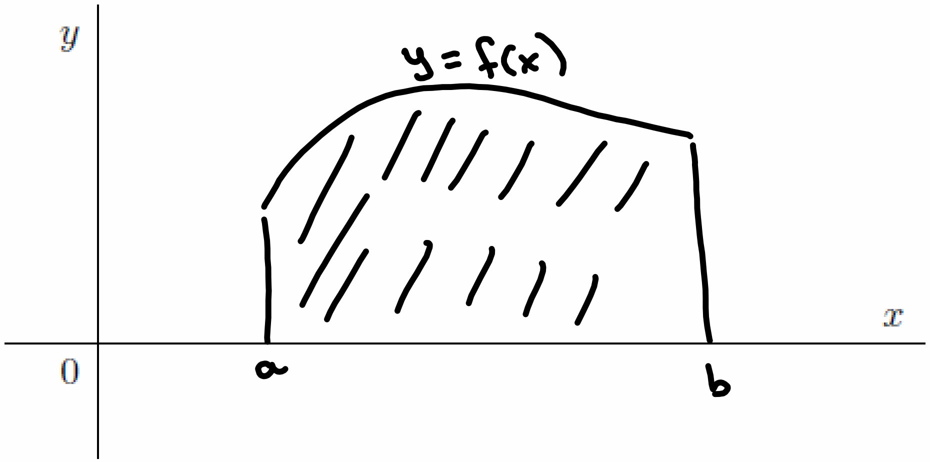
7.1: Areas Between Curves

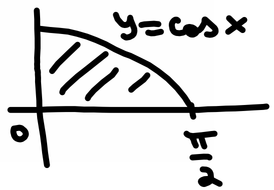
One of interpretations of definite integral

$$A = \int_a^b f(x)\,\mathrm{d}x, \quad f(x) \geq 0 \ \text{ on } \ [a,b]$$

is the area between the graph of $y = f(x)$ and the x-axis on $[a,b]$.



For example, if $f(x) = \cos x$ and $x \in [0, \frac{\pi}{2}]$ then



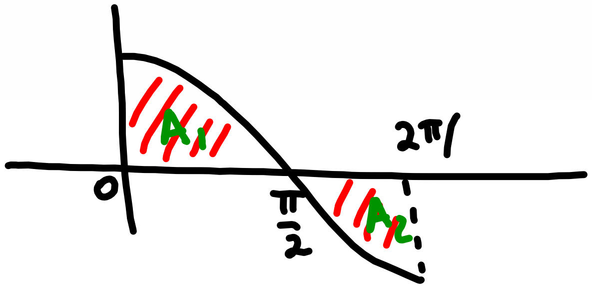
If $x \in [0, \frac{\pi}{2}]$, then $\cos x \geq 0$

$$\int_0^{\frac{\pi}{2}} \cos x\, dx = \sin x \Big|_0^{\frac{\pi}{2}} = 1 = \text{area under } y = \cos x \text{ on } [0, \tfrac{\pi}{2}]$$

If $f(x) \geq 0$ on $[a, b]$ then $\displaystyle\int_a^b f(x)\,\mathrm{d}x \geq 0$

If $f(x) \leq 0$ on $[a, b]$ then $\displaystyle\int_a^b f(x)\,\mathrm{d}x \leq 0$

The previous example on $[0, \frac{2\pi}{3}]$*:*



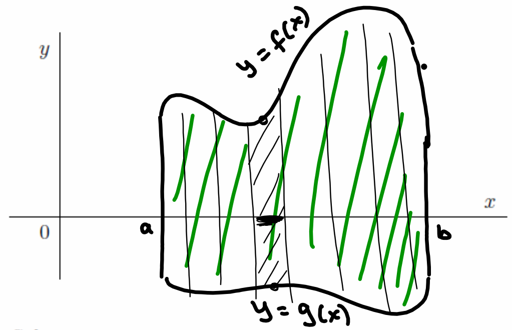
$$\int_0^{2\pi/3} \cos x \, dx \neq \text{Area}$$

$$= A_1 - A_2$$

CASE I. *Determine the area between $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$ assuming $f(x) \geq g(x)$ on $[a, b]$.*

