Dot product $\Rightarrow$ a number

11.3: Cross product $\Rightarrow$ a vector

REVIEW: Determinant of 2×2 and 3×3 matrices.

A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2$$

$$\begin{vmatrix} 0 & 0 \\ 3 & 4 \end{vmatrix} = 0 \cdot 4 - 0 \cdot 3 = 0$$

$$\begin{vmatrix} 0 & 1 \\ 0 & -2 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4 - 4 = 0$$

$$\begin{vmatrix} a & b \\ ka & kb \end{vmatrix} = akb - bka = 0$$

$(-1)^{i+j}$

A **determinant of order 3** is defined by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \overset{(-1)^{1+1}}{a_1} \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \overset{(-1)^{1+2}}{-} a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \overset{(-1)^{1+3}}{+} a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$$

or copy the first two columns onto the end and then multiply along each diagonal and add those that move from left to right and subtract those that move from right to left:

$$\begin{array}{ccc}+ & + & + & & \\ \end{array}$$
$$\left|\begin{array}{ccc|cc} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \\ c_1 & c_2 & c_3 & c_1 & c_2 \end{array}\right. = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3$$

• **THE CROSS PRODUCT IN COMPONENT FORM:**

Given $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$. Then the cross product is :

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

REMARK 1. The cross product requires both of the vectors to be three dimensional vectors.

REMARK 2. The result of a dot product is a number and the result of a cross product is a VECTOR!!!

To remember the cross product component formula use the fact that the cross product can be represented as the determinant of order 3:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \vec{i}(a_2 b_3 - b_2 a_3) - \vec{j}(a_1 b_3 - a_3 b_1) + \vec{k}(a_1 b_2 - a_2 b_1)$$

EXAMPLE 3. *If* $\mathbf{a} = \langle -2, 1, 1\rangle$, $\mathbf{b} = \langle 3, 5, 0\rangle$, *and* $\mathbf{c} = \langle -4, 2, 2\rangle$ *compute each of the following:*

(a) $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 1 & 1 \\ 3 & 5 & 0 \end{vmatrix} = \vec{i}\begin{vmatrix} 1 & 1 \\ 5 & 0 \end{vmatrix} - \vec{j}\begin{vmatrix} -2 & 1 \\ 3 & 0 \end{vmatrix} + \vec{k}\begin{vmatrix} -2 & 1 \\ 3 & 5 \end{vmatrix}$

$= \vec{i}\,(1\cdot 0 - 1\cdot 5) - \vec{j}\,(-2\cdot 0 - 1\cdot 3) + \vec{k}\,(-2\cdot 5 - 1\cdot 3)$

$= -5\vec{i} + 3\vec{j} - 13\vec{k}$

(b) $\mathbf{b}\times\mathbf{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 5 & 0 \\ -2 & 1 & 1 \end{vmatrix} = \vec{i}\begin{vmatrix} 5 & 0 \\ 1 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 3 & 5 \\ -2 & 1 \end{vmatrix}$

$= \boxed{5\vec{i} - 3\vec{j} + 13k} = -\vec{a}\times\vec{b}$

(c) $\mathbf{a}\times(5\mathbf{b}) = 5\cdot(-5\vec{i} + 3\vec{j} - 13\vec{k}) = -25\vec{i} + 15\vec{j} - 65\vec{k}$ (check it)

(d) $\mathbf{a}\times\mathbf{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{vmatrix} = \vec{0} = \langle 0,0,0\rangle$

(e) $\mathbf{a}\times\mathbf{c}$

Note that $\vec{c} = 2\vec{a} \Rightarrow \vec{a}\times\vec{c} = \vec{a}\times(2\vec{a}) = 2(\vec{a}\times\vec{a}) = \vec{0}$

Properties:

$$\mathbf{a} \times \mathbf{a} = \vec{0}$$

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

$$(\alpha \mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (\alpha \mathbf{b}) = \alpha(\mathbf{a} \times \mathbf{b}), \quad \alpha \in \mathbb{R}$$

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$$

EXAMPLE 4. *Show that if* $\mathbf{a}$ *and* $\mathbf{b}$ *are parallel then* $\mathbf{a} \times \mathbf{b} = \vec{0}$.

$\Downarrow$

there exists a constant c such that

$$\vec{b} = c\vec{a}$$

$\Downarrow$

$$\vec{a} \times \vec{b} = \vec{a} \times (c\vec{a}) = c\,(\vec{a} \times \vec{a}) = c \cdot \vec{0} = \vec{0}$$

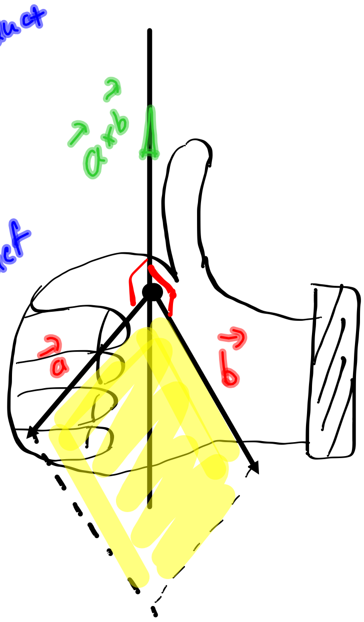
- **GEOMETRIC INTERPRETATION OF THE CROSS PRODUCT:**

Let θ be the angle between the two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, $0 \le \theta \le \pi$. Then

1. $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| \cdot |\mathbf{b}| \sin\theta =$ *the area of the parallelogram determined by* $\mathbf{a}$ *and* $\mathbf{b}$*;*
2. $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$;
3. the direction of $\mathbf{a} \times \mathbf{b}$ is determined by "right hand" rule: *if the fingers of your right hand curl through the angle* θ *from* $\mathbf{a}$ *to* $\mathbf{b}$*, then your thumb points in the direction of* $\mathbf{a} \times \mathbf{b}$.

