# Sections 6.2-6.4 Review

## 6.2: Area

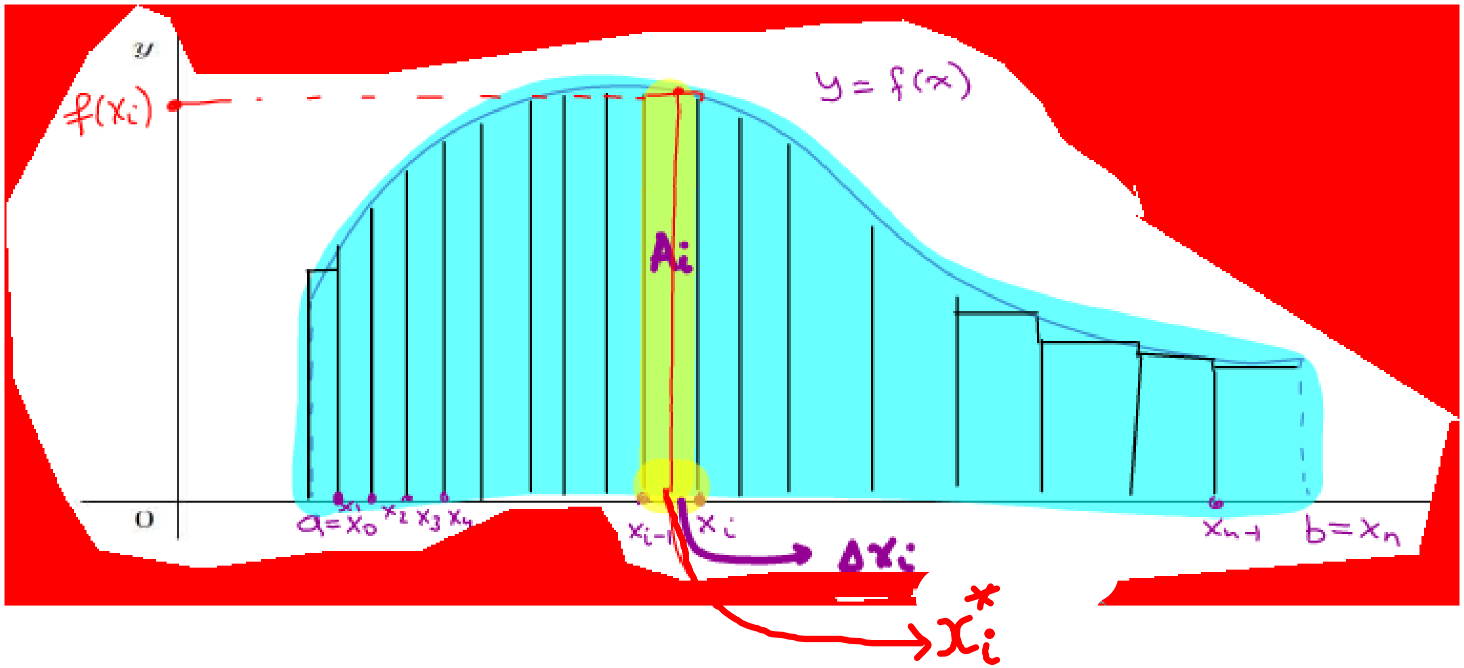
**Area problem**: Let a function $f(x)$ be positive on some interval $[a, b]$ and $D$ be the region between the function and the $x$-axis, i.e.

$$D = \{(x, y)|\ a \le x \le b, 0 \le y \le f(x)\}.$$

Then the area of $D$ is

$$A(D) = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

Here $P$ is a partition of the interval $[a, b]$, $\Delta x_i = x_i - x_{i-1}$, and $x_i^*$ is any point in the $i$-th subinterval.

