## 6.5: The substitution rule

The Substitution Rule for indefinite integrals: If $u = g(x)$ is a differentiable function, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Note that $\boxed{du = g'(x)dx}$ is differential.

The correct substitution depends on the integral:

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C = \frac{(g(x))^{n+1}}{n+1} + C$$

$$\int (g(x))^n \underbrace{g'(x)\,dx} \overset{u=g(x)}{=} \qquad \int e^{g(x)}g'(x)\,dx \qquad \int \frac{g'(x)}{g(x)}\,dx$$

$$\int \cos(g(x))g'(x)\,dx \qquad \int \sin(g(x))g'(x)\,dx \qquad \int \sec^2(g(x))g'(x)\,dx$$

$$\int \sec(g(x))\tan(g(x))g'(x)\,dx \qquad \int \csc^2(g(x))g'(x)\,dx \qquad \int \csc(g(x))\cot(g(x))g'(x)\,dx$$

EXAMPLE 1. *Evaluate each of the following integrals:*

1. $\int x \underbrace{(x^2 + 2016)}_{u(x)}{}^{2016}\,dx = \int \underbrace{(x^2+2016)^{2016}}_{u^{2016}} \underbrace{x\,dx}_{\frac{du}{2}} = \frac{1}{2}\int u^{2016}\,du$

$u = x^2 + 2016$

$du = (x^2 + 2016)'\,dx$

$du = \underbrace{2x\,dx} \Rightarrow x\,dx = \frac{du}{2}$

$$= \frac{1}{2}\,\frac{u^{2016+1}}{2016+1} + C$$

$$= \frac{u^{2017}}{4034} + C$$

$$= \frac{(x^2+2016)^{2017}}{4034} + C$$

2. $\int 18x^2 \sqrt[4]{6x^3+5}\, dx$ $= \int \underbrace{\sqrt[4]{6x^3+5}}_{\sqrt[4]{u}} \cdot \underbrace{18x^2 dx}_{du} = \int u^{\frac{1}{4}} du$

$u = 6x^3+5$

$du = 18x^2 dx$

$$= \frac{u^{\frac{1}{4}+1}}{\frac{1}{4}+1} + C = \frac{u^{\frac{5}{4}}}{\frac{5}{4}} + C$$

$$= \frac{4}{5}\sqrt[4]{u^5} + C = \boxed{\frac{4}{5}\sqrt[4]{(6x^3+5)^5} + C}$$

3. $\int \cos(3x)\sin^{10}(3x)\,dx = \int \underbrace{\sin^{10}(3x)}_{u^{10}}\ \underbrace{\cos(3x)\,dx}_{\frac{du}{3}} = \frac{1}{3}\int u^{10}\,du$

$u = \sin(3x)$

$du = 3\cos(3x)\,dx$

$\cos 3x\,dx = \frac{du}{3}$

$$= \frac{u^{11}}{3\cdot 11} + C = \frac{u^{11}}{33} + C$$

$$= \frac{\sin^{11}(3x)}{33} + C$$

4. $\int (8x-1)e^{4x^2-x}\,dx = \int \underbrace{e^{4x^2-x}}_{e^u} \underbrace{(8x-1)\,dx}_{du} = \int e^u\,du$

$u = 4x^2 - x$

$du = (8x-1)\,dx$

$= e^u + C = e^{4x^2-x} + C$

5. $\int \sec^2(5y)(\overbrace{5 - \tan(5y)}^{u})^5\, dy = -\frac{1}{5}\int u^5\, du = -\frac{1}{5}\frac{u^6}{6} + C$

$= -\frac{1}{30}(5 - \tan(5y))^6 + C$

$\boxed{\begin{array}{l} u = 5 - \tan 5y \\ du = -5\sec^2(5y)\, dy \\ \sec^2(5y)\, dy = -\frac{1}{5}du \end{array}}$

6. $\int \frac{x}{7x^2+12}\,\mathrm{d}x = \frac{1}{14}\int \frac{du}{u} = \frac{1}{14}\ln|u| + C = \frac{1}{14}\ln(7x^2+12)+C$

$u = 7x^2+12 > 0$

$du = 14x\,dx \Rightarrow x\,dx = \frac{du}{14}$

7. $\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\int \frac{du}{u} = -\ln|u| + C$

$u = \cos x$

$du = -\sin x \, dx$

$\sin x \, dx = -du$

$$\boxed{\int \tan x \, dx = -\ln|\cos x| + C}$$

8. $\int \sin(7y-12)\,dy$ $\quad \underline{\underline{u=7y-12}}$, $du=7dy$ $\quad \frac{1}{7}\int \sin u\,du = -\frac{\cos u}{7}+C = -\frac{\cos(7y-12)}{7}+C$