direction of cross product

magnitude of cross product



FACT[1]: $\mathbf{a} \| \mathbf{b} \quad \Leftrightarrow \quad \mathbf{a} \times \mathbf{b} = 0.$

By Ex. 4 If $\vec{a} \parallel \vec{b} \Rightarrow \vec{a} \times \vec{b} = \vec{0}$

Conversely, if $\vec{a} \times \vec{b} = \vec{0} \Rightarrow |\vec{a} \times \vec{b}| = 0$

$$\Downarrow$$

$|\vec{a}| \cdot |\vec{b}| \cdot \sin\theta = 0$ }

If $\vec{a}, \vec{b}$ are nonzero vectors } $\Rightarrow$

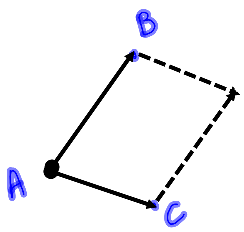
$\Rightarrow \sin\theta = 0 \Rightarrow \theta = 0, \pi$

$$\Downarrow$$

$\vec{a} \parallel \vec{b}$.

EXAMPLE 5. *Given the points* $A(1,0,0)$, $B(1,1,1)$ *and* $C(2,-1,3)$.*Find*

(a) *the area of the triangle determined by these points.*



$$A_\Delta = \frac{1}{2} A_{\square} = \frac{1}{2} |\vec{AB} \times \vec{AC}|$$

$$\vec{AB} = \langle 1-1, 1-0, 1-0 \rangle = \langle 0, 1, 1 \rangle$$

$$\vec{AC} = \langle 2-1, -1-0, 3-0 \rangle = \langle 1, -1, 3 \rangle$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & 1 \\ 1 & -1 & 3 \end{vmatrix} = \hat{\imath} \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix} + \hat{k} \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} =$$

$$= \hat{\imath}\,(1\cdot 3 - 1(-1)) - \hat{\jmath}\,(0\cdot 3 - 1\cdot 1) + \hat{k}\,(0\cdot(-1) - 1\cdot 1) =$$

$$= \langle 4, 1, -1 \rangle$$

$$|\vec{AB} \times \vec{AC}| = |\langle 4, 1, -1 \rangle| = \sqrt{4^2 + 1^2 + (-1)^2} = \sqrt{18} = \sqrt{2\cdot 9} = 3\sqrt{2}$$

$$A_\Delta = \frac{1}{2} \cdot 3\sqrt{2} = \boxed{\frac{3\sqrt{2}}{2}}$$

(b) *Find a unit vector* $\hat{\mathbf{n}}$ *orthogonal to the plane that contains the points* A, B, C.

$\Downarrow$

$\hat{n} \perp$ to any vector parallel to that plane

In particular, $\hat{n} \perp \vec{AB}$ & $\hat{n} \perp \vec{AC}$

Thus $\hat{n} \parallel (\vec{AB} \times \vec{AC})$

$$\hat{n} = \pm \frac{\vec{AB} \times \vec{AC}}{|\vec{AB} \times \vec{AC}|} \overset{(a)}{=} \pm \frac{\langle 4, 1, -1 \rangle}{3\sqrt{2}}$$

$$\hat{n} = \pm \left(\frac{4}{3\sqrt{2}} \hat{\imath} + \frac{1}{3\sqrt{2}} \hat{\jmath} - \frac{1}{3\sqrt{2}} \hat{k} \right)$$

• **SCALAR TRIPLE PRODUCT** of the vectors **a**, **b**, **c** is

$$\mathbf{a}\cdot\underbrace{(\mathbf{b}\times\mathbf{c})}^{\vec{d}}. \Rightarrow \text{number}$$

Note that the scalar triple product is a NUMBER.

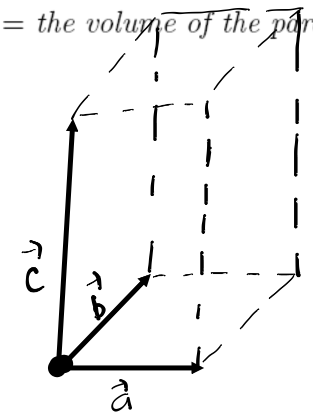
FACTS:

1. $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$

2. If $\mathbf{a} = \langle a_1, a_2, a_3\rangle$, $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ and $\mathbf{c} = \langle c_1, c_2, c_3\rangle$ then $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

3. $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$ = *the volume of the parallelepiped determined by* **a**, **b**, **c**.

abs. value



EXAMPLE 6. *Determine if the vectors*

$$\mathbf{a} = \langle 0, -9, 18 \rangle, \quad \mathbf{b} = \langle 1, 4, -7 \rangle, \quad \mathbf{c} = \langle 2, -1, 4 \rangle$$

are coplanar.[2] = i.e. they lie in the same plane

$\vec{a}, \vec{b}, \vec{c}$ coplanar $\Leftrightarrow$ $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 0 & -9 & 18 \\ 1 & 4 & -7 \\ 2 & -1 & 4 \end{vmatrix} = 0 \begin{vmatrix} \ldots \end{vmatrix} - (-9) \begin{vmatrix} 1 & -7 \\ 2 & 4 \end{vmatrix} + 18 \begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix}$$

$$= 9 \cdot (4 - (-14)) + 18(-1 - 8)$$

$$= 9 \cdot 18 - 18 \cdot 9 = 0 \Rightarrow \text{YES!}$$