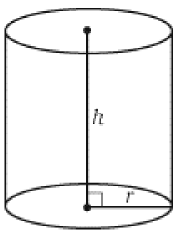
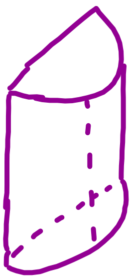
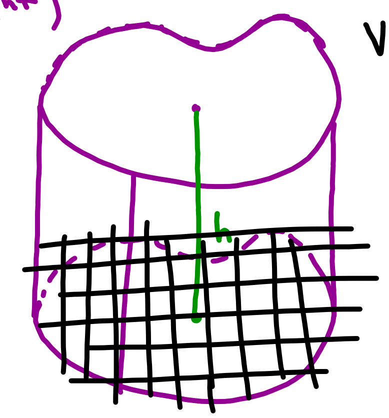
7.2: VOLUME

A simple type of solid: right **cylinder**



$V = \underbrace{\pi r^2}_{\text{(Base Area)}} \underbrace{h}_{\text{(height)}}$



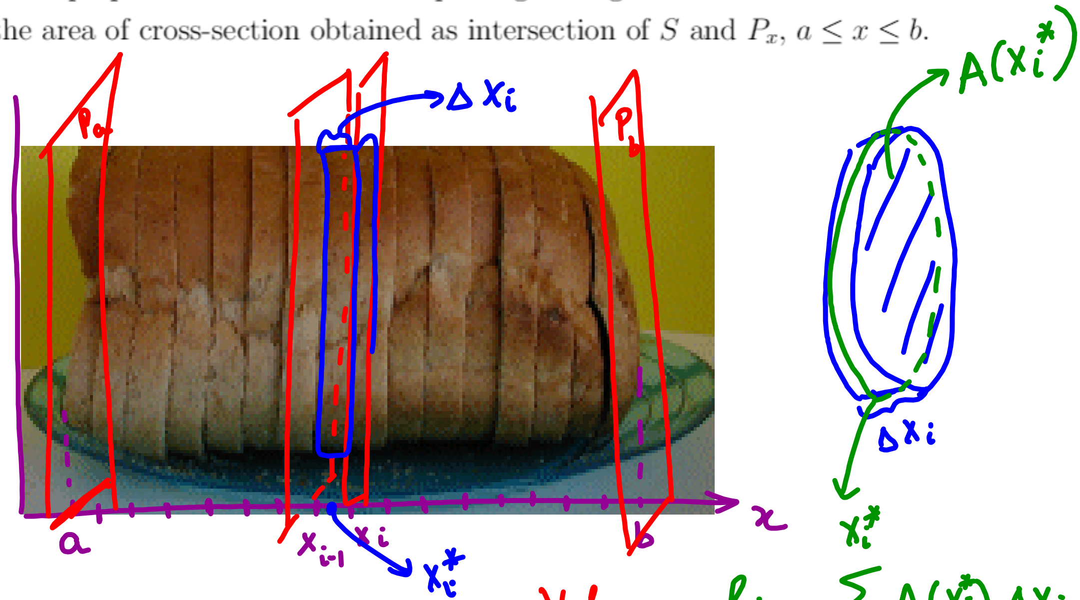


$V = (\text{Base Area}) \cdot h$

Let S be any solid. The intersection of S with a plane is a plane region that is called a **cross-section** of S.

P_x ia a plane perpendicular to x-axis and passing through x.

$A(x)$ is the area of cross-section obtained as intersection of S and P_x, $a \leq x \leq b$.



$$\text{Volume} = \lim_{\|P\|\to 0} \underbrace{\sum_i A(x_i^*)\,\Delta x_i}_{\text{Riemann sum}}$$

(Think of slicing a loaf of bread.)

