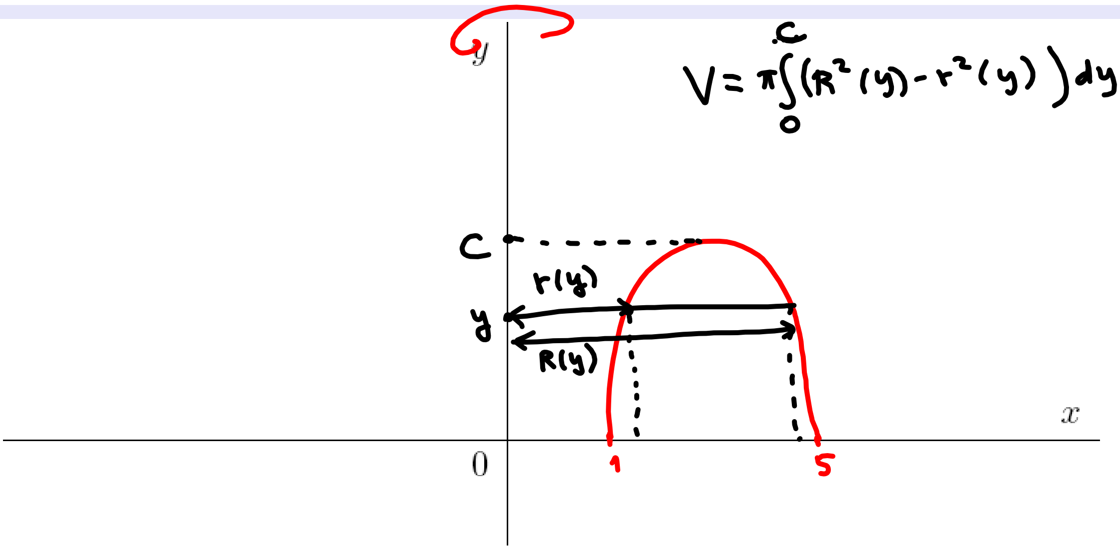
7.3: Volumes by Cylindrical Shells

EXAMPLE 1. *Determine the volume of the solid obtained by rotating the region bounded by* $y = 4(x-1)(x-5)^2$ *and the x-axis about the y-axis.*



$$V = \pi\int_0^c \left(R^2(y) - r^2(y)\right)dy$$

Let us list the troubles in application washer method here:

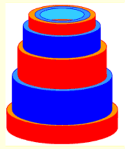
$$V = \pi\int_0^c \left(R^2(y) - r^2(y)\right)dy$$

1. It is difficult to get a formula for R(y) and r(y) in terms of y.
2. To find the upper bound of integration we need to find abs. max of the given curve on [1,5].

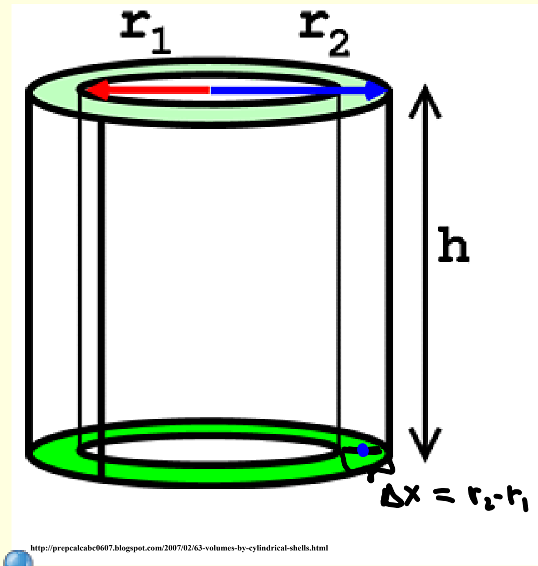
How to avoid these troubles?

Unlike the washer method in shell method the cross sections are always **parallel** to the axis of revolution.