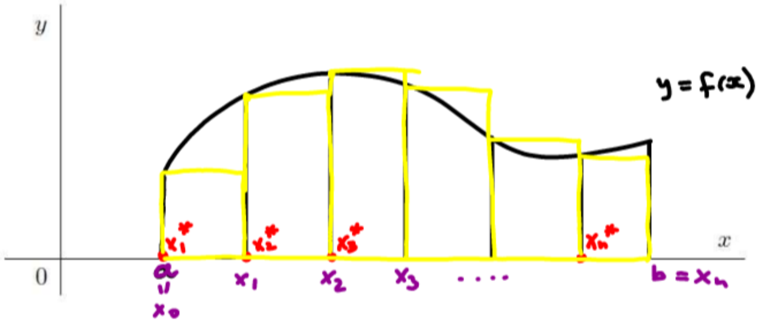


**Riemann Sum** for a function $f(x)$ on the interval $[a, b]$ is a sum of the form:

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i.$$

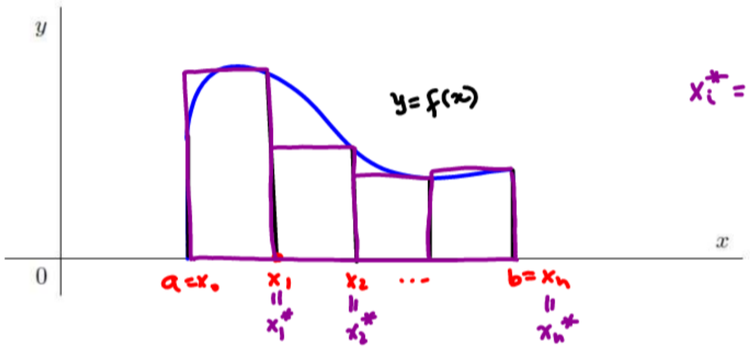
Consider a partition has equal subintervals: $x_i = a + i\Delta x$, where $\Delta x = \dfrac{b-a}{n}$.

LEFT-HAND RIEMANN SUM : $L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x = \sum_{i=1}^{n} f(a + (i-1)\Delta x)\Delta x$



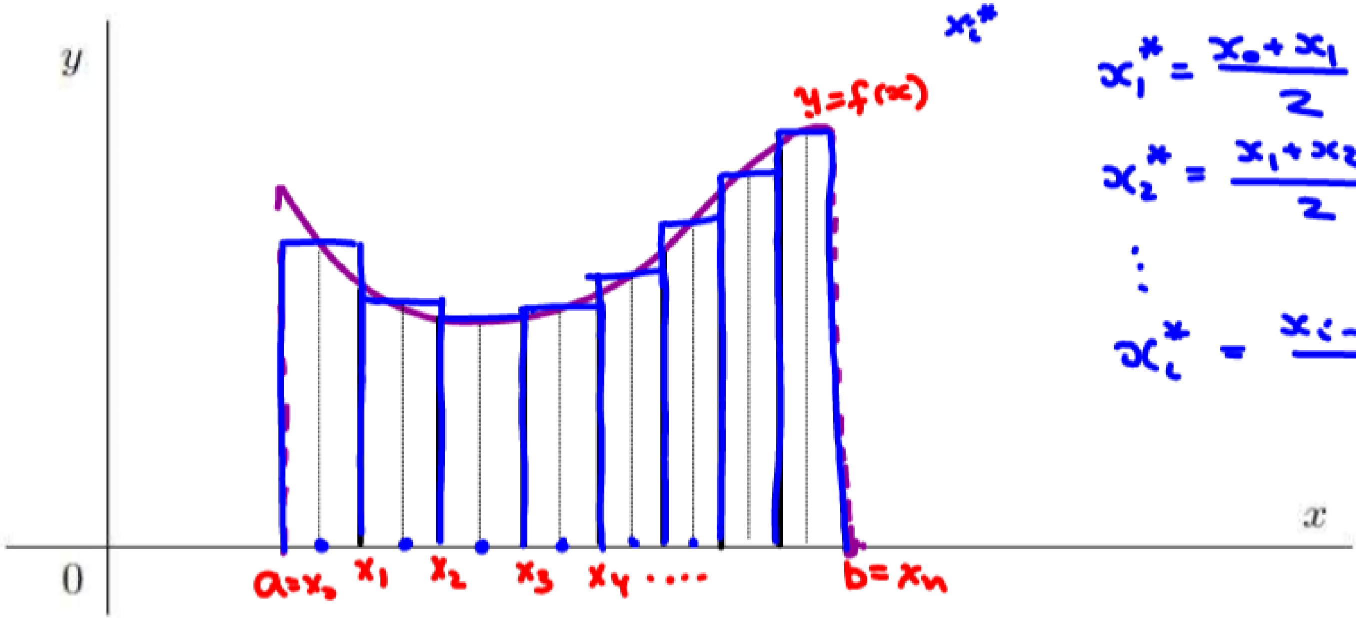

$x_1^* = x_0$
$x_2^* = x_1$
$x_3^* = x_2$
. . . .
$x_i^* = x_{i-1} = a+(i-1)\Delta x$

RIGHT-HAND RIEMANN SUM : $R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} f(a+i\Delta x)\Delta x$