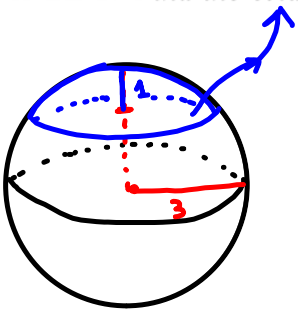
$\|P\| = \max_i \Delta x_i$

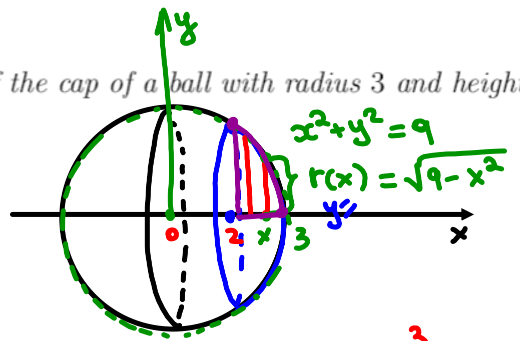
DEFINITION 1. *Let S be a solid that lies between the planes P_a and P_b. Then the volume of S is*

$$V = \lim_{\|P\| \to 0} \sum_{i=1}^{n} A(x_i^*) \Delta x_i = \int_a^b A(x)\, dx$$

Important to remember: $A(x)$ is the area of a moving cross-section obtained by slicing through x **perpendicular** to the x-axis.

EXAMPLE 2. *Find the volume of the cap of a ball with radius 3 and height 1.*





$$V = \int_2^3 A(x)\, dx$$

$$A(x) = \pi r_{(x)}^2 = \pi\left(\sqrt{9-x^2}\right)^2 = \pi(9-x^2)$$

$$V = \int_2^3 \pi(9-x^2)\, dx = \pi\left(9x - \frac{x^3}{3}\right)\Big|_2^3 = \frac{8\pi}{3}$$

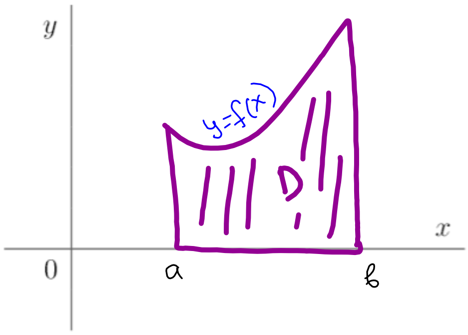
shape of cross-sections

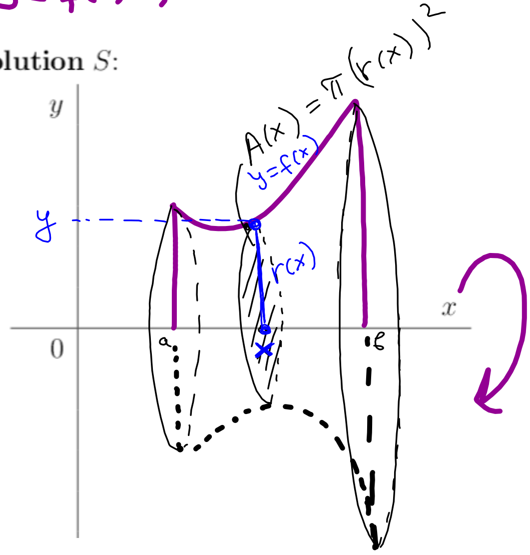
Volumes of Solids of Revolution (Disk Method)

Consider the plane region D bounded by the curves $y = f(x), y = 0, x = a, x = b$, i.e.

$$D = \{ (x,y) \mid 0 \le y \le f(x) \ , \ a \le x \le b \}$$

Rotate D about a given axis to get the **solid of revolution** S:





PROBLEM: `Determine the volume of solid of revolution.`

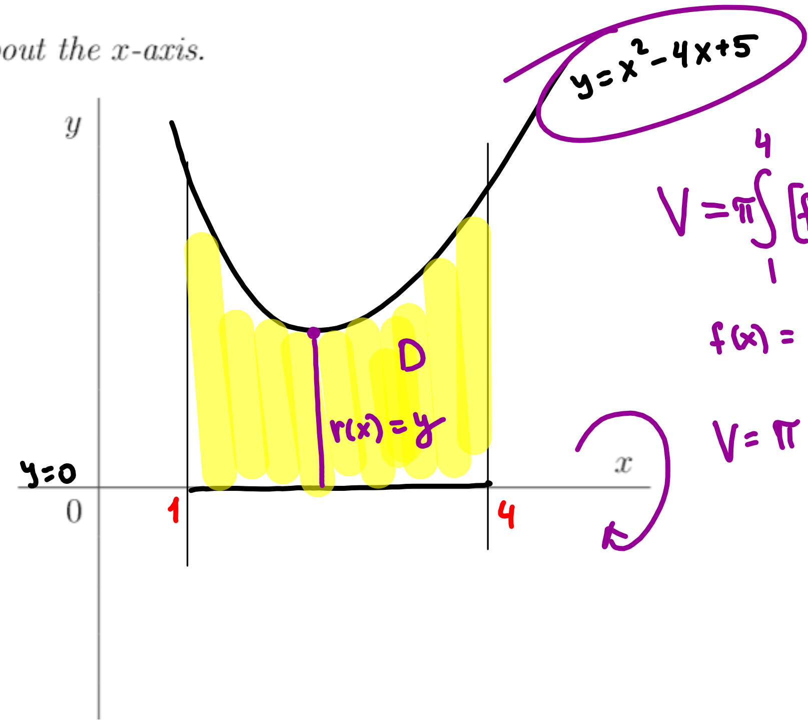
Solution: Using cross-sectional areas (**disk method**)

$$V = \int_a^b A(x)\,dx = \int_a^b \pi [\underbrace{r(x)}_{y=f(x)}]^2\,dx = \pi \int_a^b [f(x)]^2\,dx$$

EXAMPLE 3. *Determine the volume of the solid obtained by rotating the region*

$$D = \{(x,y) : 1 \le x \le 4, 0 \le y \le \underbrace{x^2 - 4x + 5}\}$$

about the x-axis.



$x_v = -\frac{-4}{2} = +2$

$y_v = 4 - 4 + 5 = 5 > 0$

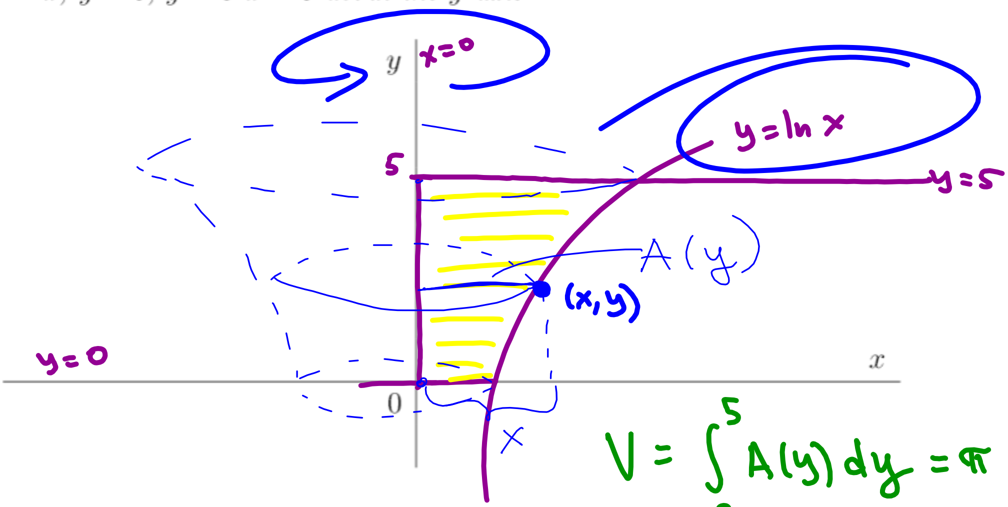
$$V = \pi \int_1^4 [f(x)]^2 dx$$

$f(x) = x^2 - 4x + 5$

$$V = \pi \int_1^4 (x^2 - 4x + 5)^2 dx =$$

$$= \frac{78\pi}{5}$$

EXAMPLE 4. *Determine the volume of the solid obtained by rotating the region enclosed by* $y = \ln x$, $y = 0$, $y = 5$ $x = 0$ *about the* y*-axis.*



