10.5: Power Series

DEFINITION 1. *A power series about $x = a$ (or centered at $x = a$), or just* **power series**, *is any series that can be written in the form*

center

$$\sum_{n=0}^{\infty} c_n (x-a)^n,$$

coefficients

where a and c_n are numbers. The c_n's are called the **coefficients** *of the power series.*

THEOREM 2. *For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are only 3 possibilities:*

1. *The series converges only for $x = a$.* $R=0$
2. *The series converges for all x.* $R=\infty$
3. *There is $R > 0$ such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$. We call such R the* **radius of convergence.**

   strictly less

   $x \in (a-R, a+R)$

REMARK 3. In case 1 of the theorem we say that $R = 0$ and in case 2 we say that $R = \infty$

DEFINITION 4. *An* **interval of convergence** *is the interval of all x's for which the power series converges.*

EXAMPLE 5. *Assume that it is known that the series* $\sum_{n=0}^{\infty} c_n(x+3)^n$ *converges when* $x = -10$ *and diverges when* $x = 6$. *What can be said about the convergence or divergence of the following series:*

$\sum_{n=0}^{\infty} c_n 2^n$

$x+3=2$
$x=-1$

converges, because $\sum c_n (x+3)^n$ converges at $x=-1$.

$\sum_{n=0}^{\infty} c_n (-11)^n$

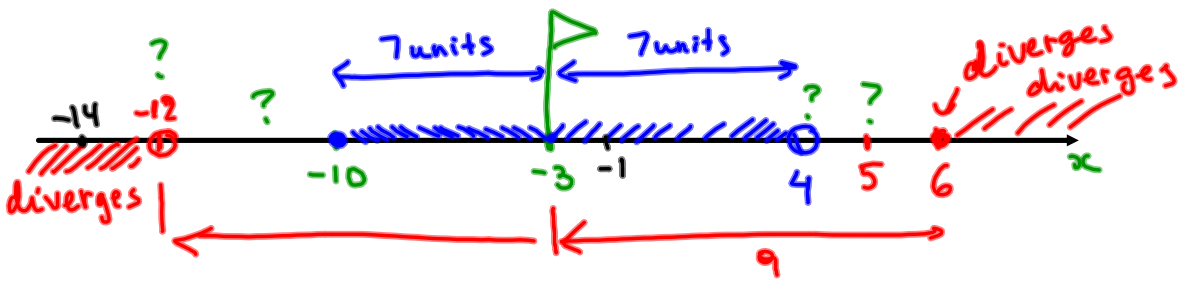
$x+3=-11$
$x=-14$

diverges. $\sum c_n (x+3)^n$ diverges at $x=-14$

$\sum_{n=0}^{\infty} c_n 8^n$

$x+3=8$
$x=5$

we don't know
need more information.



EXAMPLE 6. *Given* $\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}(x+3)^n$.

(a) *Find the radius of convergence.*

Apply ratio test $\quad |a_n| = \frac{n \cdot |x+3|^n}{4^n}$

$$|a_{n+1}| = \frac{(n+1)\,|x+3|^{n+1}}{4^{n+1}}$$

$$L = \lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)\,\cancel{|x+3|^n} \cdot |x+3|}{4 \cdot \cancel{4^n}} \cdot \frac{\cancel{4^n}}{n\,\cancel{|x+3|^n}}$$

$$= \frac{|x+3|}{4} \lim_{n\to\infty} \frac{n+1}{n} = \frac{|x+3|}{4} < 1 \Rightarrow |x+3| < 4 \Rightarrow \boxed{R=4}$$

(b) *Find the interval of convergence.*

$$|x+3| < 4$$

$$-4 < x+3 < 4$$

$$-4-3 < x < 4-3$$

$$-7 < x < 1$$

end points

Check end points

$x = -7$

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} (-4)^n = \sum_{n=1}^{\infty} \frac{n\,4^n}{4^n}$$

$= \sum_{n=1}^{\infty} n$ diverges by Divergence Test ($\lim_{n\to\infty} n \neq 0$).

$x = 1$

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} \cdot 4^n = \sum_{n=1}^{\infty} (-1)^n n$$

the series diverges by Divergence Test, because $\lim_{n\to\infty} (-1)^n n$ DNE

$$\left((-1)^n n = \begin{cases} n, & n \text{ is even} \\ -n, & n \text{ is odd} \end{cases} \right)$$

Conclusion: interval of convergence : $(-7, 1)$

EXAMPLE 7. *Given* $\sum_{n=1}^{\infty} \frac{2^n}{n}(3x-6)^n$.

