10.7: Taylor and Maclaurin Series

Problem: Assume that a function $f(x)$ has a power series representation about $x = a$:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

and $f(x)$ has derivatives of every order. Find formula for c_n in terms of f.

Solution. We have

$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots + c_n(x-a)^n + \dots$

Putting $x = a$ in $f(x)$, we get $\boxed{\frac{f(a)}{0!} = c_0}$ $\quad 0! = 1$

$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots + \dots + nc_n(x-a)^{n-1} + \dots$

Substituting $x = a$ we have $\boxed{\frac{f'(a)}{1} = c_1}$

Similarly,

$f''(x) = \underbrace{2c_2} + \underbrace{3 \cdot 2 c_3 (x-a)} + \dots + \underbrace{n(n-1)c_n(x-a)^{n-2}} + \dots$

then $x = a \Rightarrow f''(a) = 2c_2$ and $\boxed{c_2 = \frac{f''(a)}{2 \cdot 1}}$

$f'''(x) = 3 \cdot 2 c_3 + \_\_\_ c_4(x-a) + \dots + n(n-1)(n-2)c_n(x-a)^{n-3} + \dots$

then $x = a \Rightarrow f''(a) = 3 \cdot 2 \cdot c_3$ and $\boxed{c_3 = \frac{f'''(a)}{3 \cdot 2 \cdot 1}}$

Continuing in this manner, you can see the pattern:

$f^{(n)}(a) = n(n-1)(n-2)\dots \cdot 2\, c_n \Rightarrow c_n = \frac{f^{(n)}(a)}{n(n-1)(n-2)\dots \cdot 2 \cdot 1}$

$$\boxed{c_n = \frac{f^{(n)}(a)}{n!}}$$

The Taylor series for $f(x)$ about $x = a$:

$0! = 1$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n =$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

Split the Taylor series as follows:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \underbrace{\sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n}_{\substack{T_N(x) \\ N-\text{th degree} \\ \textbf{Taylor polynomial}}} + \underbrace{\sum_{n=N+1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}_{\substack{R_N(x) \\ \textbf{Remainder}}}$$

THEOREM 1. *If* $\lim_{n\to\infty} R_N(x) = 0$ *when* $|x-a| < R$ *then*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \qquad |x-a| < R.$$

REMARK 2. In all examples that we will be looking at, we assume that $f(x)$ has a power series expansion, i.e.
$\lim_{n\to\infty} R_N(x) = 0$ for some R. (This means you don't need to show it.)

EXAMPLE 3. *Find Taylor series for $f(x) = e^{3x}$ at $x = \overset{=a}{1}$. What is the associated radius of convergence?*

$$f(x) = e^{3x} = \sum_{n=0}^{\infty} c_n (x-1)^n$$

where $c_n = \dfrac{f^{(n)}(1)}{n!}$

$$f(x) = e^{3x}$$
$$f'(x) = 3e^{3x} \qquad c_n = \frac{3^n e^3}{n!}$$
$$f''(x) = 3 \cdot 3 e^{3x} = 3^2 e^{3x}$$
$$f'''(x) = 3^3 e^{3x}$$
$$\vdots$$
$$f^{(n)}(x) = 3^n e^{3x} \Rightarrow f^{(n)}(1) = 3^n e^3$$

Answer: $f(x) = \displaystyle\sum_{n=0}^{\infty} \overbrace{\frac{3^n e^3}{n!}(x-1)^n}^{a_n}$,

where $|x-1| < R$.

Radius R:

$$L = \lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|} =$$

$$= \lim_{n\to\infty} \frac{3^{\cancel{n+1}} \cancel{e^3}}{\underbrace{\cancel{(n+1)}!}_{n+1}} \cdot \frac{|x-1|^{\cancel{n+1}} \cdot \cancel{n!}}{\cancel{3^n}\, \cancel{e^3}\, \cancel{|x-1|^n}}$$

$$= |x-1| \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1 \quad \text{for all } x.$$

$R = \infty$.