$x_i^* = x_i$

MIDPOINT RIEMANN SUM : $M_n = \sum_{i=1}^{n} f\left(\underbrace{\frac{x_i + x_{i-1}}{2}}_{x_i^*}\right)\Delta x = \Delta x \sum_{i=1}^{n} f\left(a + \frac{2i-1}{2}\Delta x\right)$




$$x_1^* = \frac{x_0 + x_1}{2}$$

$$x_2^* = \frac{x_1 + x_2}{2}$$

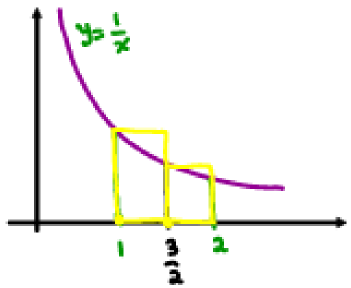
$$\vdots$$

$$x_i^* = \frac{x_{i-1} + x_i}{2}$$

$$\frac{x_i + x_{i-1}}{2} = \frac{a + i\Delta x + a + (i-1)\Delta x}{2} = \frac{2a + \Delta x(i + i - 1)}{2}$$

$$= a + \frac{2i-1}{2}\Delta x$$

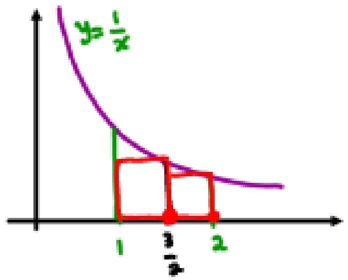
EXAMPLE 2. *Given* $f(x) = \frac{1}{x}$ *on* $[1, 2]$ $(a = b)$. *Calculate* $L_2, R_2, M_2$. $\Rightarrow n = 2$ $\quad \Delta x = \frac{b-a}{2}$




$L_2$ left-hand R.S. $\quad \Delta x = \frac{2-1}{2} = \frac{1}{2}$

$L_2 = f(1)\Delta x + f(\frac{3}{2})\Delta x$

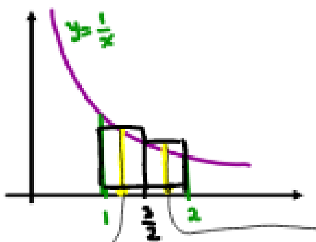
$= (f(1) + f(\frac{3}{2}))\Delta x = (1 + \frac{2}{3})\frac{1}{2} = \frac{5}{3} \cdot \frac{1}{2} = \boxed{\frac{5}{6}}$




$R_2$ right-hand R.S

$R_2 = (f(\frac{3}{2}) + f(2))\Delta x$

$= (\frac{2}{3} + \frac{1}{2})\frac{1}{2} = \frac{7}{6} \cdot \frac{1}{2} = \boxed{\frac{7}{12}}$




$x_1^* = \frac{1 + \frac{3}{2}}{2} = \frac{5}{4}$ $\quad x_2^* = \frac{\frac{3}{2} + 2}{2} = \frac{7}{4}$

$M_2$ midpoint R. S.

$M_2 = (f(x_1^*) + f(x_2^*))\Delta x$

$= (f(\frac{5}{4}) + f(\frac{7}{4}))\frac{1}{2}$

$= (\frac{4}{5} + \frac{4}{7})\frac{1}{2}$

$= {}^{2}\cancel{4}(\frac{1}{5} + \frac{1}{7})\frac{1}{\cancel{2}_{1}}$

$= 2 \cdot \frac{12}{35} = \boxed{\frac{24}{35}}$

EXAMPLE 2. *Represent area bounded by $f(x)$ on the given interval using Riemann sum. Do not evaluate the limit.*

(a) $f(x) = x^2 + 2$ *on* $[0, 3]$ *using right endpoints.*

$a = 0$, $b = 3 \Rightarrow \Delta x = \frac{b-a}{n} = \frac{3}{n}$

$\|P\| = \max\limits_{1 \le i \le n} \Delta x_i = \Delta x = \frac{3}{n} \to 0 \iff n \to \infty$

$x_i^* = x_i = a + i\Delta x = 0 + i\frac{3}{n} = \frac{3i}{n}$

$f(x_i^*) = f\left(\frac{3i}{n}\right) = \left(\frac{3i}{n}\right)^2 + 2 = \frac{9i^2}{n^2} + 2$

$$A = \lim_{n \to \infty} \frac{3}{n} \sum_{i=1}^{n} \left(\frac{9i^2}{n^2} + 2\right)$$

