10.1: Sequences

A **sequence** is a list of numbers written in a definite order.

General sequence terms are denotes as follows:

$$\begin{array}{cccc} a_1 & - & first & term \\ a_2 & - & second & term \\ & \vdots & & \\ a_n & - & n^{th} & term \\ a_{n+1} & - & (n+1)^{th} & term \\ & \vdots & & \end{array}$$

Notice that, in general, $a_{n+1} \neq a_n + 1$.

A sequence can be defined as a function whose domain is the set of positive integers:

NOTATION: $\{a_1, a_2, \ldots, a_n, a_{n+1}, \ldots\}$, $\{a_n\}$, $\{a_n\}_{n=1}^{\infty}$.

EXAMPLE 1. *Write down the first few terms of the following sequences:*

(a) $\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty}$

(b) $\left\{\frac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty}$ $= \{\underset{n=0}{-1}, \underset{n=1}{\frac{1}{2}}, \underset{n=2}{-\frac{1}{4}}, \frac{1}{8}, -\frac{1}{16}, \ldots\} = \{$

$=$

(c) The Fibonacci sequence $\{f_n\}$ is defined recursively:

$$f_1 = 1, \quad f_2 = 1, \quad f_n = f_{n-1} + f_{n-2}, \quad n \geq 3.$$

$\{f_n\}_{n=1}^{\infty} = \{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$

$f_3 = f_2 + f_1 \qquad f_4 = f_3 + f_2$

EXAMPLE 2. *Find a general formula for the sequence:*

(a) $\left\{\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \ldots\right\} = \left\{\frac{1}{3+2k}\right\}_{k=0}^{\infty} = \left\{\frac{1}{2k+1}\right\}_{k=1}^{\infty}$

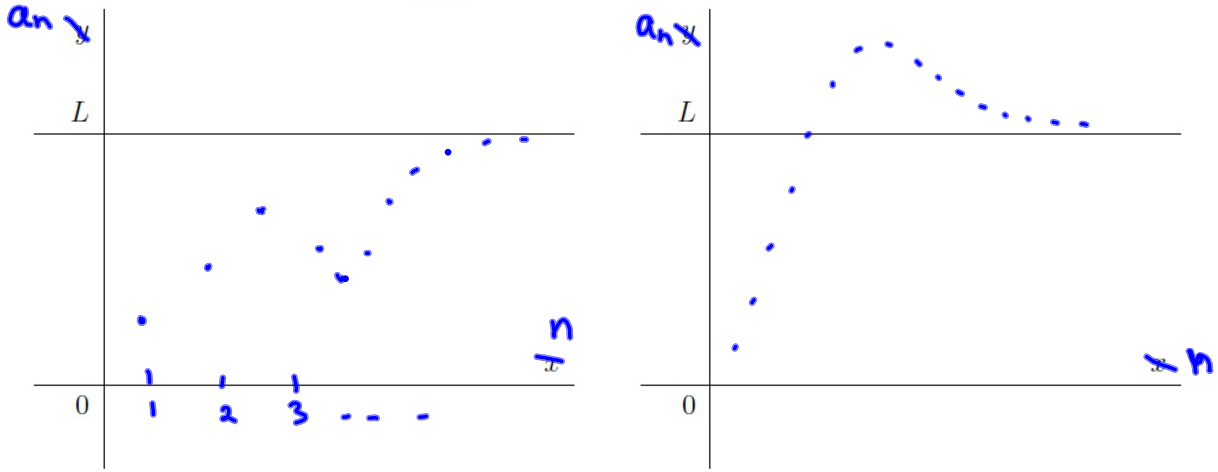
$= \left\{\frac{1}{2k-1}\right\}_{k=2}^{\infty}$

(b) $\left\{-\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \ldots\right\} = \left\{\frac{(-1)^{k+1}}{k^2}\right\}_{k=2}^{\infty}$