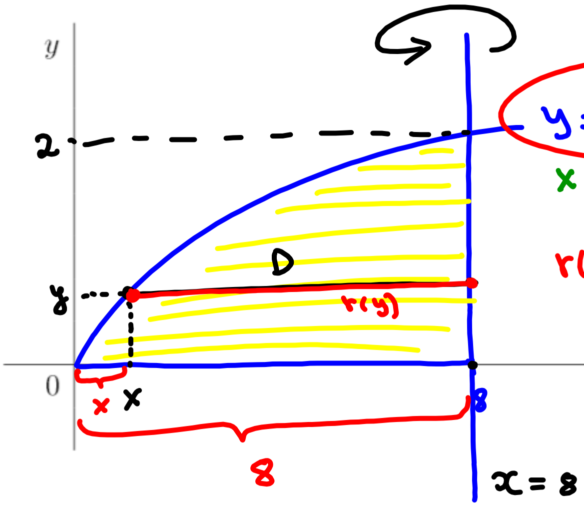
$$V = \int_0^5 A(y)\,dy = \pi \int_0^5 [\underbrace{r(y)}_{x}]^2\,dy$$

$y = \ln x \Rightarrow x = e^y$

$$V = \pi \int_0^5 (e^y)^2\,dy = \pi \int_0^5 e^{2y}\,dy$$

$$= \frac{\pi}{2} e^{2y}\Big|_0^5 = \frac{\pi}{2}(e^{10} - 1)$$

EXAMPLE 5. *Set up an integral for the volume of the solid obtained by rotating the region enclosed by the curves* $y = \sqrt[3]{x}, x = 8, y = 0$ *about the line* $x = 8$.



$y = \sqrt[3]{x}$

$x = y^3$

$r(y) = 8 - x = 8 - y^3$

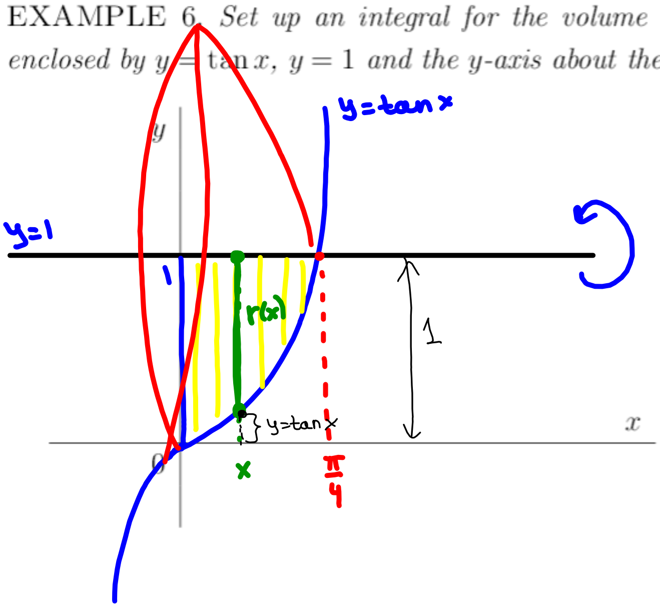
$$V = \int_0^{\sqrt[3]{8}} A(y)\,dy$$

$$= \pi \int_0^2 [r(y)]^2\,dy$$

$$V = \pi \int_0^2 (8 - y^3)^2\,dy$$

$$\frac{576\pi}{7}$$

EXAMPLE 6. *Set up an integral for the volume of the solid obtained by rotating the region enclosed by $y = \tan x$, $y = 1$ and the y-axis about the line $y = 1$.*



$\tan\left(\frac{\pi}{4}\right) = 1$

$$V = \int_0^{\pi/4} A(x)\,dx = \pi \int_0^{\pi/4} r^2(x)\,dx$$

$$r(x) = 1 - y = 1 - \tan x$$

$$\boxed{V = \pi \int_0^{\pi/4} (1 - \tan x)^2\,dx}$$

$$\boxed{\pi(1 - \ln 2)}$$

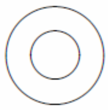
SUMMARY (Disk Method)

- Rotation about a horizontal axis ($y = k$): $V = \displaystyle\int_a^b A(x)\,\mathrm{d}x$
- Rotation about a vertical axis ($x = k$): $V = \displaystyle\int_a^b A(y)\,\mathrm{d}y$
- *Cross sections are orthogonal to the axis of rotating.*

