10.3: The Integral and Comparison Tests; Estimating Sums

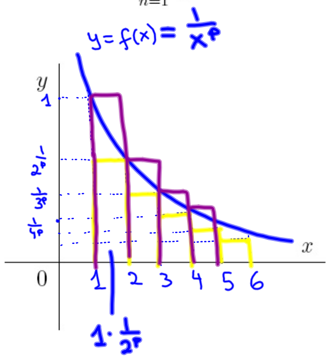
QUESTION: For what values of p the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent?

If $p<0$, then $\lim_{n\to\infty} \frac{1}{n^p}$ DNE, i.e. the series Diverges by DT.

If $p=0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 1$, i.e. the series diverges by DT.

If $p>0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$ and Divergent Test Fails.

If $p = 1$ then $\sum_{n=1}^{\infty} \frac{1}{n}$ is called **harmonic series**.



$f(x) = \frac{1}{x^p}$ $(x \in [1,\infty))$

$f(n) = \frac{1}{n^p}$ (if n is integer, $n \geq 1$)

$$\frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots \leq \int_1^{\infty} f(x)\,dx \leq 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} - 1 \leq \int_1^{\infty} \frac{1}{x^p}\,dx \leq \sum_{n=1}^{\infty} \frac{1}{n^p}$$

If $p \leq 1$ then $\int_1^{\infty} \frac{dx}{x^p}$ is divergent, so is $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent. In particular, harmonic series is divergent.

THE INTEGRAL TEST *Suppose f is a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent.* In other words:

If $\int_1^{\infty} f(x)\,dx$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges

and $\int_1^{\infty} f(x)\,dx$ diverges $\Rightarrow \sum_{n=1}^{\infty} a_n$ diverges.

Note: we apply the Integral Test to positive Series only: $a_n > 0$ for all $n \geq 1$

FACT: The p-series, $\sum_{n=1}^{\infty} \frac{1}{n^p}$, converges if $p > 1$ and diverges if $p \leq 1$.

EXAMPLE 1. *Determine if the following series is convergent or divergent:*

(a) $\sum_{n=1}^{\infty} \frac{1000}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1000}{n^{3/2}}$

We know that $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a p-series with $p = \frac{3}{2} > 1$

So $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent $\overset{\text{Th.8 (Sec.10.2)}}{\Longrightarrow} \sum_{n=1}^{\infty} \frac{1000}{n^{3/2}}$ is convergent.

(b) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

$\lim_{n\to\infty} \frac{1}{n \ln n} = 0 \Rightarrow$ Divergence Test fails. Use another test.

Apply Integral Test

THE INTEGRAL TEST *Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent.* In other words:

Let $f(x) = \frac{1}{x \ln x}$ on $[2, \infty)$.

Then $f(n) = \frac{1}{n \ln n}$.

Also $f(x) > 0$ on $[2, \infty)$, because $x > 0$ and $\ln x > 0$ for all $x \geq 2$.

$f(x)$ is continuous on $[2, \infty)$.

(Way 1. Derivative $f'(x) < 0$ on $[2, \infty)$

Way 2. Use Definion $a_n > a_{n+1}$ for all $n \geq 2$

Way 3. Notice that both $y = x$ and $y = \ln x$ are increasing on $[2, \infty)$. So, $y = x \ln x$ is increasing and then $y = \frac{1}{x \ln x}$ is decreasing.) $\longrightarrow$

$$\int_2^\infty f(x)\,dx = \int_2^\infty \frac{dx}{x\ln x} =$$

$$= \lim_{t\to\infty} \int_2^t \frac{dx}{x\ln x} = \lim_{t\to\infty} \ln(\ln x)\Big|_2^t$$

$$= \lim_{t\to\infty} \left(\ln(\ln t) - \ln(\ln 2)\right) = \infty$$

diverges.

$u = \ln x$

$du = \frac{dx}{x}$

$$\int \frac{dx}{x\ln x} = \int \frac{du}{u} = \ln u$$

$$= \ln(\ln x)$$

Since $\int_2^\infty f(x)\,dx$ diverges,

$\sum_{n=2}^\infty \frac{1}{n\ln n}$ is divergent by the Integral Test.

A disadvantage of the Integral Test: it does force us to do improper integrals which are in some cases may be impossible to determine the convergence of. For example, consider

$$\sum_{n=0}^{\infty} \frac{1}{4^n + n^4}$$

??? $\int_0^{\infty} \frac{dx}{4^x + x^4}$

$$S_n = \sum_{k=0}^{n} \frac{1}{4^k + k^4}$$

$\{S_n\}_{n=0}^{\infty}$ is positive sequence. (bounded from below)

$\{S_n\}_{n=0}^{\infty}$ is increasing:

$$S_{n+1} = S_n + a_{n+1}$$

Since $a_{n+1} > 0$, we get $S_{n+1} > S_n$.