(b) $f(x) = \sqrt{x^2+2}$ *on* $[0,3]$ *using left endpoints.*

$$A = \lim_{\|P\| \to 0} \Delta x \sum_{i=1}^{n} f(x_i^*)$$

$$\Delta x = \frac{3}{n} \qquad \|P\| \to 0 \iff n \to \infty$$

$$x_i^* = x_{i-1} = a + (i-1)\Delta x = \frac{3(i-1)}{n}$$

$$f(x_i^*) = \sqrt{\left(\frac{3(i-1)}{n}\right)^2 + 2} = \sqrt{\frac{9(i-1)^2}{n^2} + 2}$$

$$A = \lim_{n \to \infty} \frac{3}{n} \sum_{i=1}^{n} \sqrt{\frac{9(i-1)^2}{n^2} + 2}$$

EXAMPLE 4. *The following limits represent the area under the graph of $f(x)$ on an interval $[a,b]$. Find $f(x), a, b$.*

(a) $\underbrace{\lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n}\sqrt{1+\frac{3i}{n}}}_{R_n}$ $f(x_i^*)$

vs $\lim_{n\to\infty} \Delta x \sum_{i=1}^{n} f(x_i^*)$

$\Delta x = \frac{b-a}{n} = \frac{3}{n} \Rightarrow b-a=3$

$f(x)=\sqrt{1+x}$

$x_i^* = x_i = \frac{3i}{n} = a + i\Delta x = a + \frac{3i}{n} \Rightarrow a=0 \Rightarrow b=3$

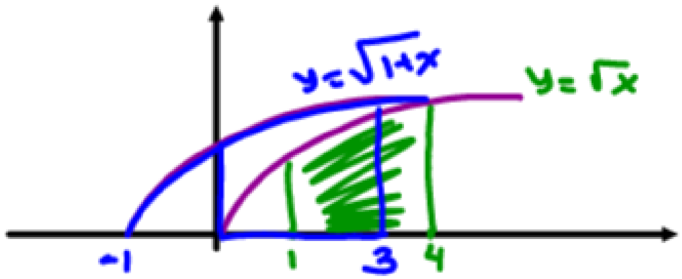
$\boxed{f(x)=\sqrt{1+x},\ a=0,\ b=3}$

another solution

$f(x)=\sqrt{x}$

$x_i^* = x_i = 1+\frac{3i}{n} = a+\frac{3i}{n}$

$a=1, \quad b=4$



(b) $\lim_{n\to\infty} \frac{10}{n} \sum_{i=1}^{n} \frac{1}{1+\left(\underbrace{7+\frac{10i}{n}}_{x_i^*}\right)^3}$ vs $\lim_{n\to\infty} \Delta x \sum_{n\to\infty}^{n} f(x_i^*)$

$\Delta x = \frac{b-a}{n} = \frac{10}{n} \Rightarrow b-a=10$

$f(x) = \frac{1}{1+x^3}$

$x_i^* = 7 + \frac{10i}{n} = x_i = a + i\Delta x = a + \frac{10i}{n}$

$\Rightarrow a = 7, \Rightarrow b = 17$

$\boxed{f(x) = \frac{1}{1+x^3}, \quad a=7, \quad b=17}$

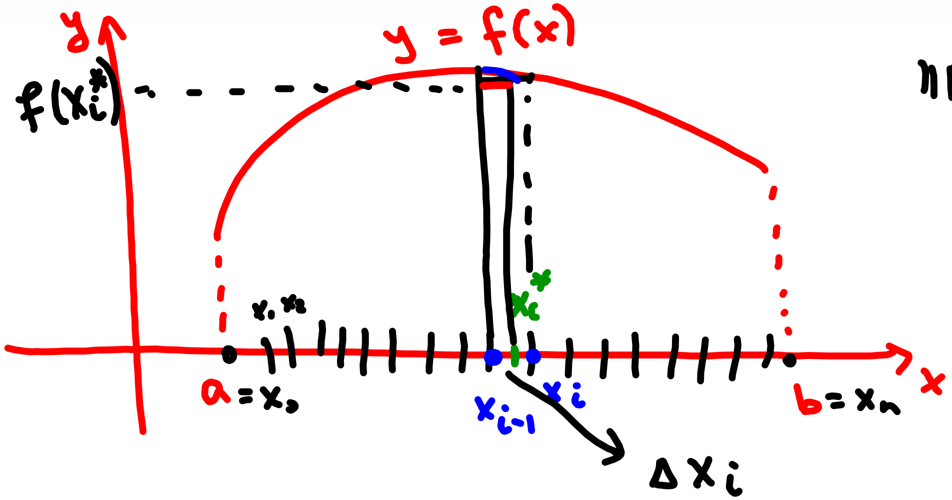
$f(x)$ , $x \in [a,b]$

## 6.3: The Definite Integral

DEFINITION 4. *The* **definite integral of** $f$ **from** $a$ **to** $b$ *is*

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i \qquad \text{R.S.}$$

*if this limit exists. If the limit does exist, then* $f$ *is called* **integrable** *on the interval* $[a, b]$.

