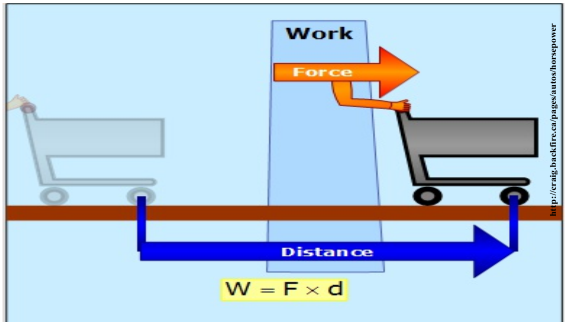
7.4: Work

PROBLEM: Find the amount of work that is done by a force in moving an object.

- Case 1: <u>constant force.</u>



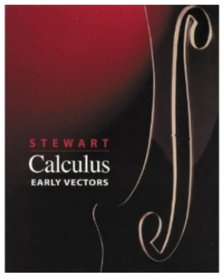
Work W done in moving an object a distance d meters is given by

$$W = Fd.$$

In the SI metric system: $[J] = [N][m]$

In the British engineering system: [ft][lb]. Also $1 ft\text{-}lb \approx 1.36J$.

EXAMPLE 1. *How much work is done in lifting your Calculus book (2.1kg) off the floor to put it on a desk that is* 0.6*m high.*

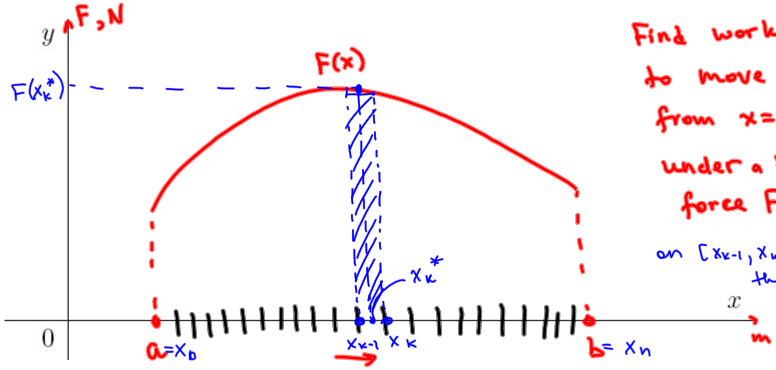


$F = mg \uparrow$ 0.6 m

$$\left.\begin{aligned} W &= F \cdot d = (mg) \cdot d \\ m &= 2.1 \text{ kg} \\ d &= 0.6 \text{ m} \\ g &= 9.8 \text{ m/s}^2 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow W = 2.1 \cdot 0.6 \cdot 9.8 = 12.348 \text{ J}$$

- Case 2: non constant force. (It requires integration.)



Find work done to move an object from x = a to x=b under a non-constant force F(x).

on $[x_{k-1}, x_k]$ we assume that the force is equal to $F(x_k^*)$.

$$W_k = F(x_k^*)(x_k - x_{k-1}) = \overbrace{F(x_k^*)}^{F}\overbrace{\Delta x_k}^{d}$$

$$W \approx \sum_{k=1}^{n} W_k = \sum_{k=1}^{n} F(x_k^*)\Delta x_k$$

Finally, $W = \lim_{\|P\|\to 0} \overbrace{\sum_{k=1}^{n} F(x_k^*)\Delta x_k}^{\text{Riemann sum}}$ where $\|P\| = \max_k |\Delta x_k|$

Thus, *work done in moving an object from $x = a$ to $x = b$ is*

$$\boxed{W = \int_a^b F(x)\, dx}$$

EXAMPLE 2. *When a particle is at distance x feet from the origin, a force of $\overbrace{3x^2 + 2x}^{F(x)}$ pounds acts on it. How much work is done in moving it from $x = \overset{a}{1}$ to $x = \overset{b}{3}$ along the x-axis?*

$$W = \int_a^b F(x)\,dx = \int_1^3 (3x^2+2x)\,dx = \ldots = 34 \text{ ft-lb}.$$

EXAMPLE 3. *A spring has a natural length of* 1*m. If a* 50*N force is required to keep it stretched to a length* 3*m, how much work is done in stretching the spring from* 2*m to* 5*m?*

Solution By Hooke's law the force required to stretch a spring x units **beyond** its natural length is

$$F(x) = kx$$

Reformulate the problem in Example 3:

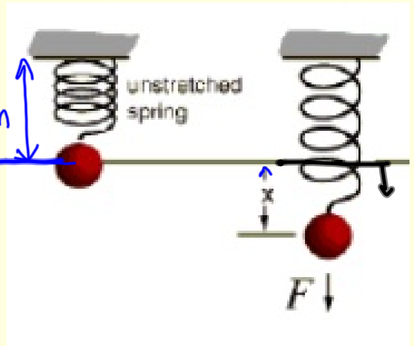
If a 50N force is required to keep a spring stretched ___m beyond its natural length, how much work is done in stretching the spring from ___m to ___m beyond its natural length ?