EXAMPLE 4. *Find Taylor series for $f(x) = \ln x$ at $x = 1$. What is the associated radius of convergence?*

$$\ln x = \sum_{n=0}^{\infty} c_n (x-1)^n$$

$$f(x) = \ln x \qquad f(1) = \ln 1 = 0$$

$$f'(x) = \frac{1}{x} \qquad f'(1) = \frac{1}{1} = 0!$$

$$f''(x) = \left(\frac{1}{x}\right)' = -\frac{1}{x^2} \qquad f''(1) = -\frac{1}{1^2} = -1!$$

$$f'''(x) = \left(-\frac{1}{x^2}\right)' = \frac{2}{x^3} \qquad f'''(1) = \frac{2}{1^3} = 2!$$

$$f^{(4)}(x) = \left(\frac{2}{x^3}\right)' = -\frac{2\cdot 3}{x^4} \qquad f^{(4)}(1) = -\frac{1\cdot 2\cdot 3}{1^4} = -3!$$

$$f^{(5)}(x) = \left(-\frac{2\cdot 3}{x^4}\right)' = \frac{2\cdot 3\cdot 4}{x^5} \qquad f^{(5)}(1) = \frac{1\cdot 2\cdot 3\cdot 4}{1^4} = 4!$$

$$c_n = \frac{f^{(n)}(1)}{n!} \quad (n \geq 1)$$

$$f^{(n)}(1) = (-1)^{n+1}(n-1)!$$

$$c_n = \frac{(-1)^{n+1}(n-1)!}{n!}$$

$$c_n = \frac{(-1)^{n+1}\cancel{(n-1)!}}{\cancel{(n-1)!}\, n}$$

$$c_n = \frac{(-1)^{n+1}}{n}$$

$$c_0 = f(1) = 0$$

So, $$f(x) = \ln x = c_0 + \sum_{n=1}^{\infty} c_n (x-1)^n$$

$$\ln x = \sum_{n=1}^{\infty} \underbrace{\frac{(-1)^{n+1}}{n}(x-1)^n}_{a_n}$$

$$L = \lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|}$$

$$= \lim_{n\to\infty} \frac{|x-1|^{\cancel{n+1}}}{n+1} \cdot \frac{n}{\cancel{|x-1|^n}}$$

$$= |x-1| \lim_{n\to\infty} \frac{n}{n+1} = |x-1| < 1$$

$R = 1.$

EXAMPLE 5. *Find Taylor series for* $\ln(1+x)$ *centered at* $x = 0$. (In other words, find McLauren series of $\ln(1+x)$).

Method 1 Use Taylor Formula (procedure similar to Ex. 4)

Method 2 Use result of Ex. 4

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n, \text{ where } |x-1|<1$$

Then

$$\ln(x+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x+1-1)^n, \text{ where } |x+1-1|<1$$

$$\ln(x+1) = \sum_{n=1}^{\infty} \underbrace{\frac{(-1)^{n+1}}{n}}_{c_n} x^n, \text{ where } |x|<1.$$

What is the associated radius of convergence? $R=1$ (the same as for $\ln x$).

However, note that the decompositions for $\ln x$ and $\ln(1+x)$ have different intervals of convergence.

Determine $f^{(100)}(0)$ *using the obtained power series expansion.*

$$c_n = \frac{(-1)^{n+1}}{n} \Rightarrow c_{100} = \frac{(-1)^{100+1}}{100} = -\frac{1}{100}$$

$$c_n = \frac{f^{(100)}(0)}{n!} \qquad c_{100} = \frac{f^{(100)}(0)}{100!}$$

$$\frac{f^{(100)}(0)}{100!} = -\frac{1}{100}$$

$$f^{(100)}(0) = -\frac{100!}{100} = -\frac{99!\cdot 100}{100}$$

$$\left.\frac{d^{100}(\ln(1+x))}{dx}\right|_{x=0} = \boxed{f^{(100)}(0) = -99!}$$

The Maclaurin series is the Taylor series about $x = 0$ (i.e. a=0):

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

EXAMPLE 6. *Find the Maclaurin series for $f(x)$:* $a=0$

(a) $f(x) = e^x$

$f^{(n)}(x) = e^x \quad \forall n \in \mathbb{N}$

$f^{(n)}(0) = 1 \Rightarrow c_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}$

$e^x = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

Note $R=\infty$ (see Ex. 3)

(b) $f(x) = e^{-x}$

$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$

(c) $f(x) = x^5 e^{-2x^2}$

$= x^5 \sum_{n=0}^{\infty} \frac{(-2x^2)^n}{n!} = x^5 \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^n}{n!} x^{2n}$