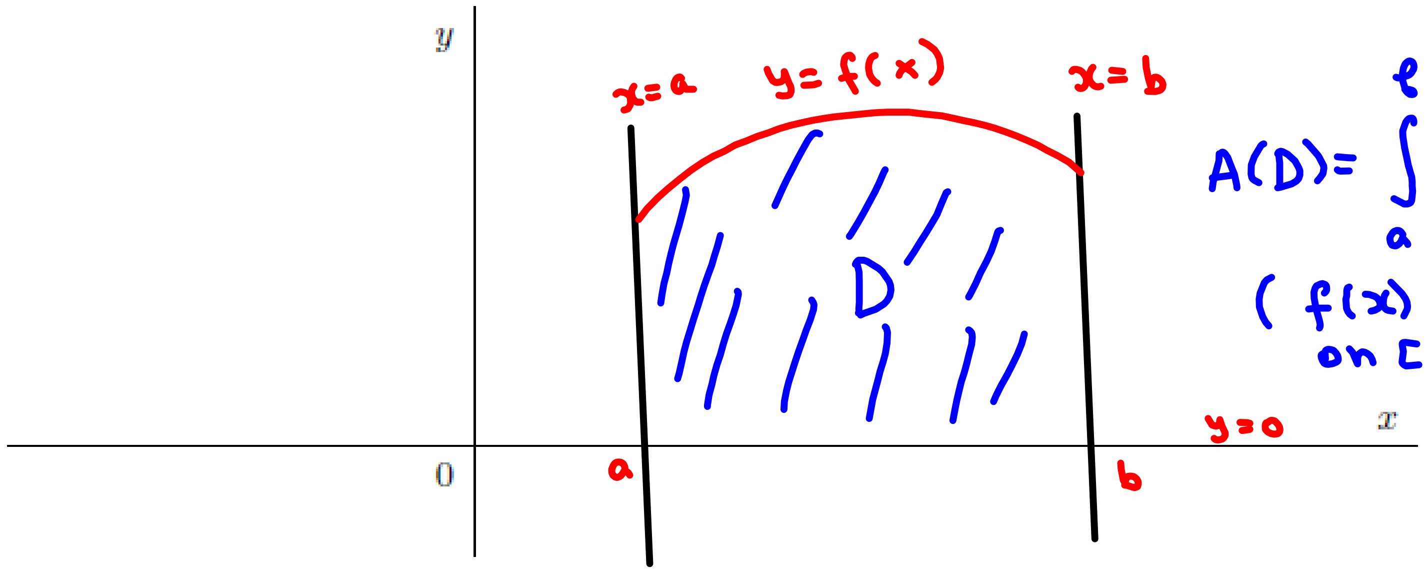



$$\|P\| = \max_{0 \le i \le n} \Delta x_i \to 0$$

$f(x) \geq 0$

If $f(x) > 0$ on the interval $[a, b]$, then the definite integral is the area bounded by the function $f$ and the lines $y = 0$, $x = a$ and $x = b$.

$$D = \{ (x, y) \mid a \leq x \leq b \;,\; 0 \leq y \leq f(x) \}$$




$$A(D) = \int_a^b f(x)\,dx$$

$(f(x) \geq 0$ on $[a, b])$

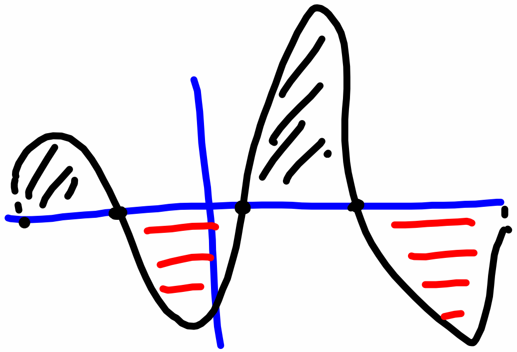
In general, a definite integral can be interpreted as a difference of areas:

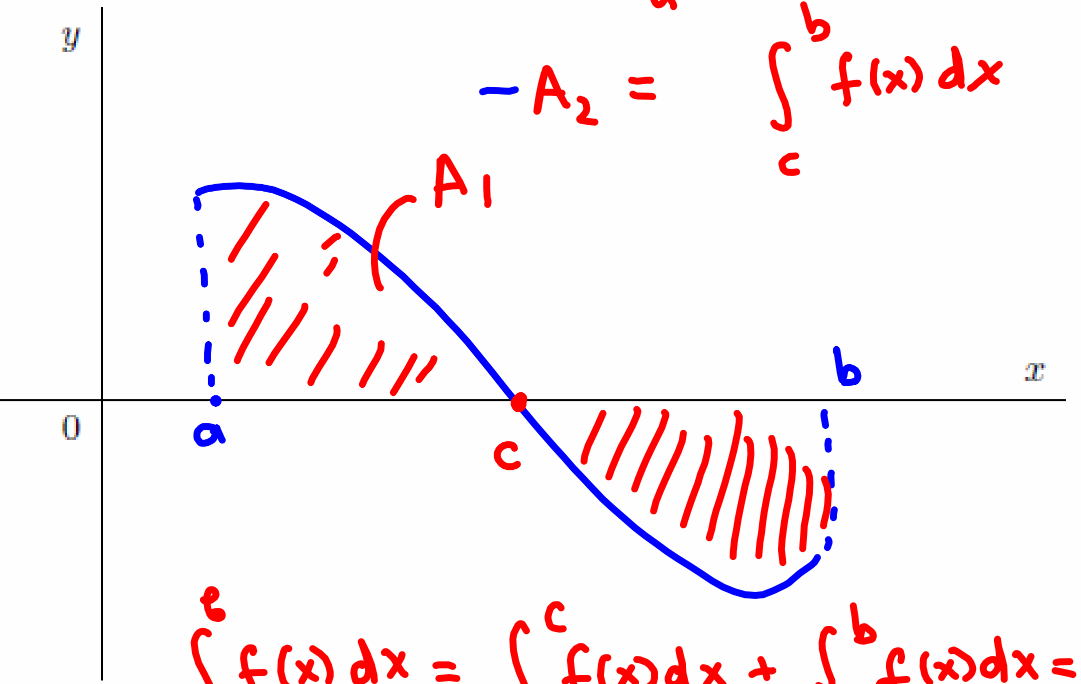
$$\int_a^b f(x)\,\mathrm{d}x = A_1 - A_2$$

where $A_1$ is the area of the region above the $x$ and below the graph of $f$ and $A_2$ is the area of the region below the $x$ and above the graph of $f$.

$A_1 = \int_a^c f(x)\,dx$

$-A_2 = \int_c^b f(x)\,dx$

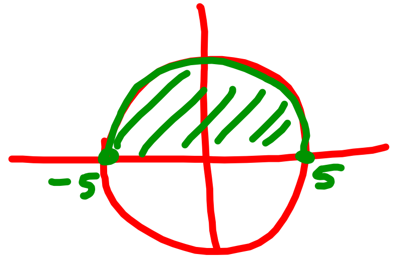





$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx = A_1 - A_2$

EXAMPLE 5. *Evaluate* $\displaystyle\int_{-5}^{5}(\sqrt{25-x^2})\,dx = A(\quad) = \frac{1}{2}\pi\cdot 5^2 = \boxed{\frac{25\pi}{2}}$

$$\int_{-5}^{5} f(x)\,dx$$



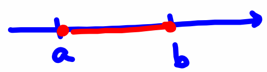

$y = f(x)$ , or $\quad y = \sqrt{25-x^2}$

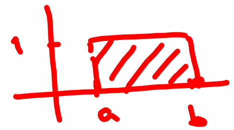
$y^2 = 25 - x^2 \quad (y \geq 0)$

$x^2 + y^2 = 25 \quad (y \geq 0)$

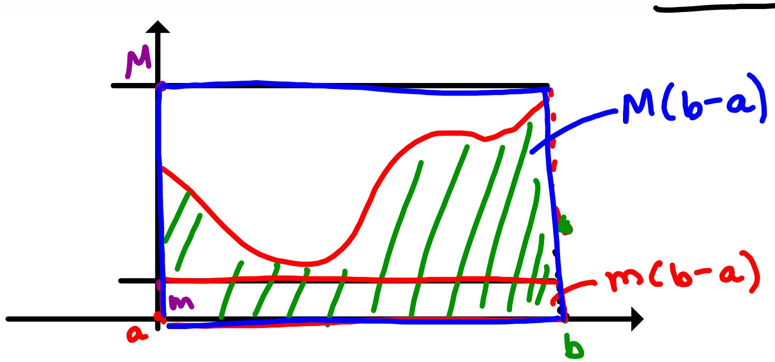
**Properties of Definite Integrals:**

- $\displaystyle\int_a^b \mathrm{d}x = b - a$




- $\displaystyle\int_a^b f(x) \pm g(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x$
- $\displaystyle\int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x$, where $c$ is any constant
- $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$, where $a \leq c \leq b$
- $\displaystyle\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$
- $\displaystyle\int_a^a f(x)\,\mathrm{d}x = 0$
- If $f(x) \geq 0$ for $a \leq x \leq b$, then $\displaystyle\int_a^b f(x)\,\mathrm{d}x \geq 0$
- If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\displaystyle\int_a^b f(x)\,\mathrm{d}x \geq \int_a^b g(x)\,\mathrm{d}x$
- If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $\displaystyle m(b-a) \leq \int_a^b f(x)\,\mathrm{d}x \leq M(b-a)$.