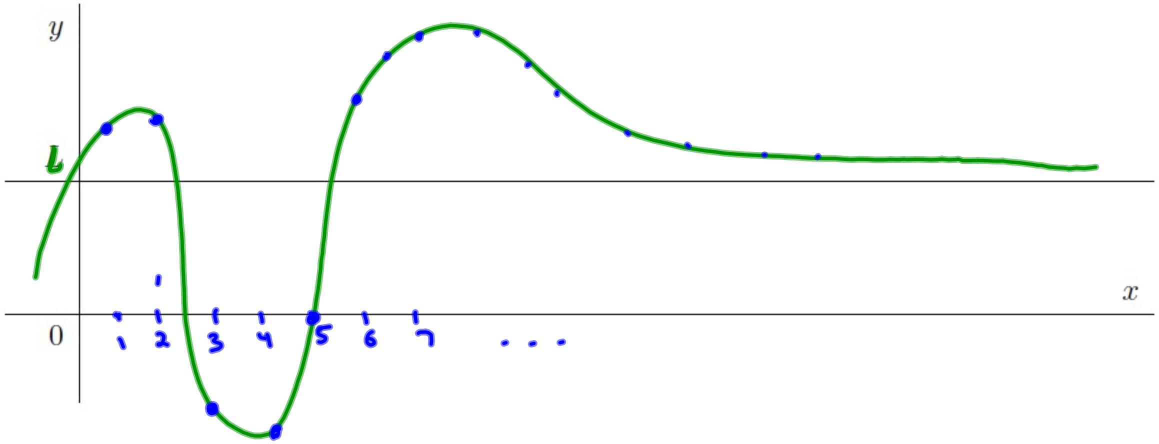
$= \left\{\frac{(-1)^k}{(k+1)^2}\right\}_{k=1}$

DEFINITION 3. *If* $\lim_{n\to\infty} a_n$ *exists then we say that the sequence* $\{a_n\}$ **converges** *(or is* **convergent.***) Otherwise, we say the sequence* $\{a_n\}$ **diverges** *(or is* **divergent.***)*

Graphs of two sequences with $\lim_{n\to\infty} a_n = L$.



THEOREM 4. *If* $\lim_{x\to\infty} f(x) = L$ *and* $f(n) = a_n$*, then* $\lim_{n\to\infty} a_n = L$.



Limit laws for convergent sequences

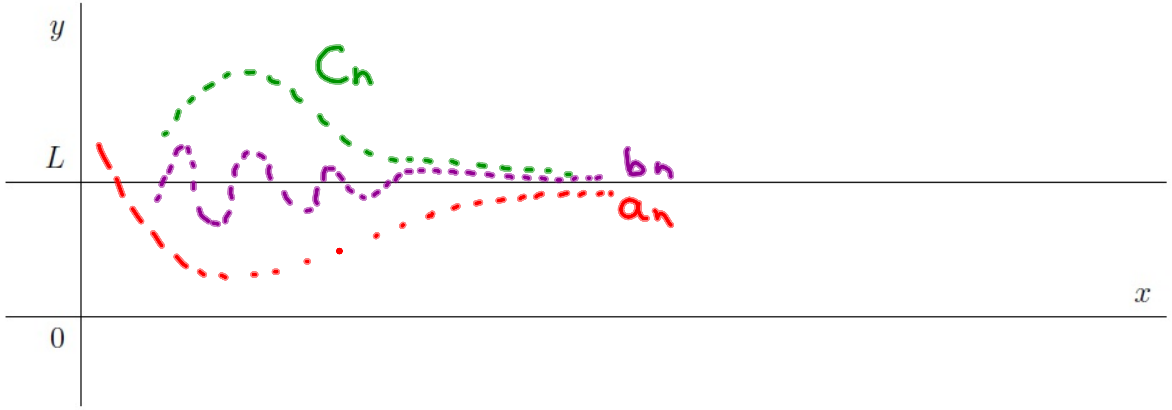
THEOREM 5. *Suppose that c is a constant and the limits*

$$\lim_{n\to\infty} a_n \qquad \text{and} \qquad \lim_{n\to\infty} b_n$$

exist. Then

1. $\lim_{n\to\infty} [a_n \pm b_n] = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$
2. $\lim_{n\to\infty} [ca_n] = c \lim_{n\to\infty} a_n$
3. $\lim_{n\to\infty} [a_n b_n] = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ *if* $\lim_{n\to\infty} b_n \neq 0$
5. $\lim_{n\to\infty} c = c$

THEOREM 6. *(**The Squeeze Theorem**) If* $a_n \le b_n \le c_n$ *for* $n \ge n_0$ *and* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, *then* $\lim_{n\to\infty} b_n = L$.



