10.2: SERIES

$\{a_k\}_{k=1}^{\infty}$

A series is a sum of sequence:

general / common term

$$\sum_{m=1}^{\infty} a_m = \sum_{n=1}^{\infty} a_n = \sum_{k=1}^{\infty} a_k = \underline{a_1 + a_2 + a_3 + \ldots + a_n + \ldots}$$

a_k is called a general (common) term of the given series. For a given sequence[1] $\{a_k\}_{k=1}^{\infty}$ define the following:

[1] $k = 1$ for convenience, it can be anything

$s_1 = a_1$
$s_2 = a_1 + a_2$
$s_3 = a_1 + a_2 + a_3$
. . . . . .
$s_n = a_1 + a_2 + a_3 + \ldots + a_n$
. . . . .

$\{s_n\}_{n=1}^{\infty}$ the sequence of partial sums

The s_n's are called **partial sums** and they form a sequence $\{s_n\}_{n=1}^{\infty}$.
We want to consider the limit of $\{s_n\}_{n=1}^{\infty}$:

$$S = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

sum of the series

If $\{s_n\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum_{k=1}^{n} a_k$ is *convergent*. The number s is called the **sum** of the series.[2]

If $\{s_n\}_{n=1}^{\infty}$ is divergent then the series $\sum_{k=1}^{\infty} a_k$ is *divergent*.

[2] When we write $\sum_{k=1}^{n} a_k = s$ we mean that by adding sufficiently many terms of the series we can get as close as we like to the number s.

GEOMETRIC SERIES

$$\overset{a\cdot r^0}{a} + ar + ar^2 + \ldots + \underbrace{ar^{n-1}} + \ldots = \sum_{n=1}^{\infty} a r^{n-1} \underset{m=n-1}{=} \sum_{m=0}^{\infty} a r^m \qquad (a \neq 0)$$

Each term is obtained from the preceding one by multiplying it by the common ratio r.

THEOREM 1. *The geometric series is convergent if* $\underbrace{|r| < 1}_{-1<r<1}$ *and its sum is*

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

If $|r| \geq 1$, *the geometric series is divergent.*

$\sum_{n=0}^{\infty} ar^n$ converges for all $-1<r<1$

Otherwise, it is divergent

Proof. $S_n = \sum_{k=0}^{n} a r^k$, or

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1} + ar^n$$
$$- \quad rS_n = \quad ar + ar^2 + \ldots + ar^{n-1} + ar^n + ar^{n+1}$$

$$S_n - rS_n = a - ar^{n+1}$$
$$S_n(1-r) = a(1-r^{n+1})$$
$$S_n = \frac{a(1-r^{n+1})}{1-r}$$

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^{n+1})}{1-r} = \frac{a}{1-r} \lim_{n\to\infty} (1-r^{n+1})$$

By Ex. 9 (Section 10.1) we know that

$\{r^n\}$ converges if $\underbrace{r=1}_{\text{series diverges}}$ or $|r|<1$

If $|r|<1$ then $r^n \xrightarrow[n\to\infty]{} 0$, so

$$S = \frac{a}{1-r}(1-0), \text{ or } \boxed{S = \frac{a}{1-r}.} \quad \square$$

Title: Mar 20-11:17 PM (2 of 9)

EXAMPLE 2. *Determine whether the following series converges or diverges. If it is converges, find the sum. If it is diverges, explain why.*

(a) $\sum_{n=1}^{\infty} 5 \cdot \left(\frac{2}{7}\right)^n = \sum_{n=1}^{\infty} \underbrace{5 \cdot \frac{2}{7}}_{a} \cdot \left(\frac{2}{7}\right)^{n-1} = \frac{\frac{10}{7}}{1-\frac{2}{7}} = \frac{\frac{10}{7}}{\frac{5}{7}} = \underbrace{2}_{\text{sum of the series}}$

geometric series

$r = \frac{2}{7}$ converges

$-1 < \frac{2}{7} < 1$

(b) $\sum_{n=0}^{\infty} \frac{(-4)^{3n}}{5^{n-1}} = \sum_{n=0}^{\infty} \frac{\left((-4)^3\right)^n \cdot 5}{5^n} = \underbrace{\sum_{n=0}^{\infty} 5 \cdot \left(-\frac{64}{5}\right)^n}_{\text{geometric series, where}}$

$a=5$, $r = -\frac{64}{5} < -1$

so, the series diverges

(c) $\underset{n=0}{1} - \underset{n=1}{\frac{3}{2}} + \frac{9}{4} - \frac{27}{8} + \ldots =$

$\sum_{n=0}^{\infty} (-1)^n \left(\frac{3}{2}\right)^n = \sum_{n=0}^{\infty} \left(-\frac{3}{2}\right)^n$

geometric series, where

$a=1$, $r = -\frac{3}{2} < -1$,

so the series is divergent.

