## 6.5: The substitution rule

$=$ antiderivative

The Substitution Rule for indefinite integrals: If $u = g(x)$ is a differentiable function, then

$$\int f(\underbrace{g(x)}_{u})\underbrace{g'(x)\,\mathrm{d}x}_{du} = \int f(u)\,\mathrm{d}u \qquad \text{u-sub}$$

Note that $\boxed{\mathrm{d}u = g'(x)\mathrm{d}x}$ is differential.

$$\int \Big(\underbrace{\frac{x^2}{2} + 3}_{u}\Big)^7 \underbrace{x\,dx}_{du}$$

$\frac{x^2}{2} + 3 = u$

$du = u'(x)dx = xdx$

$$\int u^7 du = \frac{u^8}{8} + C = \frac{\left(\frac{x^2}{2}+3\right)^8}{8} + C$$

$$\int (\underbrace{x^2+3}_{u})^7 \underbrace{x\,dx}_{\frac{du}{2}} = \ldots$$

$du = 2x\,dx$

The correct substitution depends on the integral:

$$\int (g(x))^n g'(x)\,\mathrm{d}x \qquad \int e^{g(x)} g'(x)\,\mathrm{d}x \qquad \int \frac{g'(x)}{g(x)}\,\mathrm{d}x$$

$$\int \cos(g(x))g'(x)\,\mathrm{d}x \qquad \int \sin(g(x))g'(x)\,\mathrm{d}x \qquad \int \sec^2(g(x))g'(x)\,\mathrm{d}x$$

$$\int \sec(g(x))\tan(g(x))g'(x)\,\mathrm{d}x \qquad \int \csc^2(g(x))g'(x)\,\mathrm{d}x \qquad \int \csc(g(x))\cot(g(x))g'(x)\,\mathrm{d}x$$

$u = g(x)$

$du = g'(x)dx$

$$\int u^n du = \frac{u^{n+1}}{n+1} + C = \frac{(g(x))^{n+1}}{n+1} + C$$

EXAMPLE 1. *Evaluate each of the following integrals:*

1. $\int x(x^2+2012)^{2012}\,dx = \int \underbrace{(x^2+2012)}_{u}{}^{2012} \underbrace{x\,dx}_{\frac{du}{2}} = \int u^{2012} \frac{du}{2}$

$u = x^2+2012$

$du = u'(x)dx = 2x\,dx \Rightarrow x\,dx = \frac{du}{2}$

$= \frac{1}{2}\int u^{2012}\,du$

$= \frac{1}{2}\frac{u^{2013}}{2013} + C$

$= \frac{u^{2013}}{4026} + C$

$= \frac{(x^2+2012)^{2013}}{4026} + C$

2. $\int 18x^2\sqrt[4]{6x^3+5}\,dx = \int \sqrt[4]{6x^3+5}\;18x^2dx$

$u = 6x^3+5 \Rightarrow du = 18x^2dx$

$= \int \sqrt[4]{u}\,du = \int u^{\frac{1}{4}}du = \frac{u^{\frac{1}{4}+1}}{\frac{1}{4}+1} + C = \frac{u^{5/4}}{5/4} + C$

$= \frac{4}{5}u^{5/4} + C = \frac{4(6x^3+5)^{5/4}}{5} + C$

3. $\int \cos(3x)\sin^{10}(3x)\,dx = \int \underbrace{[\sin(3x)]^{10}}_{u^{10}} \underbrace{\cos 3x\,dx}_{\frac{du}{3}} = \frac{1}{3}\int u^{10}du$

$u = \sin(3x)$

$du = (\sin(3x))'dx$

$du = 3\cos(3x)dx \Rightarrow \cos(3x)dx = \frac{du}{3}$

$= \frac{1}{3}\frac{u^{11}}{11} + C$

$= \frac{u^{11}}{33} + C = \frac{\sin^{11}(3x)}{33} + C$

4. $\int (8x-1)e^{4x^2-x}\,dx = \int e^u\,du = e^u + C = e^{4x^2-x} + C$

$u = 4x^2-x$

$du = \underbrace{(8x-1)}_{u'(x)}dx$

5. $\displaystyle\int \sec^2(5y)(5-\tan(5y))^5\,dy = \int \underbrace{(5-\tan(5y))^5}_{u^5} \underbrace{\sec^2(5y)\,dy}_{-\frac{du}{5}}$

$u = 5-\tan(5y)$

$du = -5\sec^2(5y)$

$$= -\frac{1}{5}\int u^5\,du = -\frac{1}{5}\,\frac{u^6}{6} = -\frac{u^6}{30}+C$$

$$= -\frac{(5-\tan(5y))^6}{30}+C$$

6. $\displaystyle\int \frac{x}{7x^2+12}\,dx = \int \frac{1}{7x^2+12}\,x\,dx = \int \frac{1}{u}\,\frac{du}{14} = \frac{1}{14}\int\frac{du}{u}$

