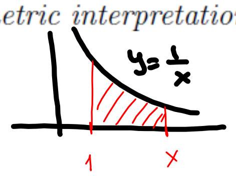
6.6: The Logarithm Defined As An Integral

DEFINITION 1. *The* **natural logarithmic function** *is the function defined by*

$$\ln x = \int_1^x \frac{1}{t}\,\mathrm{d}t, \quad x > 0. \tag{1}$$

EXAMPLE 2. *Using a geometric interpretation of the definite integral show that*

1. $\ln x > 0$ *if* $x > 1$
2. $\ln x < 0$ *if* $0 < x < 1$
3. $\ln x = 0$ *if* $x = 1$

① 

If $x > 1$ then $\ln x =$ area under $y = \frac{1}{x}$ on $[1, x]$.

Hence, $\ln x > 0$ if $x > 0$.

② If $0 < x < 1$, then $\ln x = \int_1^x \frac{dt}{t} = -\int_x^1 \frac{dt}{t} = -$area under $y = \frac{1}{x}$ on $[x, 1]$. Hence $\ln x < 0$ if $0 < x < 1$.

③ Area of a segment is 0. (or $\ln 1 = \int_1^1 \frac{dt}{t} = 0$.

Applying the Fundamental Theorem of Calculus (Part I) to (1), we get

$$\frac{d}{dx}(\ln x) = \frac{d}{dx}\left(\int_1^x \frac{dt}{t}\right) = \frac{1}{x}, \text{ i.e. } \boxed{(\ln x)' = \frac{1}{x}}$$

Laws of Logarithms

If x and y are positive numbers and r is a rational number, then

1. $\underbrace{\ln(xy)} = \ln x + \ln y$
2. $\ln \frac{x}{y} = \ln x - \ln y$
3. $\ln(x^r) = r \ln x$

Proof. ① We know $\frac{d}{dx}(\ln x) = \frac{d}{dx}\left(\int_1^x \frac{1}{t}\, dt\right) = \frac{1}{x}$

By FTC (part 1)

y is fixed here

$$\frac{d}{dx}\left(\ln(xy)\right) = \frac{d}{dx}\int_1^{xy} \frac{1}{t}\, dt = \frac{1}{xy} \cdot y = \frac{1}{x}$$

$$\frac{d}{dx}(xy) = y$$

$$\frac{d}{dx}(\ln x) - \frac{d}{dx}(\ln(xy)) = \frac{1}{x} - \frac{1}{x}$$

$$\frac{d}{dx}(\ln x - \ln(xy)) = 0$$

$$(1) \quad \boxed{\ln x - \ln(xy) = C} \quad (C \text{ is a constant}).$$

If $x = 1$, then $\ln 1 - \ln(1 \cdot y) = C$

$0 - \ln y = C$, or

$$(2) \quad \boxed{C = -\ln(y)}$$

Combining (1) & (2), we get

$\ln x - \ln(xy) = -\ln y$, or

$\ln(xy) = \ln x + \ln y$. $\square$

2. $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$.

<u>Lemma</u>. $\ln \frac{1}{y} = -\ln y$.

<u>Proof</u>. $0 = \ln 1 = \ln\left(\frac{1}{y} \cdot y\right) \overset{1.}{=} \ln \frac{1}{y} + \ln y$

Hence, $\ln \frac{1}{y} = -\ln y$. $\square$

Proof of 2.

Then 2. follows from 1:

$$\ln \frac{x}{y} = \ln\left(x \cdot \frac{1}{y}\right) \overset{1.}{=} \ln x + \ln \frac{1}{y} \overset{\text{Lemma}}{=} \ln x + (-\ln y)$$

$$= \ln x - \ln y. \quad \square$$

3. r is a rational number

Prove: $\ln(x^r) = r \ln x$.

r is rational, if $r = \frac{m}{n}$ for some integer number m and n.

Lemma: If m is an integer, then $\ln x^m = m \ln x$.

Proof. Case 1. $m > 0$ then

$$\ln x^m = \ln(\underbrace{x \cdot x \cdot \ldots \cdot x}_{m}) \overset{1.}{=} \underbrace{\ln x + \ln x + \ldots + \ln x}_{m} = m \ln x.$$

Case 2. $m = 0$, then

$$\ln x^m = \ln x^0 = \ln 1 = 0 = 0 \cdot \ln x = m \cdot \ln x$$

Case 3. $m < 0$, i.e. $m = -n$ for some $n > 0$.

Then

$$\ln x^m = \ln x^{-n} = \ln\left(\frac{1}{x}\right)^n \overset{\text{Case 1}}{=} n \ln \frac{1}{x} = n(-\ln x)$$

$$= -n \ln x = m \ln x. \quad \square$$

Proof of 3.

r is rational, hence $r = \frac{m}{n}$ for some integers m, n.

$$\ln x^r = \ln x^{\frac{m}{n}} = \underbrace{\frac{1}{n} \cdot n}_{1} \ln x^{\frac{m}{n}} \overset{\text{Lemma}}{=} \frac{1}{n} \ln \left(x^{\frac{m}{n}}\right)^n$$

$$= \frac{1}{n} \ln x^{\frac{m}{n} \cdot n} = \frac{1}{n} \ln x^m \overset{\text{Lemma}}{=}$$

$$\frac{m}{n} \ln x = r \ln x. \quad \square$$