TELESCOPING SUM

Let b_n be a given sequence. Consider the following series:

$$\sum_{n=1}^{\infty} (b_n - b_{n+1})$$ general term: $a_n = b_n - b_{n+1}$

Partial sum

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} b_k - b_{k+1}$$

$$= \underbrace{b_1 - \cancel{b_2}}_{k=1} + \underbrace{\cancel{b_2} - \cancel{b_3}}_{k=2} + \underbrace{\cancel{b_3} - \cancel{b_4}}_{k=3} + \dots + \underbrace{\cancel{b_{n-1}} - \cancel{b_n}}_{k=n-1} + \underbrace{\cancel{b_n} - b_{n+1}}_{k=n}$$

$$= b_1 - b_{n+1}$$

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} (b_1 - b_{n+1}) = b_1 - \underbrace{\lim_{n\to\infty} b_{n+1}}$$

if this limit exists, then the telescoping series converges.

Question: Are these series telescoping?

$$\sum_{n=1}^{\infty} b_{n+1} - b_n, \quad \sum b_n - b_{n-1}, \quad \sum b_{n+2} - b_n$$

$$\sum b_n - b_{n+3}$$

Answer: YES

EXAMPLE 3. *Determine whether the following series converges or diverges. If it is converges, find the sum. If it is diverges, explain why.*

(a) $\sum_{n=1}^{\infty}\left(\underbrace{\sin\frac{1}{n}}_{b_n}-\underbrace{\sin\frac{1}{n+1}}_{b_{n+1}}\right)$ telescoping

$$S_n=\sum_{k=1}^{n}\left(\sin\frac{1}{k}-\sin\frac{1}{k+1}\right)=\sin 1-\cancel{\sin\frac{1}{2}}+\cancel{\sin\frac{1}{2}}-\cancel{\sin\frac{1}{3}}+\dots+\cancel{\sin\frac{1}{n}}-\sin\frac{1}{n+1}$$

$$\lim_{n\to\infty} S_n=\sin 1-\lim_{n\to\infty}\sin\frac{1}{n+1}=\sin 1-\sin 0=\sin 1.$$

$S=\sin 1$ (the series converges).

(b) $\sum_{n=1}^{\infty}\ln\frac{n+1}{n+2}=\sum_{n=1}^{\infty}\left(\ln(n+1)-\ln(n+2)\right)$ telescoping

$$S_n=\sum_{k=1}^{n}\left(\ln(k+1)-\ln(k+2)\right)=\ln 2-\cancel{\ln 3}+\cancel{\ln 3}-\cancel{\ln 4}+\dots+\cancel{\ln(n+1)}-\ln(n+2)$$

$$\lim_{n\to\infty} S_n=\ln 2-\underbrace{\lim_{n\to\infty}\ln(n+2)}_{DNE}$$ $\Rightarrow$ the series diverges

(c) $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\overset{PFD}{=}\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$ telescoping

$$\frac{1}{n(n+1)}=\frac{A}{n}+\frac{B}{n+1}=\frac{1}{n}-\frac{1}{n+1}$$

$$1=A(n+1)+Bn$$

$n=0 \quad A=1$

$n=-1 \quad B=-1$

$$\lim_{n\to\infty} S_n=1-\lim_{n\to\infty}\frac{1}{n+1}=1=S$$

So, the series converges and its sum is 1.

necessary, but not sufficient condition for convergence of series.