Since $-|a_n| \le a_n \le |a_n|$, we get

COROLLARY 7. *If* $\lim_{n\to\infty} |a_n| = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

$$-|a_n| \le a_n \le |a_n|$$

$$\downarrow_{n\to\infty} \qquad \downarrow_{n\to\infty}$$

$$0 \qquad\qquad 0$$

$$\lim_{n\to\infty} |a_n| = 0 \Rightarrow \lim_{n\to\infty} -|a_n| = -\lim_{n\to\infty} |a_n| = -0 = 0 \Rightarrow$$

$$\lim_{n\to\infty} a_n = 0 \quad (\text{by S.T.})$$

convergent

EXAMPLE 8. *Determine if* $\{a_n\}_{n=1}^{\infty}$ *converges or diverges. If converges, find its limit.*

(a) $a_n = \dfrac{n+1}{2n+3}$

rational function

(a) $\lim_{n\to\infty} a_n = \frac{1}{2}$

(see section 2.6)

(b) $a_n = \dfrac{3n^2-1}{10n+5n^2}$

(b) $\lim_{n\to\infty} a_n = \frac{3}{5}$

(c) $a_n = \arctan(2n)$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \arctan(2n) = \lim_{u\to\infty} \arctan u = \lim_{n\to\infty} \arctan n$

$u = 2n \to \infty$ as $n\to\infty$

$= \frac{\pi}{2}$ (convergen

(d) $a_n = \ln(2n+4) - \ln n$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\ln(2n+4) - \ln(n)\right) = \lim_{n\to\infty} \ln \frac{2n+4}{n} = \ln 2$

$\{a_n\}$ is convergent

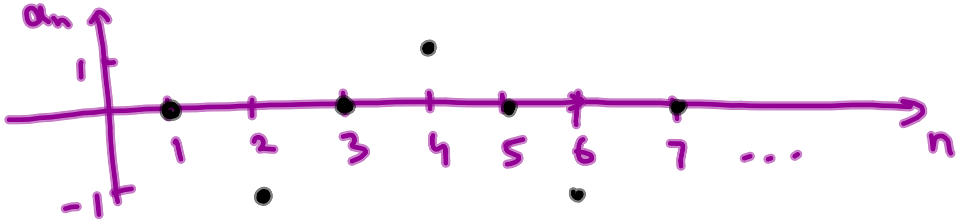
(e) $a_n = \cos(2\pi n)$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos(2\pi n) = \lim_{n\to\infty} 1 = 1$. So $\{a_n\}$ is convergent.

(f) $b_n = \cos\frac{\pi n}{2}$

$b_1 = \cos\frac{\pi}{2} = 0$, $b_2 = \cos\pi = -1$, $b_3 = \cos\frac{3\pi}{2} = 0$, $b_4 = 1$

$b_5 = 0$, $b_6 = -1$, $b_7 = 0$, $b_8 = 1$, ...



Terms of the sequence $\{a_n\}_{n=1}^{\infty}$ oscilate between -1 and 1. Thus, $\{a_n\}_{n=1}^{\infty}$ does not approach any number.

So, $\{a_n\}_{n=1}^{\infty}$ is divergent.

