125} \approx \boxed{0.18266}$$

Note that using calculator we get

$$\ln 1.2 \approx 0.182321$$