(a) *Find the radius of convergence.*

By Ratio test:

$$|a_n| = \frac{2^n}{n} \cdot |3x-6|^n$$

$$|a_{n+1}| = \frac{2^{n+1}}{n+1} |3x-6|^{n+1}$$

$$L = \lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|} = \lim_{n\to\infty} \frac{\cancel{2^n} \cdot 2 \, \cancel{|3x-6|^n} \cdot |3x-6| \cdot n}{(n+1) \, \cancel{2^n} \, \cancel{|3x-6|^n}}$$

$$= 2|3x-6| \underbrace{\lim_{n\to\infty} \frac{n}{n+1}}_{1} = 2|3x-6| < 1$$

$$2 \cdot 3 |x-2| < 1$$

$$|x-2| < \frac{1}{6}$$

$$R = \frac{1}{6}$$

(b) *Find the interval of convergence.*

$$|x-2| < \frac{1}{6} \Rightarrow -\frac{1}{6} < x-2 < \frac{1}{6} \Rightarrow$$

$$2-\frac{1}{6} < x < 2+\frac{1}{6} \Rightarrow \underbrace{\frac{11}{6} < x < \frac{13}{6}}$$ corresponds to $L<1$

Check end points: $(L=1)$

$x = \frac{11}{6}$

$x = \frac{13}{6}$

At $x = \frac{11}{6}$ we have

$$\sum_{n=1}^{\infty} \frac{2^n}{n} \underbrace{\left(3 \cdot \frac{11}{6} - 6\right)^n}_{\left(3\left(\frac{11}{6} - 2\right)\right)^n \; 3^n \cdot \left(-\frac{1}{6}\right)^n} = \sum_{n=1}^{\infty} \frac{\cancel{2^n \cdot 3^n} \cdot (-1)^n}{n \, \cancel{6^n}} = \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{n}}_{\text{converges by AC Test (see ex...)}}$$

At $x = \frac{13}{6}$ we have

$$\sum_{n=1}^{\infty} \frac{2^n}{n} \left(3 \cdot \frac{13}{6} - 6\right)^n = \sum_{n=1}^{\infty} \frac{2^n 3^n}{n} \underbrace{\left(\frac{13}{6} - 2\right)^n}_{\frac{1}{6}} = \sum_{n=1}^{\infty} \frac{\cancel{6^n}}{n} \cdot \cancel{\left(\frac{1}{6}\right)^n} = \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\text{divergent}}$$

harmonic $\leftarrow$ $\left\{\begin{array}{l}\text{p-series} \\ (p=1).\end{array}\right.$

Interval of convergence $\left[\frac{11}{6}, \frac{13}{6}\right)$

EXAMPLE 8. *Given* $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(3n+1)!}(x+8)^n$.

(a) *Find the radius of convergence.*

$$|a_n| = \frac{|x+8|^n}{(3n+1)!} \; ; \; |a_{n+1}| = \frac{|x+8|^{n+1}}{(3(n+1)+1)!} = \frac{|x+8|^{n+1}}{(3n+4)!}$$

Use Ratio Test:

$$L = \lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|} = \lim_{n\to\infty} \frac{|x+8|^{\cancel{n+1}}}{(3n+4)!} \cdot \frac{(3n+1)!}{\cancel{|x+8|^n}} = |x+8| \lim_{n\to\infty} \frac{\cancel{(3n+1)!}}{\cancel{(3n+1)!}\cdot(3n+2)(3n+3)(3n+4)}$$

$$L = |x+8| \lim_{n\to\infty} \frac{1}{(3n+2)(3n+3)(3n+4)} = 0 < 1$$ for all x,

Note: $L<1$ for all x, i.e. the given power series converges absolutely for all x. i.e. $R = \infty$

(b) *Find the interval of convergence.* $(-\infty, \infty)$.

EXAMPLE 9. *Given* $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{9^{n-1}}(x+8)^n$.

(a) *Find the radius of convergence.*

$$L = \lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|} = \lim_{n\to\infty} \frac{(\underbrace{2(n+1)}_{2n+2})! \, |x+8|^{\cancel{n+1}} \cdot 9^{n-1}}{9^{(n+1)-1} \, (2n)! \, \cancel{|x+8|^n}}$$

$$= |x+8| \lim_{n\to\infty} \frac{\cancel{(2n)!}\,(2n+1)(2n+2)\cdot \cancel{9^{n-1}}}{\cancel{9^n}\,\cancel{(2n)!}} = \frac{|x+8|}{9} \lim_{n\to\infty} (2n+1)(2n+2)$$

$= \infty > 1$ (if $x \neq -8$).

So, series diverges for all $x \neq -8$

$\boxed{R = 0}$

(b) *Find the interval of convergence.* $\{-8\}$ singleton $[8, 8]$