$(-1)^n = \pm 1$

(g) $a_n = \dfrac{3+(-1)^n}{n^2}$

$$\frac{3-1}{n^2} \leq \frac{3+(-1)^n}{n^2} \leq \frac{3+1}{n^2}$$

$$\frac{2}{n^2} \leq a_n \leq \frac{4}{n^2}$$

$$\frac{2}{n^2} \xrightarrow{n\to\infty} 0 \qquad \frac{4}{n^2} \xrightarrow{n\to\infty} 0$$

By Squeeze Theorem, we conclude that

$$a_n \xrightarrow[n\to\infty]{} 0.$$

So, $\{a_n\}$ is convergent.

EXAMPLE 9. *For what values of r does the sequence $\{r^n\}$ converges. For these values of r find* $\lim_{n\to\infty} r^n$.

Consider the following cases:

① $r > 1$. Then $\lim_{n\to\infty} r^n = \infty$ (see Section 4.1) properties of exponential function.

$\{r^n\}$ is divergent.

② $r = 1 \Rightarrow r^n = 1$ for all n $\Rightarrow \{r^n\}$ converges to 1.

③ $0 < r < 1 \Rightarrow \lim_{n\to\infty} r^n = 0$ (see Sec. 4.1) $\Rightarrow \{r^n\}$ converges to 0.

④ $r = 0 \Rightarrow r^n = 0^n = 0$ for all n $\Rightarrow \{r^n\}$ converges to 0.

⑤ $-1 < r < 0 \Rightarrow \lim_{n\to\infty} r^n = 0$ by Corollary 7 and case ③

So, $\{r^n\}$ converges to 0.

⑥ $r = -1 \Rightarrow r^n = (-1)^n$ diverges, because $\lim_{n\to\infty} (-1)^n$ DNE (cf. Ex. 8 part (f)).

⑦ $r < -1 \Rightarrow$ $\left.\begin{array}{l} r^n \to \infty, \text{ if } n \text{ is even} \\ r^n \to -\infty, \text{ if } n \text{ is odd} \end{array}\right\} \Rightarrow \lim_{n\to\infty} r^n$ DNE

Conclusion:

$$\lim_{n\to\infty} r^n = \begin{cases} 1, & \text{if } r = 1 \\ 0, & \text{if } -1 < r < 1 \\ \text{DNE} & \text{if } r \le -1 \text{ or } r > 1 \end{cases}$$

So, $\{r^n\}$ converges if and only if $-1 < r \le 1$.

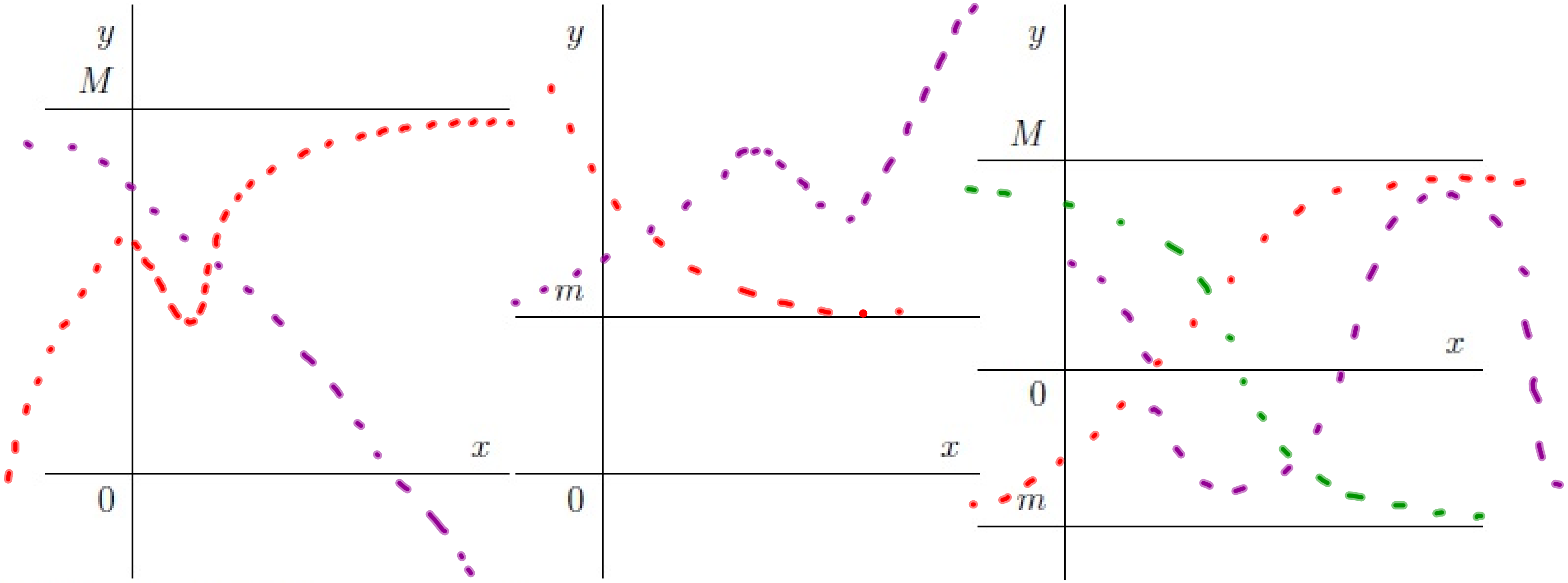
DEFINITION 10. *A sequence* $\{a_n\}$ *is* **bounded above** *if there is a number* M *s.t.*