In addition, $\{S_n\}$ is bdd from above: $\frac{1}{4^n + n^4} \leq \frac{1}{4^n}$

So, $S_n = \sum_{k=0}^{n} \frac{1}{4^k + k^4} \leq \sum_{k=0}^{n} \frac{1}{4^k} =$

$$= \frac{1\left(1 - \left(\frac{1}{4}\right)^{n+1}\right)}{1 - \frac{1}{4}}$$

$$= \frac{4\left(1 - \left(\frac{1}{4}\right)^{n+1}\right)}{3} \leq \frac{4}{3}$$

Conclusion: $\{S_n\}$ is monotonic and bdd. So, $\{S_n\}$ is convergen. Thus $\sum_{n=0}^{\infty} \frac{1}{4^n + n^4}$ converges.

THE COMPARISON TEST *Suppose that* $\sum a_n$ *and* $\sum b_n$ *are series with positive terms and* $a_n \leq b_n$ *for all* n.

- *If* $\sum b_n$ *is convergent then* $\sum a_n$ *is also convergent.*
- *If* $\sum a_n$ *is divergent then* $\sum b_n$ *is also divergent.*

EXAMPLE 2. *Determine if the following series is convergent or divergent:*

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^4+4}{n^6+6}$ positive series $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^4+4}{n^6+6} = 0$

DT fails

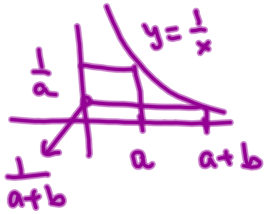
$$0 < a_n = \frac{n^4+4}{n^6+6} = \frac{n^4}{n^6+6} + \frac{4}{n^6+6} \leq \frac{n^4}{n^6} + \frac{4}{n^6}$$

$$= \frac{1}{n^2} + \frac{4}{n^6}$$

$a, b > 0$

$a+b > a > 0$

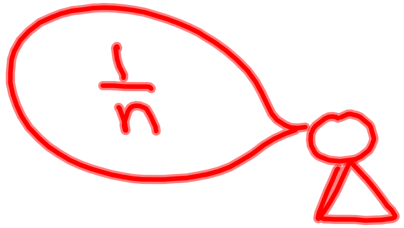
$\frac{1}{a+b} < \frac{1}{a}$



Since $\sum \frac{1}{n^2}$ and $\sum \frac{4}{n^6}$ converges as p-series ($p = 2$ and 6, respectively), we conclude that $\sum \left(\frac{1}{n^2} + \frac{4}{n^6}\right)$ converges.

Thus, by Comparison Test, the given series converges as well.

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{n^{2016}}{n^{2017} - \sin^{2016} n}$



The given series is positive

$\left(n^{2016} > 0 \text{ and } n^{2017} - \sin^{2016} n > n^{2017} - 1 > 0 \right)$

So, one can apply the Comparison Test.

$$b_n = \frac{n^{2016}}{n^{2017} - \sin^{2016} n} \geq \frac{n^{2016}}{n^{2017}} = \frac{1}{n} = a_n > 0$$

$a, b > 0$

$a - b > 0$

$a - b < a$

$\frac{1}{a-b} > \frac{1}{a}$

Since $\sum \frac{1}{n}$ is harmonic series (i.e. divergent), the given series diverges.

(c) $\sum_{n=1}^{\infty} \frac{\cos^4 n}{n^2\sqrt{n}}$ positive series.

$$\frac{\cos^4 n}{n^2\sqrt{n}} = \frac{\cos^4 n}{n^{5/2}} \le \frac{1}{n^{5/2}}$$

Since $\sum \frac{1}{n^{5/2}}$ converges (as p-series with $p = \frac{5}{2} > 1$),

the given series converges by Comparison Test.

(d) $\sum_{n=1}^{\infty} \frac{5^n+1}{4^n}$ positive series,

so, one can apply Comparison Test.

$$\sum \frac{5^n+1}{4^n} \stackrel{?}{=} \sum \frac{5^n}{4^n} + \sum \frac{1}{4^n}$$

$$\frac{5^n+1}{4^n} > \frac{5^n}{4^n} \; !$$

$$\frac{5^n+1}{4^n} > \frac{1}{4^n}$$

$$\frac{5^n+1}{4^n} = \frac{5^n}{4^n} + \frac{1}{4^n} > \left(\frac{5}{4}\right)^n > 0$$

The series $\sum_{n=1}^{\infty} \left(\frac{5}{4}\right)^n = \sum_{n=1}^{\infty} \frac{5}{4} \cdot \left(\frac{5}{4}\right)^{n-1}$ is

divergent (as geometric series with $r = \frac{5}{4} > 1$).