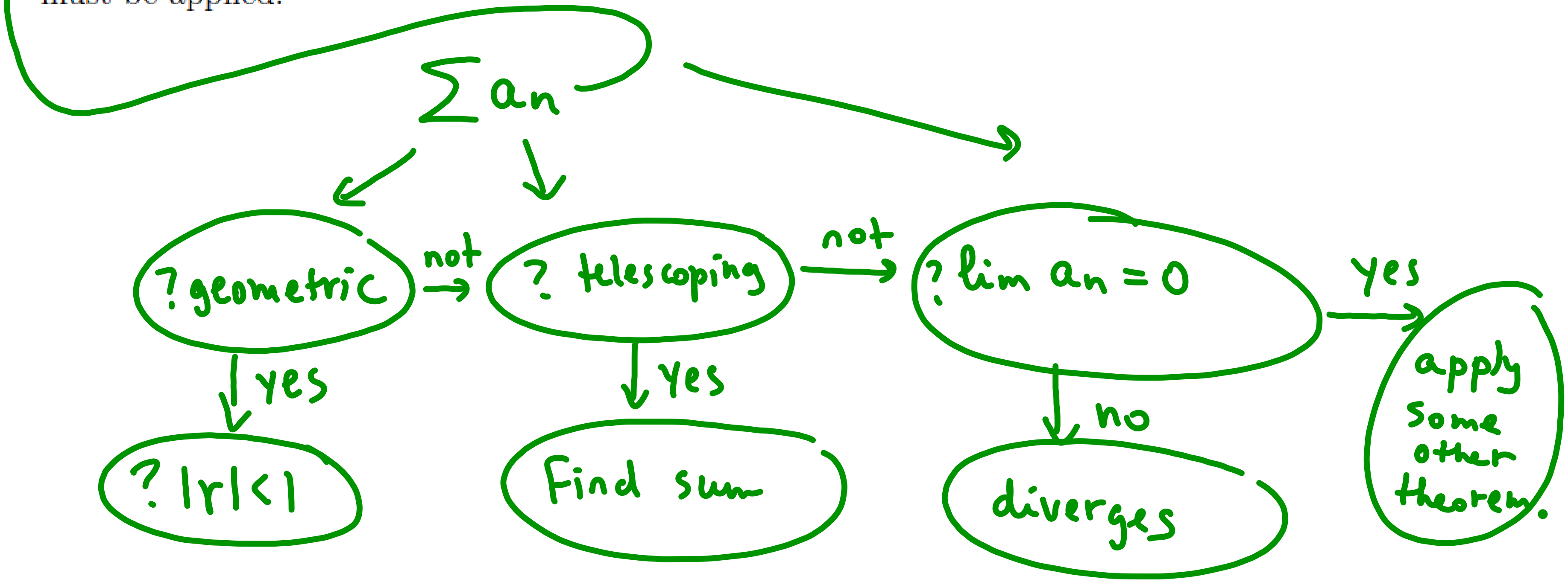
THEOREM 4. *If the series* $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

REMARK 5. The converse is not necessarily true.

THE TEST FOR DIVERGENCE:

If $\lim_{n\to\infty} a_n$ *does not exist or if* $\lim_{n\to\infty} a_n \neq 0$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

REMARK 6. If you find that $\lim_{n\to\infty} a_n = 0$ then the Divergence Test fails and thus another test must be applied.



EXAMPLE 7. *Use the test for Divergence to determine whether the series diverges.*

(a) $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{n^2}{3(n+1)(n+2)}}_{a_n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{3(n+1)(n+2)} = \frac{1}{3} \neq 0$$

Conclusion: the series diverges.

(b) $\displaystyle\sum_{n=1}^{\infty} \cos\frac{\pi n}{2}$

$\lim_{n\to\infty} \cos\frac{\pi n}{2}$ DNE (see Ex... section 10.1)

So, the series diverges by the Divergence Test.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

$$\lim_{n\to\infty} \left|\frac{(-1)^n}{n^2}\right| = \lim_{n\to\infty} \frac{1}{n^2} = 0$$

By Cor. 7 (section 10.1), this implies

$$\lim_{n\to\infty} \frac{(-1)^n}{n^2} = 0.$$

Conclusion: The Divergence Test fails here. Thus, to make a conclusion we have to use some other test (see next section).

THEOREM 8. *If* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are convergent series, then so are the series* $\sum_{n=1}^{\infty} ca_n$ *(where c is a constant),* $\sum_{n=1}^{\infty}(a_n + b_n)$, *and* $\sum_{n=1}^{\infty}(a_n - b_n)$, *and*

$$\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n, \quad \sum_{n=1}^{\infty}(a_n+b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n, \quad \sum_{n=1}^{\infty}(a_n-b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

<u>Proof</u> Let s_n and $\tilde{s}_n$ be partial sums for $\sum a_n$ and $\sum b_n$, respectively:

$$s_n = a_1 + a_2 + \dots + a_n$$

$$\tilde{s}_n = b_1 + b_2 + \dots + b_n.$$

Since $\sum a_n$ and $\sum b_n$ converge, $\lim s_n$ and $\lim \tilde{s}_n$ exist. But then by Th. 5(Section 10.1)

$$\lim_{n\to\infty} c\, s_n, \quad \lim_{n\to\infty}(s_n + \tilde{s}_n), \quad \lim(s_n - \tilde{s}_n)$$

exist.

Since cs_n, $s_n + \tilde{s}_n$, $s_n - \tilde{s}_n$ are partial sums of $\sum c a_n$, $\sum a_n + b_n$, $\sum a_n - b_n$, respectively, we conclude that these series are convergent. $\square$

EXAMPLE 9. *Determine whether the following series converges or diverges. If it is converges, find its sum.*

$$\sum_{n=1}^{\infty}\left(\frac{1}{n(n+1)}+5\cdot\left(\frac{2}{7}\right)^n\right) = 1+2=3$$

Consider

$\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1$ by Ex. 3(c)

telescoping

$\sum 5\cdot\left(\frac{2}{7}\right)^n = 2$ by Ex. 2a.

geometric

$\xRightarrow{\text{Th. 8}}$ the given series is convergent and its sum is 3.