$$a_n \le M \quad \text{for all } n.$$

A sequence $\{a_n\}$ *is* **bounded below** *if there is a number* m *s.t.*

$$m \le a_n \quad \text{for all } n.$$

If its bounded above and below, then a_n *is a* **bounded sequence**.



Positive bounded above sequence is bdd

DEFINITION 11. *A sequence* $\{a_n\}$ *is* **increasing** *if*

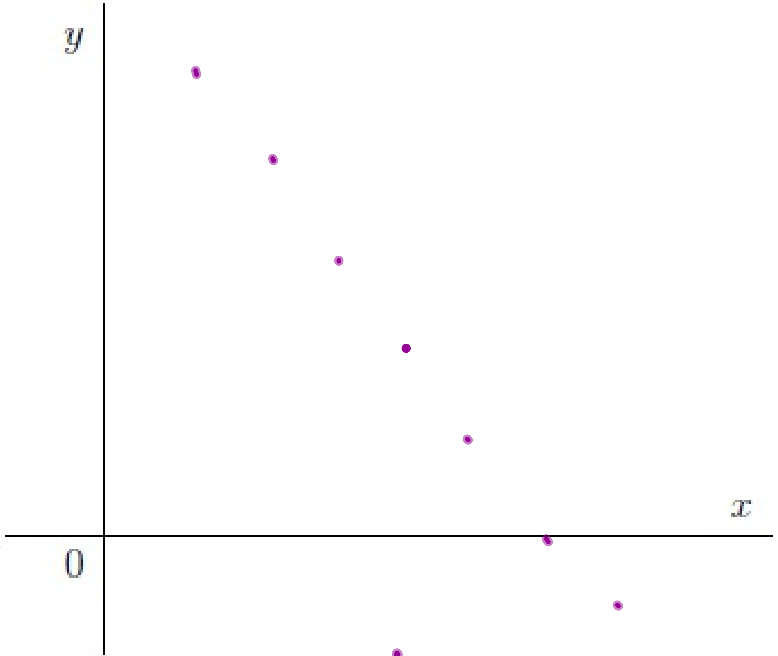
$$a_n < a_{n+1} \quad \text{for all } n.$$