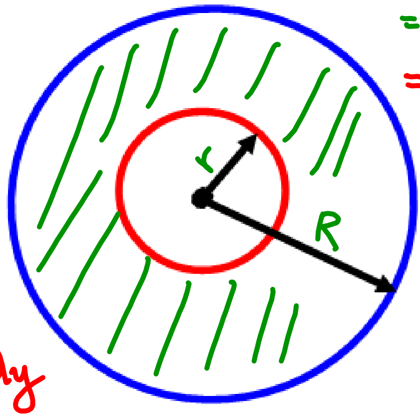
Washer Method

Use it when the cross-sections orthogonal to the axis of rotating of a solid of revolution are in the shape of a washer (ring).

The area of a ring:



$$A = \pi R^2 - \pi r^2$$
$$= \pi(R^2 - r^2)$$
$$= \pi\left(\begin{pmatrix}\text{outer}\\\text{radius}\end{pmatrix}^2 - \begin{pmatrix}\text{inner}\\\text{radius}\end{pmatrix}^2\right)$$

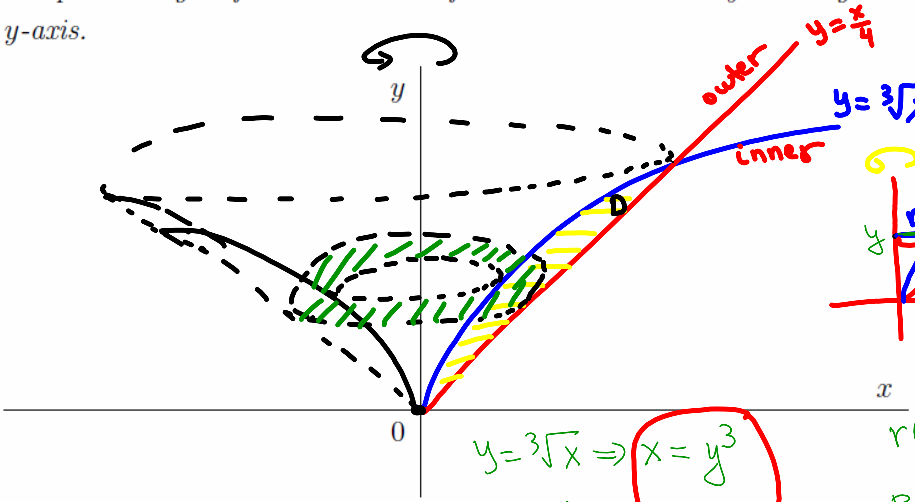


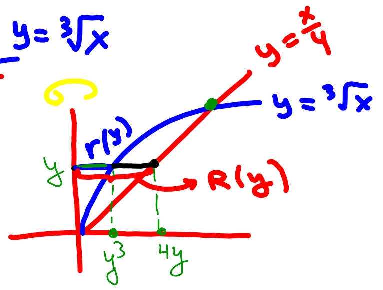
$$V = \int_a^b A(x)\,dx = \pi\int_a^b \left(R^2(x) - r^2(x)\right)dx$$

$$V = \int_c^d A(y)\,dy = \pi\int_c^d \left(R^2(y) - r^2(y)\right)dy$$

EXAMPLE 7. *Let D be the plane region that lies in the first quadrant and enclosed by $y = \sqrt[3]{x}$ and $y = \dfrac{x}{4}$.*

(a) *Set up an integral for the volume of the solid obtained by rotating the region D about the y-axis.*