I other words, find the area of the region D defined by

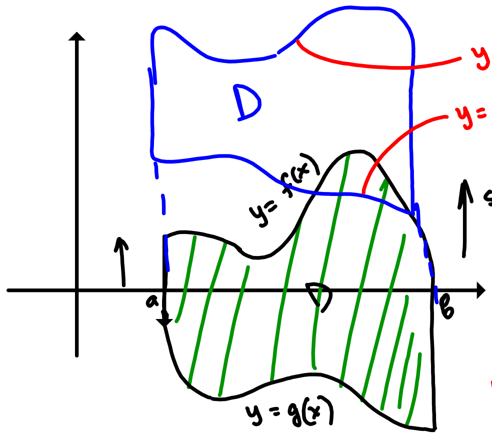
$$D = \{(x,y) \mid a \leq x \leq b,\ g(x) \leq y \leq f(x)\}$$



Solution:

$$\boxed{A = A(D) = \int_a^b f(x) - g(x)\,\mathrm{d}x}$$

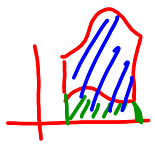
Explanation:



$y = f(x) + h \geq 0$ on $[a,b]$

$y = g(x) + h \geq 0$

shift up by h units such that the region D be in the upper half-plane.



$$A(D) = \left(\begin{array}{c}\text{Area}\\ \text{below}\\ y = f(x)+h\end{array}\right) - \left(\begin{array}{c}\text{Area}\\ \text{below}\\ y = g(x)+h\end{array}\right)$$

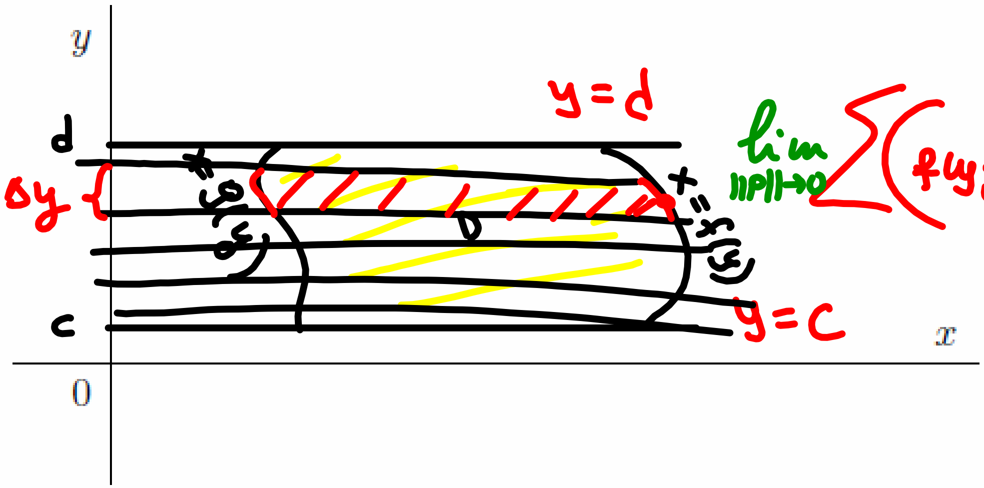
$$A(D) = \int_a^b (f(x)+h)\,dx - \int_a^b (g(x)+h)\,dx$$

$$= \int_a^b \left(f(x) + \cancel{h} - g(x) - \cancel{h}\right) dx = \int_a^b \left(f(x) - g(x)\right) dx$$

CASE II. *Determine the area between $x = f(y)$ and $x = g(y)$ on the interval $[c, d]$ assuming $f(y) \geq g(y)$ on $[c, d]$.*

I other words, find the area of the region D defined by

$$D = \{ (x, y) \mid c \leq y \leq d \ , g(y) \leq x \leq f(y) \}$$



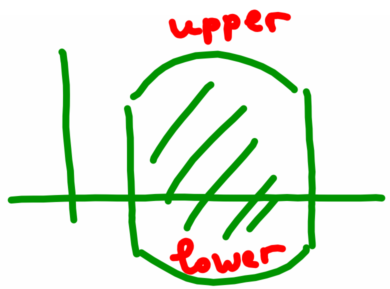
$$\lim_{\|P\| \to 0} \sum \left(f(y_i^*) - g(y_i^*) \right) \Delta \cancel{y}$$

