10.9: Applications of Taylor Polynomials

Recall that the Nth degree Taylor Polynomial is defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \underbrace{\sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n}_{\substack{T_N(x) \\ N-\text{th degree} \\ \textbf{Taylor polynomial}}} + \underbrace{\sum_{n=N+1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}_{\substack{R_N(x) \\ \textbf{Remainder}}}$$

$\underbrace{c < x < d}_{\text{interval of convergence}}$

$f(x) \approx T_N(x)$

EXAMPLE 1. *For* $f(x) = \cos x$ *find* $T_N(x)$ *for* $N = 0, 1, 2, \ldots, 8$

We know

$$f(x) = \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$= \underset{+0\cdot x^1}{\overset{x^0}{1}} - \frac{x^{2}}{2!} + \frac{x^{4}}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \ldots$$

$$T_1(x) = T_0(x) = 1, \quad \underset{T_3(x)}{\overset{\shortparallel}{T_2(x)}} = 1 - \frac{x^2}{2}, \quad \underset{T_5(x)}{\overset{\shortparallel}{T_4(x)}} = 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

$$T_7(x) = T_6(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$$

$$T_8(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{90320}$$

REMARK 2. As the degree of the Taylor polynomial increases, it starts to look more and more like the function itself (and thus, it approximates the function better).

REMARK 3. The first degree Taylor polynomial

$$T_1(x) = \overbrace{f(a) + f'(a)(x - a)}^{\text{related to tangent line}}$$

is the same as a linear approximation of f at $x = a$.

Note that $T_2(x)$ is the same as a quadratic approximation.

In general, $f(x)$ is the sum of its Taylor series if $T_N(x) \to f(x)$ as $n \to \infty$. So, $T_N(x)$ can be used as an approximation:

$$\lim_{N\to\infty} T_N(x) = f(x) \qquad\qquad f(x) \approx T_N(x).$$

$\alpha \le x \le \beta$

How to estimate the Remainder $|R_N(x)| = |f(x) - T_N(x)|$? on the interval $[\alpha, \beta]$

- Use graph of $R_N(x)$. $\max\limits_{\alpha\le x\le\beta} |R_N(x)| = \max\limits_{\alpha\le x\le\beta} |f(x) - T_N(x)|$
- If the series happens to be an alternating series, you can use the Alternating Series Theorem. $\max R_N(x) \le \max b_{N+1}(x)$
- In all cases you can use **Taylor's Inequality**:

$$|R_N(x)| \le \frac{M}{(N+1)!}|x-a|^{N+1}$$

where $\left|f^{N+1}(x)\right| \le M$ for all x in an interval containing a.

M can be found as $\max\limits_{\alpha\le x\le\beta} |f^{(N+1)}(x)|$

EXAMPLE 4. *Let* $f(x) = e^{x^2}$.

(a) *Approximate* $f(x)$ *by a Taylor polynomial of degree 3 at* $a = 0$.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\Downarrow$$

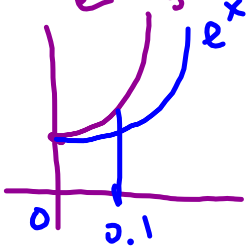
$$e^{x^2} = \underbrace{1 + x^2}_{T_2(x) = T_3(x)} + \frac{x^4}{2} + \frac{x^6}{3!} + \frac{x^8}{8!} + \dots$$

$$e^{x^2} \approx 1 + x^2.$$

Note that $|R_3(x)| = |\overset{= f(x)}{e^{x^2}} - \overset{= T_3(x)}{(1+x^2)}|$

(b) *How accurate is this approximation when* $0 \le x \le 0.1$

Method 1 Using graph: $\max\limits_{0 \le x \le 0.1} |R_3(x)| = \max\limits_{0 \le x \le 0.1} |\underbrace{e^{x^2} - (1+x^2)}_{h(x)}|$



$$\Downarrow$$

$\max |R_3(x)| = R_3(0.1)$

Find critical values:

$$h'(x) = (e^{x^2} - 1 - x^2)' = 2x e^{x^2} - 2x = 0$$

$$2x(e^{x^2} - 1) = 0$$

$x = 0$ or $e^{x^2} - 1 = 0$

$e^{x^2} = 1$

$x = 0$

Critical point $x=0$ coincides with one of the end points.

So

$$\max_{0 \le x \le 0.1} |h(x)| = \max\{h(0), h(0.1)\}$$

$h(0) = e^{0^2} - 1 - 0^2 = 0$; $h(0.1) = e^{0.01} - 1 - 0.01 \approx 5 \cdot 10^{-5}$

$$|R_3(x)| \le 5 \cdot 10^{-5}.$$

Method2 Using Taylor Inequality.

$$|R_3(x)| \le \frac{M\,|x-0|^4}{4!}$$

Note: $N=3$, $a=0$, $f(x)=e^{x^2}$

or $|R_3(x)| \le \frac{M\,x^4}{24}$, where $M = \max\limits_{0\le x\le 0.1} |f^{4}(x)|$

$f(x)=e^{x^2} \Rightarrow f'(x) = 2xe^{x^2} \Rightarrow f''(x) = 2\left(e^{x^2} + 2x^2e^{x^2}\right) \Rightarrow$

$f'''(x) = 2\left(2xe^{x^2} + 2\left(2xe^{x^2} + 2x^3e^{x^2}\right)\right)$

$f^{(4)}(x) = 2e^{x^2}\left(6 + 24x^2 + 8x^4\right)$

$= 4e^{x^2}\left(3 + 12x^2 + 4x^4\right) = g(x)$

Now $M = \max\limits_{0\le x\le 0.1} |g(x)|$

~~Find critical values of $g(x)$ on $[0, 0.1]$:~~

~~$g'(x)=0$, or~~

Note $g'(x) = 4\cdot 2xe^{x^2}(3+12x^2+4x^4)$
$+4e^{x^2}(24x + 16x^3) > 0$ on $(0, 0.1)$

So, $g(x)$ is strictly increasing on $(0, 0.1)$.

