10.4: Other Convergence Tests

Alternating Series Test

DEFINITION 1. *An alternating series is a series whose terms are alternately positive and negative. It means if $\sum a_n$ is an alternating series then either*

$$\underbrace{a_n}_{\text{general term}} = (-1)^n b_n, \qquad \underline{b_n > 0}$$

or

$$a_n = (-1)^{n+1} b_n, \qquad b_n > 0.$$

Note that $b_n = |a_n|$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n b_n = -b_1 + b_2 - b_3 + b_4 - b_5 + b_6 + \dots$$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \dots$$

Alternating Series Test: *If $\underbrace{\lim_{n\to\infty} b_n = 0}_{1.}$ and the sequence $\underbrace{\{b_n\} \text{ is decreasing}}_{2.}$ then the series $\sum (-1)^n b_n$ is convergent.*

REMARK 2. This test will only tell us when a series converges and not if a series will diverge.

EXAMPLE 3. *Determine if the following series are convergent or divergent:*

(a) $\sum_{n=1}^{\infty} \underbrace{\frac{(-1)^{n+1}}{n}}_{a_n}$ alternating series $b_n = \frac{1}{n}$

1. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$

2. $\{b_n\} = \{\frac{1}{n}\}$ is decreasing, because $\forall n \geq 1$, $n < n+1 \Rightarrow \frac{1}{n} > \frac{1}{n+1}$

So, by AST, the given series converges.

(b) $\sum_{n=1}^{\infty} \frac{\cos \pi n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, because

$$\cos \pi n = \begin{cases} 1, & \text{if } n \text{ is even} \\ -1, & \text{if } n \text{ is odd} \end{cases} = (-1)^n, n \geq 1$$

$b_n = \frac{1}{\sqrt{n}}$

1. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

2. $\{b_n\}_{n=1}^{\infty} = \{\frac{1}{\sqrt{n}}\}_{n=1}^{\infty}$ is decreasing because for all $n \geq 1$,

$\sqrt{n} < \sqrt{n+1} \Rightarrow \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}$

So, by AST $\sum_{n=1}^{\infty} \frac{\cos \pi n}{\sqrt{n}}$ is convergent.

(c) $\sum_{n=1}^{\infty} \underbrace{\frac{(-1)^n n^2}{n^2+5}}_{a_n}$ alternating series, but

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^2+5} = 1 \neq 0$.

So, AST fails here and we need to use another test.

Apply Divergence test:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n n^2}{n^2+5} = \begin{cases} 1, & \text{if } n \text{ is even} \\ -1, & \text{if } n \text{ is odd} \end{cases}$$

So, $\lim_{n\to\infty} a_n$ DNE

and the given series diverges by Divergence Test.

Absolute Convergence

arbitrary (not necessarily positive or alternating)

- A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.

- If a series $\sum a_n$ is convergent but the series of absolute values $\sum |a_n|$ is divergent then the series $\sum a_n$ is **conditionally convergent**.

For example:

- The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges absolutely, because the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series with $p=2>1$)

- The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges conditionally, because the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series).

FACT: If $\sum a_n$ converges absolutely then it is also convergent. (use it as a convergence Test).

EXAMPLE 4. *Determine if each of the following series are absolutely convergent, conditionally convergent or divergent.*

(a) $\sum_{n=1}^{\infty} \frac{1}{n^4}$ both convergent and absolutely convergent

Note: Every convergent positive series is automatically absolutely convergent.

(~~a~~ b) $\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$ not positive series (and not alternating).

Consider $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^3}$. This series is convergent

by Comparison Test:

$$\frac{|\sin n|}{n^3} \leq \frac{1}{n^3}$$

and $\sum \frac{1}{n^3}$ is convergent (p-series, $p = 3 > 1$)

So, the given series converges absolutely and then it is convergent (see Fact)

Remainder Estimate

The Alternating Series Theorem. *If* $\sum_{n=1}^{\infty}(-1)^n b_n$ *is a convergent alternating series and you used a partial sum* s_n *to approximate the sum* s *(i.e.* $s \approx s_n$*) then*

$$|R_n| \le b_{n+1}.$$

$$S = \sum_{n=1}^{\infty} (-1)^n b_n = \underbrace{-b_1 + b_2 - b_3 + b_4 - \dots + (-1)^n b_n}_{S_n} + \underbrace{(-1)^{n+1} b_{n+1} \dots}_{R_n}$$

$$S = S_n + R_n$$

$$R_n = S - S_n \qquad -b_{n+1} \le R_n \le b_{n+1}$$

$$S \in (S_n - b_{n+1},\ S_n + b_{n+1})$$

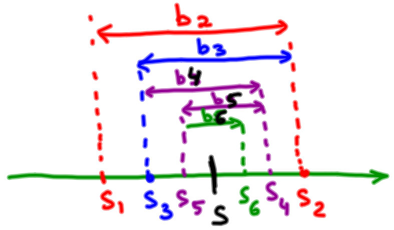
AST $\quad \sum (-1)^n b_n \quad (b_n > 0)$ ← convergent

$\lim_{n\to\infty} b_n = 0$ ✓ and $\{b_n\}$ decreasing ✓ → yes; → NO: TEST FAILS

$$\sum_{n=1}^{\infty} (-1)^n b_n = \overbrace{\underbrace{-b_1}_{S_1} + b_2}^{S_2} - b_3 + b_4 - b_5 + \dots$$

$$S_1 = -b_1$$
$$S_2 = S_1 + b_2$$
$$S_3 = S_2 - b_3$$
$$S_4 = S_3 + b_4$$

$b_1 > b_2 > b_3 > b_4 > \dots$



$$|R_1| = |S - S_1| \le b_2$$
$$|R_2| = |S - S_2| \le b_3$$
$$|R_3| = |S - S_3| \le b_4$$
$$\vdots$$

$$|R_n| = |S - S_n| \le b_{n+1}$$

EXAMPLE 5. *Given* $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$. alternating series

(a) *Show that the series converges. Does it converges absolutely?*

The series converges absolutely, because $\sum \frac{1}{n^3}$ is convergent $(p=3)$.
So by Fact, the given series converges.