`Think of cutting your solid by a cylindrical cutter centered on the axis of revolution and pushed down into the solid.`



A `cylindrical shell` is a solid bounded by two concentric circular cylinders with the same height. The volume of the cylindrical shell:



$$V = \pi r_2^2 h - \pi r_1^2 h$$

$$V = \pi (r_2^2 - r_1^2) h$$

$$= \pi 2 \cdot \frac{(r_2 + r_1)}{2} \overbrace{(r_2 - r_1)}^{\Delta x} h$$

$$= 2\pi \frac{r_2 + r_1}{2} h \, \Delta x$$

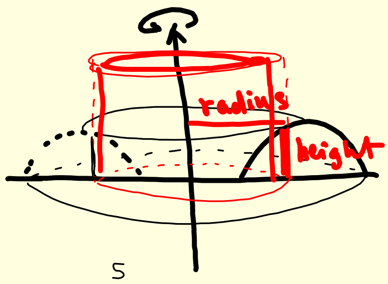
$$= \underbrace{2\pi (\text{radius})}_{\text{circumference}} (\text{height}) \, \Delta x$$

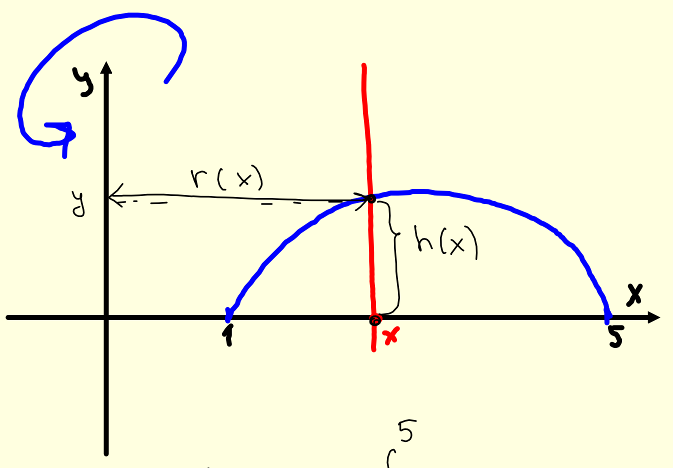
$$V = 2\pi \int_a^b (\text{radius})(\text{height}) \, dx$$

Solution of Example 1:

Determine the volume of the solid obtained by rotating the region bounded by

$y = 4(x-1)(x-5)^2$ *and the x-axis about the y-axis.*





$$V = 2\pi \int_1^5 r(x)\, h(x)\, dx$$

$$r(x) = x$$

$$h(x) = y = 4(x-1)(x-5)^2$$

$$V = 2\pi \int_1^5 x \cdot 4(x-1)(x-5)^2 = \ldots$$

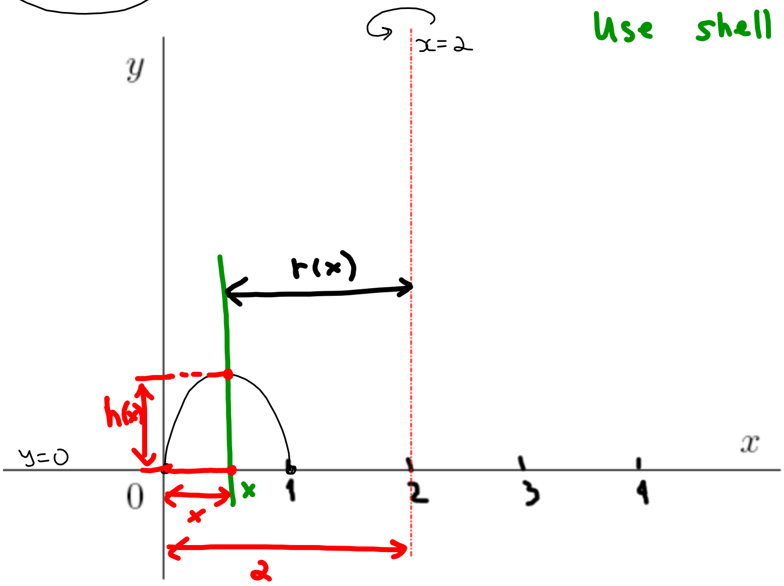
SUMMARY (Method of Cylindrical Shells)

- Area of cross sections: $A(x) = 2\pi(\text{radius})(\text{height})$
- For rotation about a *vertical* axis we use $V = \displaystyle\int_a^b A(x)\,\mathrm{d}x$.