Thus $M = \max\limits_{0\le x\le 0.1} |g(x)| = \max\{g(0), g(0.1)\}$

$g(0) = 12$

$g(0.1) = 12.607$

So, $M \approx 12.607$

Conclusion:

$$|R_3(x)| \lesssim \frac{12.607}{24} \max_{0 \le x \le 0.1} x^4$$

$$\lesssim \frac{12.607}{24} (0.1)^4$$

$$\lesssim 0.53 \cdot 10^{-4} \lesssim \underbrace{5.3 \cdot 10^{-5}}$$

EXAMPLE 5. *Find $T_2(x)$ for $f(x) = \cos x$ at $x = \pi/4$. How accurate this approximation when $\pi/6 \le x \le 2\pi/3$.*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\frac{\pi}{4})}{n!}\left(x-\frac{\pi}{4}\right)^n$$

$$T_2(x) = \sum_{n=0}^{2} \frac{f^{(n)}(\frac{\pi}{4})}{n!}\left(x-\frac{\pi}{4}\right)^n$$

$$T_2(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right) + \frac{f''(\frac{\pi}{4})}{2}\left(x-\frac{\pi}{4}\right)^2$$

$$f\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$f'\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$$

$$f''\left(\frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$$

$$T_2(x) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x-\frac{\pi}{4}\right)^2$$

To find how accurate the approximation

$$\cos x \approx T_2(x)$$

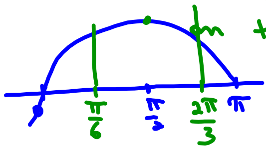
on $\left[\frac{\pi}{6}, \frac{2\pi}{3}\right]$, apply Taylor inequality:

$$|R_N(x)| \le \frac{M}{(N+1)!}|x-a|^{N+1}$$

In our case $N=2$, $a=\frac{\pi}{4}$, we can find M as

$$M = \max_{\frac{\pi}{6}\le x\le\frac{2\pi}{3}} |f^{(3)}(x)|$$ (upper bound for $f^{(N+1)}(x)$ on the interval)

Find M: $f^{(3)}(x) = \sin x$



Using graph we conclude that $\max_{\frac{\pi}{6}\le x\le\frac{2\pi}{3}} |\sin x| = 1$. So, $M = 1$.

Finally, $$|R_2(x)| \le \frac{1}{3!}\underbrace{\left|x-\frac{\pi}{4}\right|^3}_{\text{replace by } \left(\max_{\frac{\pi}{6}\le x\le\frac{2\pi}{3}}\left|x-\frac{\pi}{4}\right|\right)^3}$$

We have $\frac{\pi}{6} \le x \le \frac{2\pi}{3}$

$$\frac{\pi}{6} - \frac{\pi}{4} \le x - \frac{\pi}{4} \le \frac{2\pi}{3} - \frac{\pi}{4}$$

$$-\frac{\pi}{12} \le x - \frac{\pi}{4} \le \frac{5\pi}{12}$$

Thus, $$\max_{\frac{\pi}{6}\le x\le\frac{2\pi}{3}}\left|x-\frac{\pi}{4}\right| = \max\left\{\frac{\pi}{12}, \frac{5\pi}{12}\right\} = \frac{5\pi}{12}$$

So, $$|R_2(x)| \le \frac{1}{6}\left(\frac{5\pi}{12}\right)^3 \approx 0.373822$$

$\ln 1.2 = f(0.2)$

EXAMPLE 6. *How many terms of the Maclaurin series for $f(x) = \ln(x+1)$ do you need to use to estimate* $\ln 1.2$ *to within 0.001.*

We already know (see Sec. 10.7)

$$f(x) = \ln(1+x) = \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}}_{\text{Alternating}} \qquad b_n$$

$$\left| R_n(\overset{=0.2}{x}) \right| \le b_{n+1}(x) \le 0.001$$

We need to find n such that

$$b_{n+1}(0.2) \le 0.001$$

$$\frac{(0.2)^{n+2}}{n+2} \le 0.001$$

$0.2 = \frac{1}{5}$

$(0.2)^{n+2} = \frac{1}{5^{n+2}}$

$$\frac{1}{5^{n+2}(n+2)} \le \frac{1}{1000}$$

$$5^{n+2}(n+2) \ge 1000$$

$\checkmark\ n=0$ $\quad 5^2 \cdot 2 \ge 1000$ (false)

$\checkmark\ n=1$ $\quad 5^3 \cdot 3 = 125 \cdot 3 \ge 1000$ (false)

$\checkmark\ n=2$ $\quad 5^4 \cdot 4 = 625 \cdot 4 \ge 1000$ (True)

We need 3 terms of Maclaurin series for the required accuracy.

$$\ln(1.2) = f(0.2) = f\left(\frac{1}{5}\right) = \frac{1}{5} - \frac{1}{25 \cdot 2} + \frac{1}{125 \cdot 3}$$

$$\approx 0.18266$$

Note that using a calculator, one can get $\ln(1.2) \approx 0.182321$