Way2: use AST.

(b) *Use s_6 to approximate the sum of the series.*

$$s_6 = -1 + \frac{1}{8} - \frac{1}{27} + \frac{1}{64} - \frac{1}{125} + \frac{1}{216} \approx -0.89978$$

(c) *Determine the upper bound on the error in using s_6 to approximate the sum.*

$$|R_6| \leq b_7$$

$$|R_6| \leq \frac{1}{7^3} \approx 0.0029$$

Note that (b) & (c) $\Rightarrow$ $S \approx -0.89978 \pm 0.0029$

EXAMPLE 6. *Given* $\sum_{n=1}^{\infty} \frac{(-1)^{n+3}}{n^5}$. $= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5}$

(a) *Show that the series converges.*

Way 1 Use absolute convergence (see Ex. 5(a))

Way 2 Apply AST : $b_n = \frac{1}{n^5}$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^5} = 0$

$\{b_n\} = \{\frac{1}{n^5}\}$ is decreasing $\left(n < n+1 \Rightarrow n^5 < (n+1)^5 \Rightarrow \frac{1}{n^5} > \frac{1}{(n+1)^5} \text{ for all } n \geq 1 \right)$

(b) *Approximate the sum of the series with error less than* 10^{-5}.

First, find n such that $|R_n| < 10^{-5}$

By Alternating Series Theorem $|R_n| < b_{n+1}$, or

$|R_n| < \frac{1}{(n+1)^5}$

$\frac{1}{(n+1)^5} < 10^{-5}$

$(n+1)^5 > \frac{1}{10^{-5}}$

$(n+1)^5 > 10^5$

$n+1 > 10 \Rightarrow n > 9$

$S_{10} = 1 - \frac{1}{2^5} + \frac{1}{3^5} - \dots - \frac{1}{10^5} \approx 0.972116$

arbitrary

RATIO TEST For a series $\sum a_n$ define

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|}$$

- If $L < 1$ then the series is absolutely convergent (which implies the series is convergent.)
- If $L > 1$ then the series is divergent.
- If $L = 1$ then the series may be divergent, conditionally convergent or absolutely convergent (test fails).

$\sum \frac{1}{n}$ divergent | $\sum \frac{(-1)^n}{n}$ conditionally convergent | $\sum \frac{1}{n^5}$, $\sum \frac{(-1)^n}{n^5}$ absolutely convergent

$$L = \lim_{n\to\infty} \frac{n^5}{(n+1)^5} = 1.$$

$$L = \lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

No conclusion can be made using RT.

Note: Usually RT is useful when general term of series involves factorials $n!$ an exponents a^n $(4^n, 3^{n^2})$

EXAMPLE 7. *Determine if the following series are convergent or divergent:*

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-10)^n}{4^{2n+1}(n+1)}$

$$|a_n| = \frac{10^n}{4^{2n+1}(n+1)}$$

$$|a_{n+1}| = \frac{10^{n+1}}{4^{2(n+1)+1}((n+1)+1)} = \frac{10^{n+1}}{4^{2n+3}(n+2)}$$

$$L = \lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|} = \lim_{n\to\infty} \frac{10^{n+1}}{4^{2n+3}(n+2)} \cdot \frac{4^{2n+1}(n+1)}{10^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{10^n} \cdot 10 \cdot \cancel{4^{2n+1}}}{\cancel{4^{2n+1}} \cdot 4^2 \cdot \cancel{10^n}} \cdot \frac{n+1}{n+2} = \frac{10}{16} \lim_{n\to\infty} \frac{n+1}{n+2} = \frac{10}{16} < 1$$

The series converges absolutely by RT and then it is convergent.

Factorials: $n \geq 1$ integer

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$$

$$0! = 1$$

$$(n+1)! = \underbrace{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n}_{n!} \cdot (n+1) = n!\,(n+1)$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{5^n}$

$$|a_n| = \frac{n!}{5^n} \qquad |a_{n+1}| = \frac{(n+1)!}{5^{n+1}} = \frac{n!(n+1)}{5 \cdot 5^n}$$

$$\lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|} = \lim_{n\to\infty} \frac{\cancel{n!}(n+1)}{5 \cdot \cancel{5^n}} \cdot \frac{\cancel{5^n}}{\cancel{n!}} = \frac{1}{5} \lim_{n\to\infty} (n+1) = \infty > 1$$

The series diverges by RT.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n+1)!}{n!10^n}$

$$|a_n| = \frac{(2n+1)!}{n!\ 10^n}$$

$$|a_{n+1}| = \frac{(2(n+1)+1)!}{(n+1)! \cdot 10^{n+1}} = \frac{(2n+3)!}{(n+1)!\ 10^{n+1}}$$

$$= \frac{(2n+1)!\,(2n+2)(2n+3)}{n!\,(n+1)\cdot 10^n \cdot 10}$$

$$L = \lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|} = \lim_{n\to\infty} \frac{\cancel{(2n+1)!}\,(2n+2)(2n+3)}{\cancel{n!}\,(n+1)\,\cancel{10^n}\cdot 10} \cdot \frac{\cancel{n!}\,\cancel{10^n}}{\cancel{(2n+1)!}}$$

$$= \lim_{n\to\infty} \frac{2(n+1)(2n+3)}{10(n+1)} = \lim_{n\to\infty} \frac{2n+3}{5} = \infty > 1$$

The series diverges by Ratio Test.