  For rotation about a *horizontal* axis we use $V = \displaystyle\int_c^d A(y)\,\mathrm{d}y$

  Note: Exactly opposite of washer method.
- For the limits of integration we take the only range of x or y covering one side of the solid (not the complete range).

EXAMPLE 2. *Find the volume of the solid obtained by rotating the region enclosed by the curves* $y = x - x^2$ *and* $y = 0$ *about the line* $x = 2$.



Use shell method

$$V = 2\pi \int_0^1 r(x)\, h(x)\, dx$$

$r(x) = 2 - x$

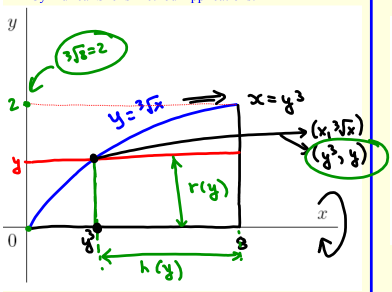
$h(x) = y = x - x^2$

$$V = 2\pi \int_0^1 (2-x)(x-x^2)\, dx$$

$$= \ldots = \frac{\pi}{2}$$

EXAMPLE 3. *Set up, but do not evaluate, the volume of the solid obtained by rotating the region enclosed by the curves $y = \sqrt[3]{x}, x = 8, y = 0$ about the x-axis using the method of cylindrical shells and then using the washer method.*

Cylindrical Shells Method Applications:



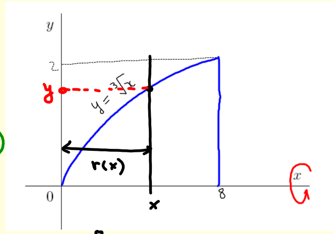
$$V = 2\pi \int_0^2 r(y)\,h(y)\,dy$$

$r(y) = y$

$h(y) = 8 - y^3$

$$\boxed{V = 2\pi \int_0^2 y(8 - y^3)\,dy}$$

Washer Method Application



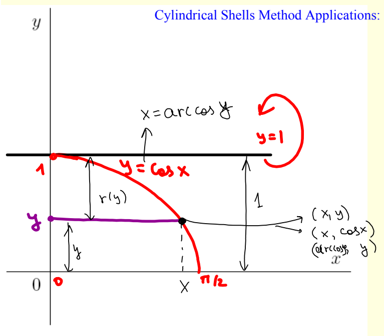
$$V = \pi \int_0^8 r^2(x)\,dx$$

$r(x) = y = \sqrt[3]{x}$

$$\boxed{V = \pi \int_0^8 (\sqrt[3]{x})^2\,dx}$$

EXAMPLE 4. *Set up, but do not evaluate, the volume of the solid obtained by rotating the region enclosed by the curves $y = \cos x, x = 0, x = \pi/2$ and $y = 0$ about the line $y = 1$ using the method of cylindrical shells and then using the washer method.*

Cylindrical Shells Method Applications:

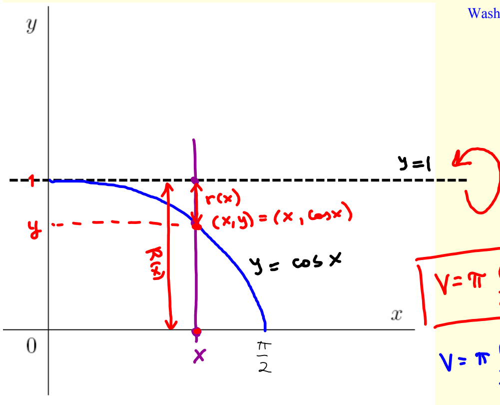


$$V = 2\pi \int_0^1 r(y)\,h(y)\,dy$$

$r(y) = 1 - y$

$h(y) = x = \arccos y$

$$V = 2\pi \int_0^1 (1-y)\arccos y\,dy$$



Washer Method Application

$$V = \pi \int_0^{\pi/2} \left(R^2(x) - r^2(x)\right)dx$$

$R(x) = 1$

$r(x) = 1 - \cos x$

$$\boxed{V = \pi \int_0^{\pi/2} \left(1^2 - (1-\cos x)^2\right)dx}$$

$$V = \pi \int_0^{\pi} \left(1 - (1 - 2\cos x + \cos^2 x)\right)dx$$

$\frac{\pi}{2}$