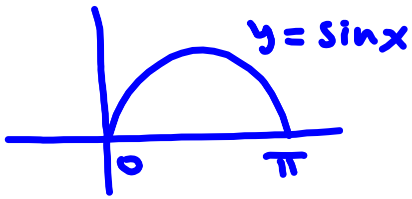


EXAMPLE 6. *Write as a single integral:*

$$\int_3^5 f(x)\,\mathrm{d}x + \int_0^3 f(x)\,\mathrm{d}x - \int_6^5 f(x)\,\mathrm{d}x + \underbrace{\int_5^5 f(x)\,\mathrm{d}x}_{=0}$$

$$= \int_0^3 f(x)\,dx + \int_3^5 f(x)\,dx + \int_5^6 f(x)\,dx = \int_0^6 f(x)\,dx$$

EXAMPLE 7. *Estimate the value of* $\displaystyle\int_0^{\pi} \underbrace{(4\sin^5 x + 3)}_{f(x)}\, dx$




$-1 \le \sin x \le 1$

If $x \in$ belongs $[0,\pi]$, then

$0 \le \sin x \le 1$

$0 \le \sin^5 x \le 1^5$

$0 \le 4\sin^5 x \le 4$

$0+3 \le 4\sin^5 x + 3 \le 4+3$

$\underbrace{3}_{m} \le 4\sin^5 x + 3 \le \underbrace{7}_{M}$

$a = 0$
$b = \pi$

$3(\pi - 0) \le \int_0^{\pi} (4\sin^5 x + 3)\,dx \le 7(\pi - 0)$

$3\pi \le \int_0^{\pi} (4\sin^5 x + 3)\,dx \le 7\pi$

## 6.4: The fundamental Theorem of Calculus

The fundamental Theorem of Calculus :

**PART I** If $f(x)$ is continuous on $[a, b]$ then $g(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on $(a, b)$ and $g'(x) = f(x)$.

EXAMPLE 8. *Differentiate* $g(x) = \int_{-4}^{x} \overbrace{e^{2t}\cos^2(1-5t)}^{f(t)}\,dt$

$$g'(x) = f(x) = e^{2x}\cos^2(1-5x)$$

EXAMPLE 9. *Let $u(x)$ be a differentiable function and $f(x)$ be a continuous one. Prove that*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\underbrace{\int_a^{u(x)} f(t)\,\mathrm{d}t}_{g(u(x))}\right) = f(u(x))u'(x).$$

Using Chain Rule

$$\frac{d}{dx}\left(g(u(x))\right) = g'(u)\,u'(x) = f(u)u'(x)$$

using Part I of FTC

$$\int_{\cancel{3}\ \sin x}^{\cancel{\sin x}\ 3} e^t \cos t \, dt$$

EXAMPLE 10. *Let $u(x)$ and $v(x)$ be differentiable functions and $f(x)$ be a continuous one. Then Prove that*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{v(x)}^{u(x)} f(t)\,\mathrm{d}t\right) = f(u(x))u'(x) - f(v(x))v'(x).$$

$$\underbrace{\frac{d}{dx}\left(\int_{v(x)}^{u(x)} f(t)\,dt\right)} = \frac{d}{dx}\left(\int_{v(x)}^{c} f(t)\,dt + \int_{c}^{u(x)} f(t)\,dt\right)$$

(where $u(x) \le c \le v(x)$

$$= -\frac{d}{dx}\left(\int_{c}^{v(x)} f(t)\,dt\right) + \frac{d}{dx}\left(\int_{c}^{u(x)} f(t)\,dt\right)$$

$$\overset{\text{by Ex. 9}}{=} -f(v(x))\,v'(x) + f(u(x))\,u'(x)$$

$$= \underbrace{f(u(x))\,u'(x) - f(v(x))\,v'(x).}$$

$F'(x) = f(x)$

**PART II** If $f(x)$ is continuous on $[a, b]$ and $F(x)$ is any antiderivative for $f(x)$ then

$$\int_a^b F'(x)\,dx = \int_a^b f(x)\,\mathrm{d}x = F(x)\Big|_a^b = F(b) - F(a).$$