$y=\sqrt[3]{x} \Rightarrow x = y^3$

$y = \frac{x}{4} \Rightarrow x = 4y$

$r(y) = y^3$

$R(y) = 4y$

Find intersection points

$y^3 = 4y$

$y^3 - 4y = 0$

$y(y^2-4) = 0$

$y(y-2)(y+2) = 0$

$y=0$ or $y=2$ or $y=-2$

in first quadrant

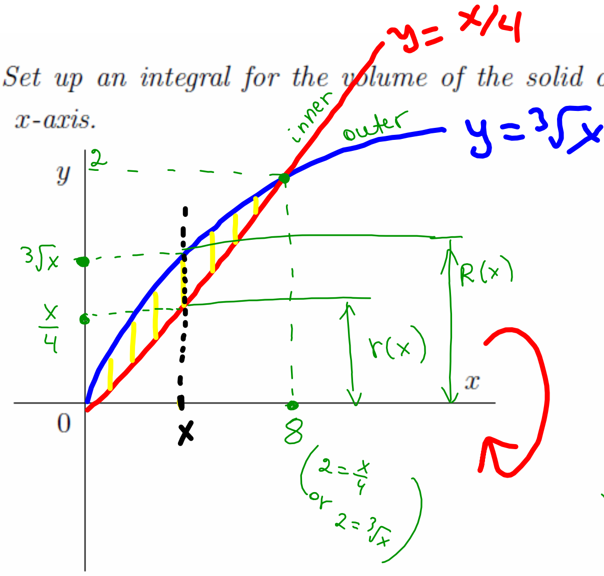
$$V = \pi\int_0^2 \left(R^2(y) - r^2(y)\right)dy$$

$$V = \pi\int_0^2 \left((4y)^2 - (y^3)^2\right)dy$$

$$\boxed{V = \pi\int_0^2 \left(16y^2 - y^6\right)dy}$$

$$\boxed{\frac{512\pi}{21}}$$

(b) *Set up an integral for the volume of the solid obtained by rotating the region D about the x-axis.*



$r(x) = \frac{x}{4}$

$R(x) = \sqrt[3]{x}$

$$V = \pi \int_0^8 \left(R^2(x) - r^2(x) \right) dx$$

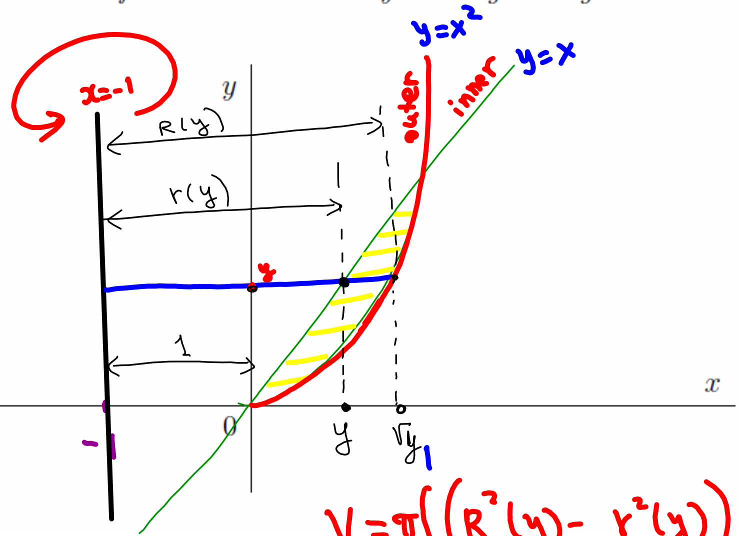
$$V = \pi \int_0^8 \left(\left(\sqrt[3]{x}\right)^2 - \left(\frac{x}{4}\right)^2 \right) dx$$

$$V = \pi \int_0^8 \left(x^{\frac{2}{3}} - \frac{x^2}{16} \right) dx$$

$$\boxed{\frac{128\pi}{15}}$$

EXAMPLE 8. *Let D be the region enclosed by $y = x$ and $y = x^2$.*

(a) *Set up an integral for the volume of the solid obtained by rotating the region D about the line $x = -1$.*