Thus, the given series also diverges.

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{4^n + n^4}$ positive series

$\sum \frac{1}{4^n}$ geom. $r = \frac{1}{4}$

$\sum \frac{1}{n^4}$ p-series $p = 4$

$$\frac{1}{4^n + n^4} < \frac{1}{4^n}$$

Since $\displaystyle\sum_{n=0}^{\infty} \frac{1}{4^n}$ converges (geom. $r = \frac{1}{4}$)

We conclude that the given series converges (by Comparison Test)

Warning: Distinguish between $\displaystyle\sum_{n=1}^{\infty} n^p$ and $\displaystyle\sum_{n=1}^{\infty} p^n$.

In some cases inequalities are useless. For example, for the series

$$\sum_{n=0}^{\infty} \frac{1}{4^n - n}$$ positive

we have

$$4^n - n < 4^n$$

$$\frac{1}{4^n - n} > \frac{1}{4^n}$$ useless (we cannot apply Comparison test here)

THE LIMIT COMPARISON TEST *Suppose that* $\sum a_n$ *and* $\sum b_n$ *are series with positive terms .* *If*

$$\lim_{n\to\infty}\left(\frac{1}{b_n}\right)\cdot a_n = \lim_{n\to\infty}\frac{a_n}{b_n} = c$$

where c *is a finite number and* $c > 0$*, then either both series converge or both diverge.*

$c = 0, \infty$, DNE mean that LCT fails.
Use another test.

EXAMPLE 3. *Determine if the following series is convergent or divergent:*

(a) $\sum_{n=1}^{\infty} \underbrace{\frac{1}{4^n - n}}_{a_n}$ Consider $\sum_{n=1}^{\infty} \underbrace{\frac{1}{4^n}}_{b_n}$ convergent (geom. series, $r=\frac{1}{4}$)

$$c = \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\left(\frac{1}{b_n}\right)\cdot(a_n) = \lim_{n\to\infty} 4^n\cdot\frac{1}{4^n-n}$$

$$= \lim_{n\to\infty}\frac{4^n}{4^n-n} \overset{\frac{\infty}{\infty}}{\underset{\text{L.R.}}{=}} \lim_{n\to\infty}\frac{(4^n\ln 4)/4^n}{(4^n\ln 4 - 1)/4^n}$$

$$= \lim_{n\to\infty}\frac{\ln 4}{\ln 4 - \frac{1}{4^n}} = \frac{\ln 4}{\ln 4 - 0} = 1 > 0$$

Conclusion: $\sum\frac{1}{4^n-n}$ converges by comparison with $\sum\frac{1}{4^n}$.

(b) $\sum_{n=2}^{\infty} \underbrace{\frac{n^2+n}{\sqrt{n^5+n^3}}}_{a_n}$ $\quad \sim \frac{n^2}{\sqrt{n^5}} = \frac{1}{\sqrt{n}}$

Apply L.C.T using $\sum_{n=2}^{\infty}\underbrace{\frac{1}{\sqrt{n}}}_{b_n}$, which is divergent p-series $(p=\frac{1}{2}<1)$.

$$c = \lim_{n\to\infty}\frac{1}{b_n}\cdot a_n = \lim_{n\to\infty}\frac{\sqrt{n}\cdot(n^2+n)}{\sqrt{n^5+n^3}} = \lim_{n\to\infty}\sqrt{\underbrace{\frac{n(n^2+n)^2}{n^5+n^3}}_{\text{rational function}}} = 1 > 0$$

So, the given series also diverges.

Illustration: Why c in the Limit Comparison Test must be positive and finite:

Ex.1 We know that $\sum \overbrace{\frac{1}{n}}^{a_n}$ is divergent

and $\sum \underbrace{\frac{1}{n^2}}_{b_n}$ is convergent

However

$$c = \lim_{n\to\infty} \frac{1}{b_n} a_n = \lim_{n\to\infty} n^2 \cdot \frac{1}{n} = \lim_{n\to\infty} n = \infty,$$ i.e. LCT fails here and no conclusion can be made.