$u = 7x^2+12$

$du = 14x\,dx \Rightarrow x\,dx = \frac{du}{14}$

$$= \frac{1}{14}\ln|u|+C$$

$$= \frac{1}{14}\ln(7x^2+12)+C$$

7. $\displaystyle\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int\frac{du}{u} = -\ln|u|+C$

$u = \cos x$

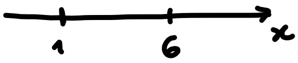
$du = -\sin x\,dx$

$$= -\ln|\cos x|+C$$

**The Substitution Rule for definite integrals:** If $u = g(x)$ is a differentiable function, then

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u$$

$u = g(x) = 5x$




$x=1 \Rightarrow u=5$

$x=6 \Rightarrow u=30$


EXAMPLE 2. *Evaluate each of the following integrals:*

1. $\displaystyle\int_{e^2}^{e^6} \frac{(\ln x)^4}{x}\,\mathrm{d}x$

$u = \ln x$

$du = \frac{dx}{x}$

$x = e^6 \Rightarrow u = \ln e^6 = 6$

$x = e^2 \Rightarrow u = \ln e^2 = 2$

$$\int_2^6 u^4\,du = \frac{u^5}{5}\Big|_2^6 = \frac{1}{5}(6^5 - 2^5)$$

OR

$u = \ln x$

$du = \frac{dx}{x}$

$$\int_{\cdots}^{\cdots} u^4\,du = \frac{u^5}{5}\Big|_{\cdots}^{\cdots} = \frac{(\ln x)^5}{5}\Big|_{e^2}^{e^6} = \frac{(\ln e^6)^5 - (\ln e^2)^5}{5}$$

$$= \frac{1}{5}(6^5 - 2^5)$$

## Linear u- substitution

$$\int f(ax+b)\,dx \overset{u=ax+b}{\underset{du=a\,dx}{=\!=\!=}} \frac{1}{a}\int f(u)\,du = \frac{1}{a}F(u) = \frac{1}{a}F(ax+b)$$

$dx = \frac{du}{a}$

$F'(x) = f(x)$

**Example**

$$\int \sin(7y)\,dy = -\frac{1}{7}\cos(7y) + C$$

$$\int e^{3x-8}\,dx = \frac{1}{3}e^{3x-8} + C$$

$$\int \frac{1}{1+3x}\,dx = \frac{1}{3}\ln|1+3x| + C$$

$$\int \frac{1}{(1+3x)^3}\,dx = \int (1+3x)^{-3}\,dx = \frac{1}{3}\,\frac{(1+3x)^{-3+1}}{-3+1} + C$$

$$= -\frac{1}{6(1+3x)^2} + C$$

2. $\int_{-1}^{1} \frac{1}{(1+3x)^3} - \frac{3}{1+3x}\, dx$

Note that the integrand is not continuous at $x = -\frac{1}{3}$ and $-\frac{1}{3}$ is in $[-1,1]$.

$\Downarrow$ Replace

$$\int_0^1 \frac{1}{(1+3x)^3} - \frac{3}{1+3x}\, dx = \left(-\frac{1}{6(1+3x)^2} - \ln|1+3x|\right)\Big|_0^1$$

$$= \left(-\frac{1}{6\cdot 16} - \ln 4 - \left(-\frac{1}{6} - \ln 1\right)\right) = \frac{1}{6}\left(-\frac{1}{16}+1\right) - \ln 4$$

$$= \frac{1}{\cancel{6}_2}\cdot\frac{\cancel{15}^5}{16} - \ln 4 = \frac{5}{32} - \ln 4$$

3. $\int_0^{0.5} (\sin(\pi y) - e^y)\, dy = \left(-\frac{1}{\pi}\cos(\pi y) - e^y\right)\Big|_0^{1/2}$

$$= \underbrace{-\frac{1}{\pi}\cos\frac{\pi}{2}}_{=0} - e^{\frac{1}{2}} - \left(-\frac{1}{\pi}\underbrace{\cos 0}_{=1} - \underbrace{e^0}_{=1}\right)$$

$$= -\sqrt{e} + \frac{1}{\pi} + 1$$

EXAMPLE 3. *If $\alpha$ and $\beta$ are positive numbers, show that*

$$\underbrace{\int_0^1 x^\alpha (1-x)^\beta \, dx}_{LHS} = \underbrace{\int_0^1 x^\beta (1-x)^\alpha \, dx}_{RHS}$$

$$LHS = \int_0^1 x^\alpha (1-x)^\beta \, dx \quad \begin{array}{l} u = 1-x \Rightarrow x = 1-u \\ du = -dx \\ x = 1 \Rightarrow u = 1-1=0 \\ x = 0 \Rightarrow u = 1-0=1 \end{array} \quad \int_1^0 (1-u)^\alpha u^\beta (-du)$$

$$= \int_0^1 (1-u)^\alpha u^\beta du = \int_0^1 (1-x)^\alpha x^\beta dx = \int_0^1 x^\beta (1-x)^\alpha dx = RHS$$