Intersection points:

$$\left.\begin{array}{l} y=x^2 \\ y=x \end{array}\right\} \Rightarrow x^2 = x$$

$x(x-1)=0$

$x=0$ or $x=1$ $\Rightarrow$ $y=1$

$\Downarrow$

$x = \pm\sqrt{y}$

$$V = \pi \int_0^1 \left(R^2(y) - r^2(y)\right) dy$$

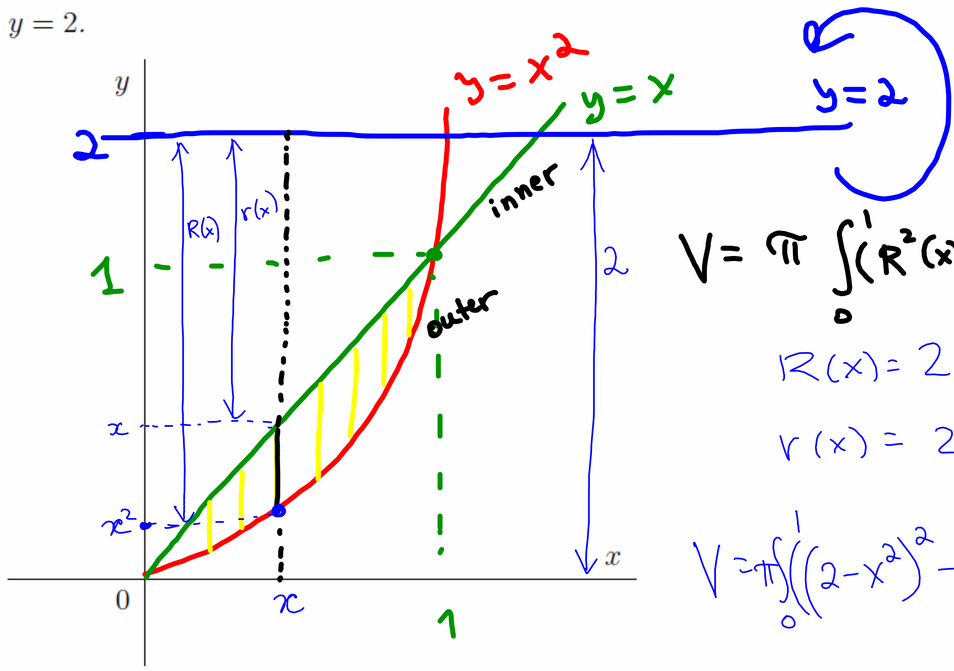
$r(y) = x+1 = y+1$

$R(y) = \sqrt{y}+1$

$$V = \pi \int_0^1 \left((\sqrt{y}+1)^2 - (y+1)^2\right) dy$$

$$\boxed{\frac{\pi}{2}}$$

(b) *Set up an integral for the volume of the solid obtained by rotating the region D about the line $y = 2$.*



$$V = \pi \int_0^1 \left(R^2(x) - r^2(x) \right) dx$$

$$R(x) = 2 - x^2$$

$$r(x) = 2 - x$$

$$V = \pi \int_0^1 \left((2-x^2)^2 - (2-x)^2 \right) dx$$

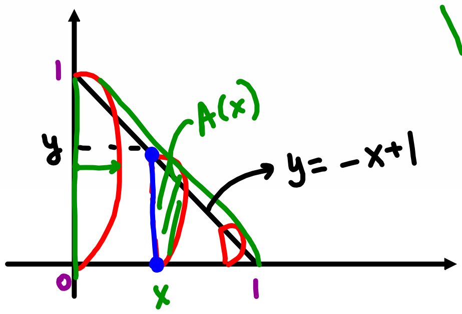
$$\frac{8\pi}{15}$$

More general case: Cross Sections other than Circles

Use the basic formula:

$$V = \int_a^b A(x)\,\mathrm{d}x \qquad \left(V = \int_c^d A(y)\,dy \right)$$

EXAMPLE 9. *The base of the solid S is the triangular region with the vertices $(0,0),(1,0)$ and $(0,1)$. Find the volume of S if the cross sections perpendicular to the x-axis are semicircles with diameters on the base.*



$$V = \int_0^1 A(x)\,dx$$

$A(x) =$ (area of semicircle with diameter equal to $y = -x+1$)

$$= \frac{1}{2}\pi \left(\frac{1-x}{2}\right)^2 = \frac{\pi}{8}(1-x)^2$$

$$V = \int_0^1 \frac{\pi}{8}(1-x)^2\,dx = \frac{\pi}{8}\int_0^1 (1-2x+x^2)\,dx$$

$$= \dots = \frac{\pi}{24}$$