Solution:

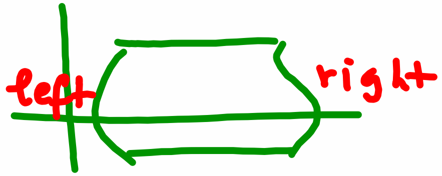
$$A = A(D) = \int_c^d f(y) - g(y) \, \mathrm{d}y$$

The above formulas in the "word" form:

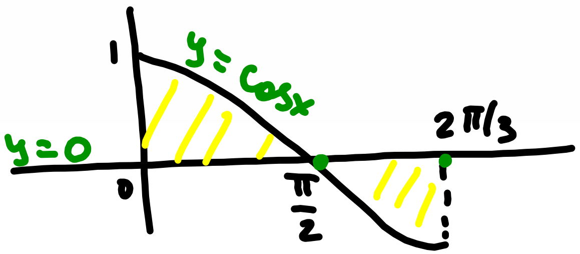
$$\text{CASE I} \quad A = \int_a^b \left(\begin{array}{c} \text{upper} \\ \text{function} \end{array} \right) - \left(\begin{array}{c} \text{lower} \\ \text{function} \end{array} \right) \mathrm{d}x$$



$$\text{CASE II} \quad A = \int_c^d \left(\begin{array}{c} \text{right} \\ \text{function} \end{array} \right) - \left(\begin{array}{c} \text{left} \\ \text{function} \end{array} \right) \mathrm{d}y$$

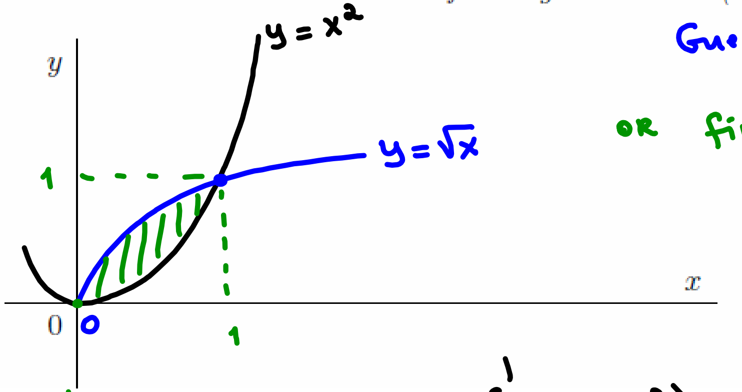


Coming back to the previous example: $f(x) = \cos x$, where $0 \le x \le 2\pi/3$ we get:



$$\text{Area} = \int_0^{\pi/2} (\cos x - 0)\, dx + \int_{\frac{\pi}{2}}^{2\pi/3} (0 - \cos x)\, dx$$

EXAMPLE 1. *Determine the area of the region enclosed (=bounded by) by $y = x^2$ and $y = \sqrt{x}$.*



Guess : intersection points are $(0,0)$ and $(1,1)$

OR find them solving

$x^2 = \sqrt{x}$

$x^4 = x \quad (x > 0)$

$x(x^3 - 1) = 0$

$x = 0$ or $x = 1$

$$A = \int_0^1 (\text{u.f.}) - (\text{l.f.})\, dx = \int_0^1 (\sqrt{x} - x^2)\, dx = \dots = \frac{1}{3}$$

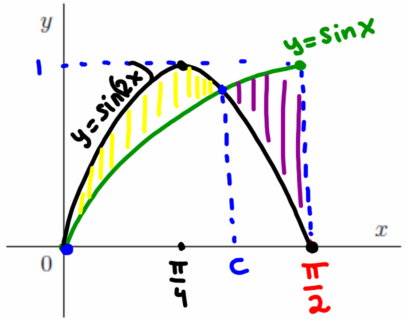
Alternative way (but, probably, not an optimal one) use the case II formula.

REMARK 2. 1. The limits of integration in the above example were determined as the intersection points of the two curves.

2. Sketch of a graph of the region is recommended (it helps to determine which of the functions is upper/right).

3. The area between two curves will always be positive

EXAMPLE 3. *Determine the area of the region enclosed by* $y = \sin x$, $y = \sin 2x$, $x = 0$, $x = \pi/2$.