$F(x) + C$

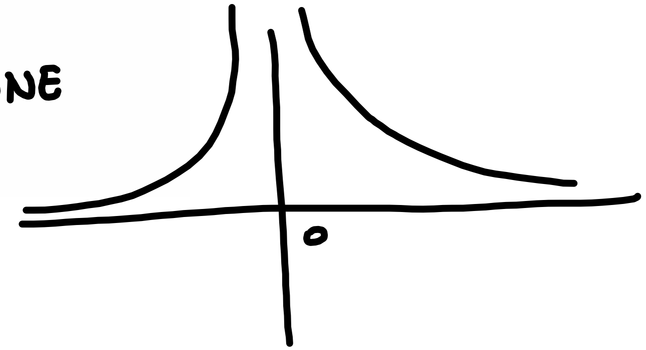
$F(b) + \cancel{C} - (F(a) + \cancel{C})$

EXAMPLE 11. *Evaluate*

1. $\int_1^5 \frac{1}{x^2}\,dx = \int_1^5 x^{-2}\,dx = \frac{x^{-2+1}}{-2+1}\Big|_1^5 = -\frac{1}{x}\Big|_1^5$

$= -\left(\frac{1}{5} - \frac{1}{1}\right) = -\left(-\frac{4}{5}\right) = \frac{4}{5}$

2. $\int_{-1}^5 \frac{1}{x^2}\,dx$ DNE

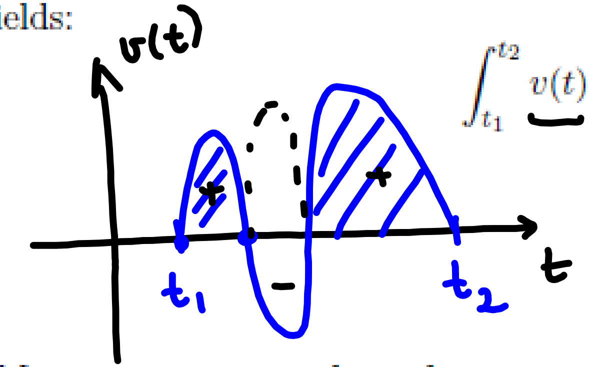



$f(x) = \frac{1}{x^2}$ has infinite discontinuity at $x=0$. So, $f(x)$ is not continuous on $[-1, 5]$

$t_1 \le t \le t_2$

## Applications of the Fundamental Theorem

If a particle is moving along a straight line then application of the Fundamental Theorem to $s'(t) = v(t)$ yields:




$$= \int_{t_1}^{t_2} s'(t)\,dt$$

$$\int_{t_1}^{t_2} v(t)\,\mathrm{d}t = s(t_2) - s(t_1) = \text{displacement}.$$

Moreover, one can show that

$$\text{total distance traveled} = \int_{t_1}^{t_2} |v(t)|\,\mathrm{d}t.$$

EXAMPLE 12. *A particle moves along a line so that its velocity at time $t$ is $v(t) = t^2 - 2t - 8$. Find the displacement and the distance traveled by the particle during the time period $1 \le t \le 6$.*

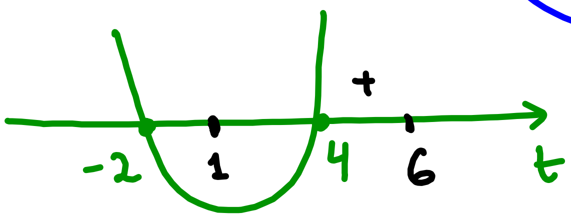
$$\text{displacement} = \int_1^6 v(t)\,dt = \int_1^6 (t^2 - 2t - 8)\,dt$$

$$= \left(\frac{t^3}{3} - t^2 - 8t\right)\Big|_1^6 = \ldots = -\frac{10}{3}$$

$$\text{total distance traveled} = \int_1^6 |v(t)|\,dt = \int_1^6 |t^2 - 2t - 8|\,dt =$$

$$t^2 - 2t - 8 = 0$$

$$(t-4)(t+2) = 0$$




$$|t^2 - 2t - 8| = \begin{cases} t^2 - 2t - 8 & \text{if } 4 \le t \le 6 \\ -(t^2 - 2t - 8) & \text{if } 1 \le t \le 4 \end{cases}$$

$$= \int_1^4 -(t^2 - 2t - 8)\,dt + \int_4^6 (t^2 - 2t - 8)\,dt$$

$$= -\left(\frac{t^3}{3} - t^2 - 8t\right)\Big|_1^4 + \left(\frac{t^3}{3} - t^2 - 8t\right)\Big|_4^6$$

$$= \ldots = \frac{98}{3}.$$