9. $\int \frac{3}{1+3x}\,dx$ $\quad \underline{\underline{u=1+3x}}$, $du=3dx$ $\quad \cancel{\frac{1}{3}}\int \frac{\cancel{3}\,du}{u} = \ln|u|+C = \ln|1+3x|+C$

10. $\int \frac{1}{(1+3x)^3}\,dx$ $\quad \frac{1}{3}\int \frac{du}{u^3} = \frac{1}{3}\int u^{-3}du$

$= \frac{1}{3}\frac{u^{-3+1}}{-3+1}+C = -\frac{1}{6u^2}+C$

$= -\frac{1}{6(1+3x)^2}+C$

In general, to evaluate

$\int f(ax+b)\,dx$ $\quad \underline{\underline{u=ax+b}}$, $du=a\,dx$, $dx=\frac{du}{a}$ $\quad \frac{1}{a}\int f(u)\,du$

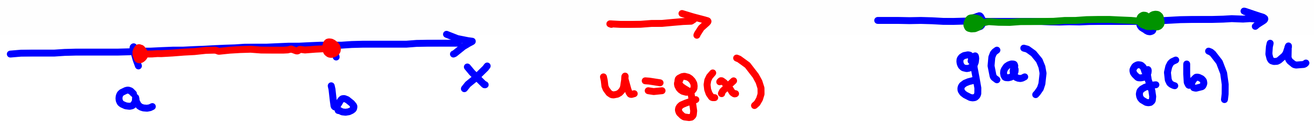
$F'(x)=f(x)$

$=\frac{1}{a}F(u)+C$

$=\frac{1}{a}F(ax+b)+C$

The Substitution Rule for definite integrals: If $u = g(x)$ is a differentiable function, then

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u$$




$$\int_{\frac{\pi}{2}}^{\pi} \cos 3x\,dx \overset{u=3x}{\underset{du=3dx}{=}} \frac{1}{3}\int_{\cdots}^{\cdots} \cos u\,du = \frac{1}{3}\sin u = \frac{1}{3}\sin(3x)\Big|_{\frac{\pi}{2}}^{\pi}$$

an extra step $\begin{cases} u(\frac{\pi}{2}) = \frac{3\pi}{2} \\ u(\pi) = 3\pi \end{cases}$

$$\frac{1}{3}\int_{\frac{3\pi}{2}}^{3\pi} \cos u\,du = \frac{1}{3}\sin u\Big|_{\frac{3\pi}{2}}^{3\pi}$$

$$= \frac{1}{3}\Big(\overset{=0}{\sin(3\pi)} - \overset{=-1}{\sin\left(\tfrac{3\pi}{2}\right)}\Big)$$

$$= \frac{1}{3}$$

EXAMPLE 2. *Evaluate each of the following integrals:*

1. $\displaystyle\int_{e^2}^{e^6} \frac{(\ln x)^4}{x}\,dx = \int_2^6 u^4\,du = \frac{u^5}{5}\Big|_2^6 = \frac{1}{5}(6^5 - 2^5)$

$= \frac{1}{5}(6^2 - 32)$

$u = \ln x$

$du = \frac{dx}{x}$

$u(e^2) = \ln e^2 = 2$

$u(e^6) = \ln e^6 = 6$

2. $\int_0^{0.5} (\sin(\pi y) - e^y)\,dy\cancel{)} = \left(-\frac{1}{\pi}\cos \pi y - e^y\right)\Big|_0^{0.5}$

$= -\frac{1}{\pi}\cos\frac{\pi}{2} - e^{0.5} - \left(-\frac{1}{\pi}\cos 0 - e^0\right)$

$= -e^{0.5} + \frac{1}{\pi} + 1$

EXAMPLE 3. *If $\alpha$ and $\beta$ are positive numbers, show that*

$$\underbrace{\int_0^1 x^\alpha (1-x)^\beta \, dx}_{\text{LHS}} = \underbrace{\int_0^1 x^\beta (1-x)^\alpha \, dx}_{\text{RHS}}$$

$$\text{LHS} = \int_0^1 x^\alpha (1-x)^\beta \, dx \overset{\substack{u = 1-x \\ x = 1-u}}{\underset{du = -dx}{=}} -\int_1^0 (1-u)^\alpha u^\beta \, du = \int_0^1 u^\beta (1-u)^\alpha \, du$$

$u(0) = 1$

$u(1) = 0$

$$= \int_0^1 x^\beta (1-x)^\alpha \, dx = \text{RHS}$$