If $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n}$

then $c = \lim_{n\to\infty} n \cdot \frac{1}{n^2} = \lim_{n\to\infty} \frac{1}{n} = 0$, i.e. LCT fails here

Ex.2 $\sum \frac{1}{n^2}$ and $\sum \frac{1}{n^3}$ both convergent, but

$$c = \lim_{n\to\infty} n^3 \frac{1}{n^2} = \lim_{n\to\infty} n = \infty.$$

$$\underbrace{\sum_{n=1}^{\infty} a_n}_{S} = \underbrace{a_1 + a_2 + \dots + a_n}_{S_n} + \underbrace{a_{n+1} + a_{n+2} + \dots}_{R_n}$$

If $\alpha < R_n < \beta$, then $\alpha < S - S_n < \beta \Rightarrow S_n + \alpha < S < S_n + \beta$

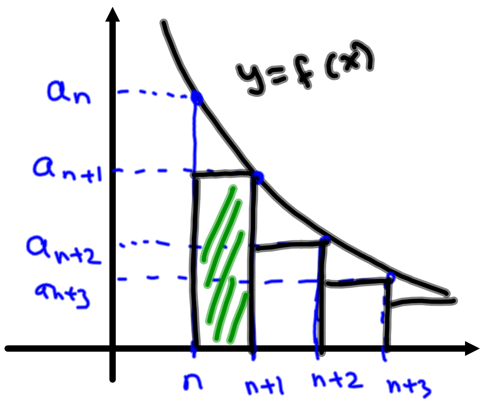
REMINDER ESTIMATE FOR THE INTEGRAL TEST

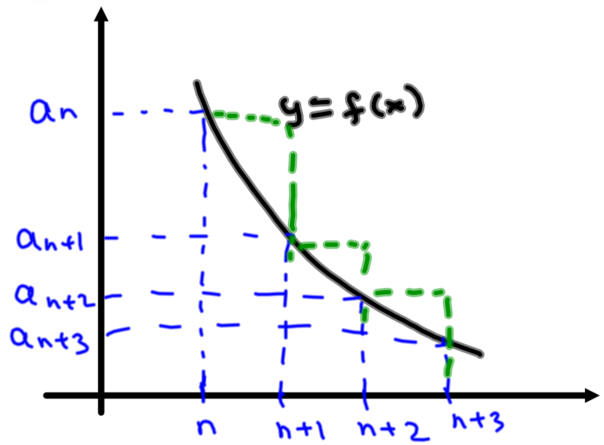
If $\underbrace{\sum a_n}_{\text{positive series}}$ *converges by the Integral Test and* $\boxed{R_n = s - s_n}$ *then*

$$\underbrace{\int_{n+1}^{\infty} f(x)\,\mathrm{d}x}_{\alpha} \le R_n \le \underbrace{\int_{n}^{\infty} f(x)\,\mathrm{d}x}_{\beta}$$

which implies

$$s_n + \int_{n+1}^{\infty} f(x)\,\mathrm{d}x \le s \le s_n + \int_{n}^{\infty} f(x)\,\mathrm{d}x$$





$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx$$

EXAMPLE 4. *Given* $\sum_{n=1}^{\infty} \frac{1}{n^3}$ convergent (p-series with $p=3$).

(a) *Approximate the sum of the series by using the sum of the first 10 terms.*

$$S_{10} = a_1 + \dots + a_{10} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \dots + \frac{1}{10^3} \approx 1.1975$$

(b) *Estimate the error.* $S \approx S_{10}$ then error is R_{10}.

Since convergence of $\sum \frac{1}{n^3}$ was verified using integral test with $f(x) = \frac{1}{x^3}$, we can apply

$$\int_{11}^{\infty} \frac{1}{x^3}dx \le R_{10} \le \int_{10}^{\infty} \frac{1}{x^3}\,dx$$

$$\int_n^{\infty} \frac{dx}{x^3} = \lim_{t\to\infty} \int_n^t \frac{dx}{x^3} = \lim_{t\to\infty} \frac{1}{-2x^2}\Big|_n^t = -\frac{1}{2}\left(\lim_{t\to\infty}\frac{1}{t^2} - \frac{1}{n^2}\right) = \frac{1}{2n^2}$$

$$\frac{1}{2\cdot 11^2} \le R_{10} \le \frac{1}{2\cdot 10^2}$$

$$\frac{1}{242} \le R_{10} \le \frac{1}{200}$$

Remark: $1.1975 + \frac{1}{242} \le S \le 1.1975 + \frac{1}{200}$

(c) *How many terms are required to ensure that the sum is accurate to within* 0.0005*?*

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_n^{\infty} f(x)\,dx$$

$$\frac{1}{2(n+1)^2} \le R_n \le \frac{1}{2n^2}$$

Find n such that

$$\frac{1}{2n^2} \le 0.0005$$

$$\frac{1}{n^2} \le 0.001$$

$$\frac{1}{n^2} \le \frac{1}{1000}$$

$$n^2 \ge 1000$$

$$n \ge \sqrt{1000}$$

$$n \gtrsim 31.6$$

Answer: We need at least 32 terms to ensure that the sum is accurate within the 0.0005.

It also says that $S \in (S_{32} - 0.0005,\ S_{32} + 0.0005)$.