Find c: Solve $\sin x = \sin 2x$ on $[0, \frac{\pi}{2}]$

$$\sin x = 2\sin x \cos x$$

$$\sin x - 2\sin x \cos x = 0$$

$$\sin x\,(1 - 2\cos x) = 0$$

$\sin x = 0$ OR $1 - 2\cos x = 0$

$x = 0$ $\qquad \cos x = \frac{1}{2}$

$$x = \frac{\pi}{3} \qquad c = \frac{\pi}{3}$$

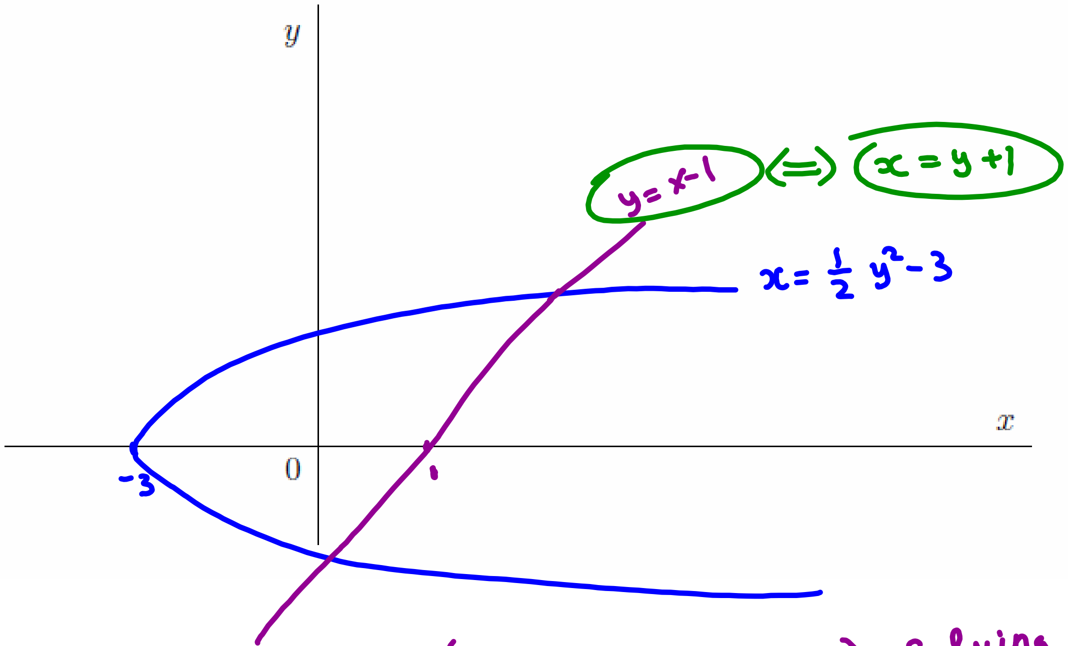
$$A = \int_0^c (\sin(2x) - \sin x)\,dx + \int_c^{\pi/2} (\sin x - \sin(2x))\,dx$$

$$= \int_0^{\pi/3} (\sin(2x) - \sin x)\,dx + \int_{\frac{\pi}{3}}^{\pi/2} (\sin x - \sin(2x))\,dx$$

$$= \left(-\frac{1}{2}\cos(2x) + \cos x\right)\Big|_0^{\pi/3} + \left(-\cos x + \frac{1}{2}\cos(2x)\right)\Big|_{\pi/3}^{\pi/2}$$

$$= \dots = \frac{1}{2}$$

EXAMPLE 4. *Determine the area of the region enclosed by* $x = \frac{1}{2}y^2 - 3$, $y = x - 1$.



Find intersection points (their y-components) solving:

$$\frac{1}{2}y^2 - 3 = y + 1$$

$$y^2 - 6 = 2y + 2$$

$$y^2 - 2y - 8 = 0$$

$$(y+2)(y-4) = 0$$

$$y = -2 \quad \text{or} \quad y = 4$$

$$A = \int_{-2}^{4} \left((\text{right f.}) - (\text{left. f.})\right) dy = \int_{-2}^{4} (y+1) - \left(\frac{1}{2}y^2 - 3\right) dy = \ldots = 18$$