A sequence $\{a_n\}$ *is* **decreasing** *if*

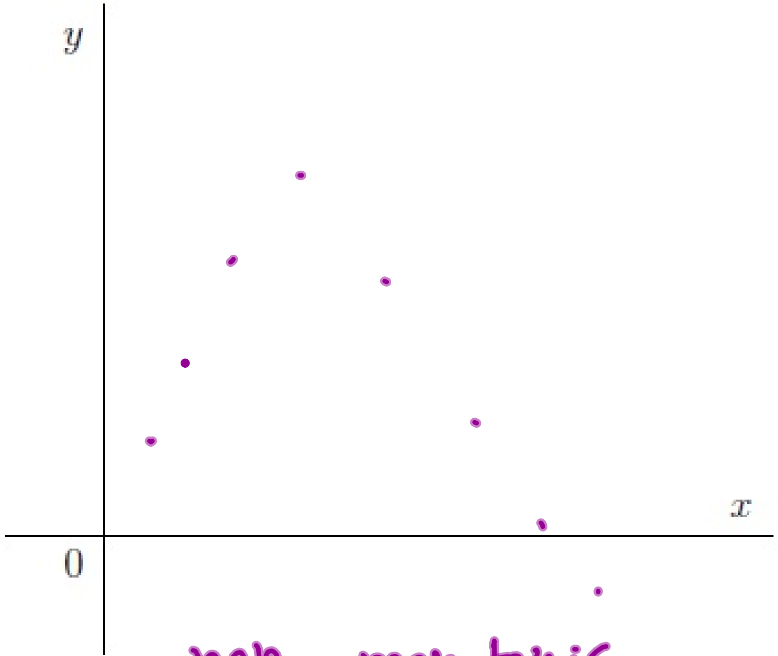
$$a_n > a_{n+1} \quad \text{for all } n.$$

A sequence is **monotonic** *if it is either increasing or decreasing.*

Note: one can apply Th. 4 to determine if a sequence is decreasing/increasing defining $f(x)$ such that $f(n) = a_n$

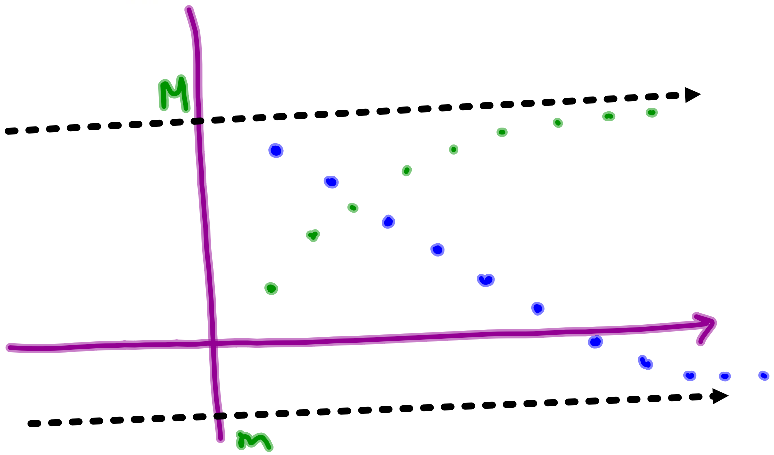


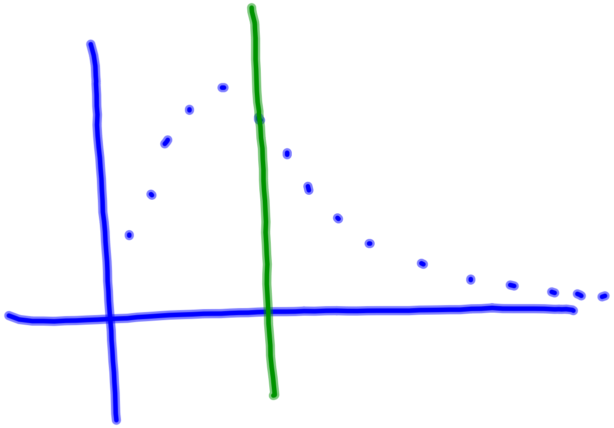
monotonic



non monotonic

THEOREM 12. MONOTONIC SEQUENCE THEOREM. *Every bounded, monotonic sequence is convergent.*

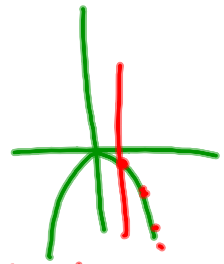




bdd & monotonic $\Rightarrow$ convergent

EXAMPLE 13. *Determine whether* $\{a_n\}$ *is increasing, decreasing or not monotonic. Then determine if* $\{a_n\}_{n=1}^{\infty}$ *is bounded.*

(a) $a_n = -n^2$, $n \geq 1$



Consider $f(x)$ such that $f(n) = a_n$, i.e.

$$f(x) = -x^2, \quad x \geq 1$$

On $[1, \infty)$ the function $f(x)$ is monotonic (decreasing), but not bdd from below, so not bdd.