$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!} x^{2n+5}$

EXAMPLE 7. *Find the Maclaurin series for $f(x)$:*

(a) $f(x) = \cos x$

$f(x) = \sum_{n=0}^{\infty} c_n x^n$, where $c_n = \frac{f^{(n)}(0)}{n!}$

	$x=0$
$f(x) = \cos x$	1
$f'(x) = -\sin x$	0
$f''(x) = -\cos x$	-1
$f'''(x) = \sin x$	0
$f^{(4)}(x) = \cos x$	1

n odd $n=2k+1$: $C_{\underbrace{2k+1}_{\text{odd}}} = \frac{0}{(2k+1)!} = 0$ for all k

n even $n=2k$: $C_{2k} = \frac{(-1)^k}{(2k)!}$

$n = 2k = 0 \Downarrow k=0$

$$f(x) = \sum_{k=0}^{\infty} \underbrace{\frac{(-1)^k}{(2k)!} x^{2k}}_{a_k}$$ even exponents

OR

$$\underbrace{\cos x}_{\text{even function,}} = f(x) = \underbrace{1}_{k=0} - \underbrace{\frac{x^2}{2!}}_{k=1} + \underbrace{\frac{x^4}{4!}}_{k=2} - \underbrace{\frac{x^6}{6!}}_{k=3} + \dots \quad \dots$$

$$L = \lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|} = \lim_{n\to\infty} \frac{|x|^{2(n+1)}}{(2(n+1))!} \cdot \frac{(2n)!}{|x|^{2n}}$$

$$= \lim_{n\to\infty} \frac{|x|^{2n+2}}{(2n+2)!} \frac{(2n)!}{|x|^{2n}} = \lim_{n\to\infty} \frac{\cancel{|x|^{2n}} \cdot |x|^2 \cdot \cancel{(2n)!}}{\cancel{(2n)!} \cdot (2n+1)(2n+2)\, \cancel{|x|^{2n}}}$$

$$= x^2 \lim_{n\to\infty} \frac{1}{(2n+1)(2n+2)} = 0 < 1$$

So $R = \infty$ (i.e. the series converges for all values of x).

Interval of convergence: $(-\infty, \infty)$

(b) $f(x) = \sin x$

Method 1: Use Taylor series formula as we did in (a).

Method 2: Use the answer from (a).

$$\cos x = \sum_{k=0}^{\infty} \left(\frac{(-1)^k}{(2k)!} x^{2k} \right) \quad (x \in (-\infty, \infty))$$

Integrate (we can do termwise integration (see sec. 10.6))

$$\int \cos x \, dx = \sum_{k=0}^{\infty} \left(\frac{(-1)^k}{(2k)!} \int x^{2k} \, dx \right)$$

$$\sin x = \sum_{k=0}^{\infty} \underbrace{\frac{(-1)^k}{(2k)!} \frac{x^{2k+1}}{2k+1}} + C$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} + C$$

To find C let $x=0$ then

$$\sin 0 = 0 + C \Rightarrow C = 0$$

So,

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \quad (x \in (-\infty, \infty))$$

or

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

know them!

Known Mclaurin series and their intervals of convergence you must have memorized:

function		*power series*		*power series*	*interval of convergence*
$\frac{1}{1-x}$	$=$	$\sum_{n=0}^{\infty} x^n$	$=$	$1+x+x^2+x^3+\dots$	$(1,1)$
e^x	$=$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$=$	$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$	$(-\infty,\infty)$
$\cos x$	$=$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	$=$	$1-\frac{x^2}{2}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots$	$(-\infty,\infty)$
$\sin x$	$=$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$=$	$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots$	$(-\infty,\infty)$
$\arctan x$	$=$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$=$	$x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$	$[-1,1]$

$|x| < 1$

EXAMPLE 8. *Find the sum of the series:*

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n}(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{\pi}{3}\right)^{2n}$

$= \cos \frac{\pi}{3} = \frac{1}{2}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{2016^n}{n!} = e^{2016}$

(c) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{2n+1}$

$= \arctan(x^2)$

where $|x^2| < 1 \Rightarrow |x| < 1$

or $x \in (-1, 1)$