10.6: Representation of Functions as Power Series (see sec. 10.2)

Problem: Find the sum of the following series: $\underbrace{\sum_{n=0}^{\infty} x^n}_{\text{geometric series}} = \frac{1}{1-x}$ $\quad (|x|<1)$

$a=1,\ r=x$

BTW, by Ratio Test

$$L = \lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|} = \lim_{n\to\infty} \frac{|x|^{n+1}}{|x|^n} = |x| < 1$$

the series converges

$R=1$

$L=1$ means $\sum_{n=0}^{\infty} 1^n = \sum_{n=0}^{1} 1$ divergent $(x=1)$

$\sum_{n=0}^{\infty} (-1)^n$ divergent $(x=-1)$.

$x=\frac{1}{2} \Rightarrow 2 = \sum_{n=0}^{\infty} \frac{1}{2^n}$

EXAMPLE 1. *Find a power series representation for $f(x)$ and determine the interval of convergence.*

(a) $f(x) = \dfrac{1}{5-x}$ $= \dfrac{1}{5\left(1-\frac{x}{5}\right)} = \dfrac{1}{5}\cdot\dfrac{1}{1-\frac{x}{5}}$

$$= \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n = \frac{1}{5}\sum_{n=0}^{\infty}\frac{x^n}{5^n} = \sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}},$$

where $\left|\frac{x}{5}\right|<1$, or $|x|<5$,

or $x \in (-5,5)$.

$$\frac{a}{1-\text{☺}} = \sum_{n=0}^{\infty} a\cdot \text{☺}^n,$$

where $|\text{☺}|<1$

(b) $f(x) = \dfrac{x}{5-x}$ $= x\cdot\dfrac{1}{5-x} \overset{(a)}{=} x\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}} = \sum_{n=0}^{\infty}\frac{x^{n+1}}{5^{n+1}},$

where $x \in (-5,5)$.

(c) $f(x) = \dfrac{1}{1+x^2}$ $= \dfrac{1}{1-(-x^2)} = \displaystyle\sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n},$

where $|-x^2|<1 \Rightarrow |x|^2<1 \Rightarrow |x|<1 \Rightarrow$

$x \in (-1,1)$.

Differentiation and Integration of power series

THEOREM 2. *If the power series* $\sum_{n=0}^{\infty} c_n(x-a)^n$ *has radius of convergence* $\underbrace{R > 0}$*, then the function* f *defined by*

$|x-a| < R$

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ *and*

- $f'(x) = \sum_{\cancel{n=0}\ n=1}^{\infty} n c_n (x-a)^{n-1}$
- $\int f(x)\,\mathrm{d}x = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$

TERMWISE DIFFERENTIATION and INTEGRATION

The radii of convergence of the power series for $f'(x)$ *and* $\int f(x)\,\mathrm{d}x$ *are both* R.

$$f'(x) = \left(\sum_{n=0}^{\infty} c_n (x-a)^n\right)' = \sum_{n=0}^{\infty} \left(c_n \underbrace{(x-a)^n}\right)'$$

$$= \sum_{n=0}^{\infty} c_n\, n\, (x-a)^{n-1}$$

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EXAMPLE 3. *Find a power series representation for $f(x)$ and determine the radius of convergence.*

(a) $f(x) = \dfrac{1}{(5-x)^2}$

$$\int f(x)dx = \int \frac{dx}{(5-x)^2} = \frac{1}{5-x} = \frac{1}{5(1-\frac{x}{5})} = \frac{1}{5} \cdot \frac{1}{1-\frac{x}{5}} =$$

$u = 5-x$

$$\int \frac{-du}{u^2} = \frac{1}{u}$$

$$= \frac{1}{5} \sum_{n=0}^{\infty} \frac{x}{5}^{n}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}}$$

where $\left|\frac{x}{5}\right| < 1$, or $|x| < 5$, or $-5 < x < 5$

So, $\displaystyle\int f(x)dx = \sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}} \quad (|x|<5)$

Differentiate both sides and apply Th.2 to the right hand side:

$$\frac{d}{dx}\left(\int f(x)\,dx\right) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{x^n}{5^{n+1}}\right)$$

$$f(x) = \sum_{n=0}^{\infty} \frac{n x^{n-1}}{5^{n+1}} = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{5^{n+1}},$$

where $|x| < 5$

(b) $f(x) = \ln(2+x)$

$$f'(x) = \frac{1}{2+x} = \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2\left(1-\left(-\frac{x}{2}\right)\right)}$$

$$f'(x) = \frac{1}{2}\sum_{n=0}^{\infty}\Big(\underbrace{-\frac{x}{2}}_{\left|\frac{x}{2}\right|<1}\Big)^n = \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^{n+1}} \quad (|x|<2)$$

By Th.2:

$$f(x) = \sum_{n=0}^{\infty}\int \frac{(-1)^n x^n}{2^{n+1}}\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}\int x^n\,dx$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}\cdot\frac{x^{n+1}}{(n+1)} + C$$

Find c:

$$\ln(2+x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{2^{n+1}(n+1)} + C \quad \text{for all } x\in(-2,2).$$

In particular, if $x=0$, we get

$$\ln 2 = 0 + C, \quad \text{so}$$

$$\ln(2+x) = \ln 2 + \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{2^{n+1}(n+1)} \quad (|x|<2).$$