Way II $\quad f'(x) = -2x < 0$ for all $x \geq 1$

So, $f(x)$ is decreasing on $[1, \infty)$ $\Rightarrow$ $\{a_n\}$ is decreasing.

Way III For all $n \geq 1$

$n < n+1 \Rightarrow n^2 < (n+1)^2 \Rightarrow$

$-n^2 > -(n+1)^2 \Rightarrow a_n > a_{n+1} \Rightarrow$

$\{a_n\}$ is decreasing.

$\{a_n\}$ is unbounded, because

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-n^2) = -\infty$$

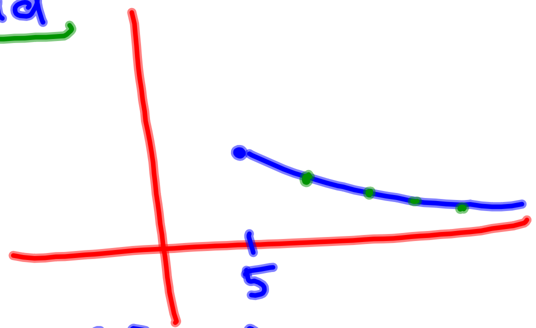
(b) $\left\{\dfrac{2}{n^2}\right\}_{n=5}^{\infty}$

Consider $f(x)$ on $[5,\infty)$ such that $f(n)=\dfrac{2}{n^2}$.

For example, $f(x)=\dfrac{2}{x^2}$.

$$\underbrace{0 \leq \frac{2}{n^2} \leq \frac{2}{5^2}}_{\downarrow} \Rightarrow \left\{\frac{2}{n^2}\right\}_{n=5}^{\infty} \text{ is } \underline{\text{bdd}}$$

because the sequence is <u>decreasing</u>. Indeed



$$f'(x) = -\frac{4}{x^3} < 0 \quad \text{for all } x \in [5,\infty).$$

Conclusion: $\left\{\dfrac{2}{n^2}\right\}$ is bdd and monotonic

(BTW it is convergent by Th. 12).

EXAMPLE 14. *Consider the sequence defined by* $a_1 = 1$, $a_{n+1} = \dfrac{1}{3 - a_n}$. *Find the first five terms of this sequence. Then Find the limit of the sequence.*

$a_1 = 1$

$a_2 = \frac{1}{3-1} = \frac{1}{2}$

$a_3 = \frac{1}{3-\frac{1}{2}} = \frac{2}{5}$

$a_4 = \frac{1}{3-\frac{2}{5}} = \frac{5}{13}$

$a_5 = \frac{1}{3-\frac{5}{13}} = \frac{13}{34}$

$1, \frac{1}{2}, \frac{2}{5}, \frac{5}{13}, \frac{13}{34}, \ldots$

$\{a_n\}$ is a positive and bdd above, so $\{a_n\}$ is bdd.

$\{a_n\}$ is decreasing.

By Th. 12 $\{a_n\}$ is convergent, i.e. there exists $L \geq 0$ (more precisely $0 \leq L \leq 1$) such that $\lim_{n\to\infty} a_n = L$.

Note that $L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{3-a_n} = \frac{1}{3-L}$

$L = \frac{1}{3-L}$

$3L - L^2 = 1$

$L^2 - 3L + 1 = 0$

$L_{1,2} = \frac{3 \pm \sqrt{3^2 - 4\cdot 1}}{2} = \frac{3 \pm \sqrt{5}}{2}$

Since $0 \leq L \leq 1$,

$\lim_{n\to\infty} a_n \ \frac{3-\sqrt{5}}{2}$.