1 4

d 2 m beyond
ng the spring

1m

0



Given

$F(2) = 50$

$k \cdot 2 = 50$

$k = 25$

$$W = \int_1^4 F(x)\,dx$$

$$W = \int_1^4 kx\,dx$$

$$W = 25\int_1^4 x\,dx = 25\left.\frac{x^2}{2}\right|_1^4$$

$$= 187.5 \text{ J}$$

EXAMPLE 4. *If the work required to stretch a spring* 1*ft beyond its natural length is* 12*ft-lb, how much work is needed to stretch it* 9 *inches beyond its natural length?*

$$\frac{9}{12}\text{ ft} = \frac{3}{4}\text{ ft}$$

Given

$$12 = \int_0^1 kx\,dx$$

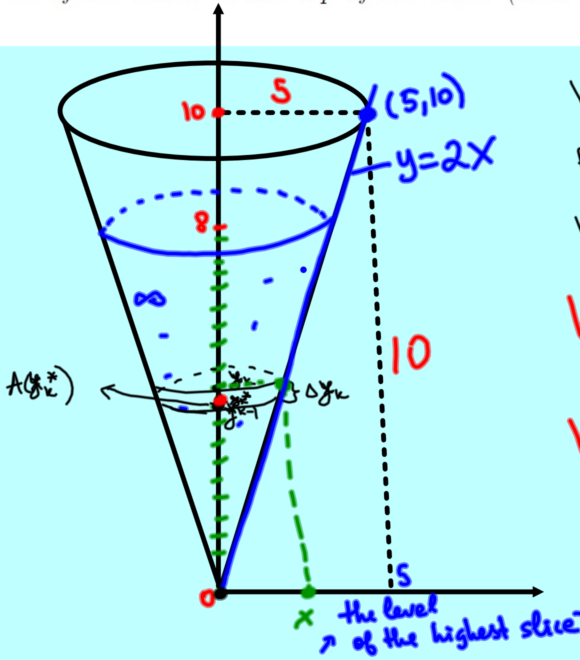
$$12 = k \cdot \frac{1}{2} \Rightarrow \boxed{k = 24}$$

$$W = \int_0^{3/4} kx\,dx$$

$$= 24\int_0^{3/4} x\,dx$$

$$= 24 \cdot \frac{1}{2} \cdot \left(\frac{3}{4}\right)^2 = \frac{27}{4} \text{ ft-lb}$$

EXAMPLE 5. *A tank has a shape of an inverted circular cone with height* 10*m and base radius* 5*m. It is filled with water to a height of* 8*m. Find the work required to empty the tank by pumping all of the water to the top of the tank. (The density of water is* 1000*kg*/*m*^{3}.)



$V_k \approx A(y_k^*)\,\Delta y_k$

$F_k \approx m_k g = \rho V_k g = \rho g\, A(y_k^*)\,\Delta y_k$

$W_k \approx F_k \cdot (10 - y_k^*)$

$$W \approx \sum_{k=1}^{n} W_k = \sum_{k=1}^{n} \rho g\, A(y_k^*)(10-y_k^*)\Delta y_k$$

$$W = \int_0^8 \rho g\, A(y)(10-y)\,dy$$

$$A(y) = \left(\begin{array}{l}\text{area of the circle}\\ \text{with radius} = x = \frac{y}{2}\end{array}\right) = \pi \frac{y^2}{4}$$

Remark: $W = \rho g \int \left(\begin{array}{l}\text{cross}\\ \text{sectional}\\ \text{area of}\\ \text{a slice}\\ \text{on level } y\end{array}\right) \cdot \left(\begin{array}{l}\text{distance}\\ \text{(depth of}\\ \text{the slice}\\ \text{from the top)}\end{array}\right) dy$

the level of the lowest slice

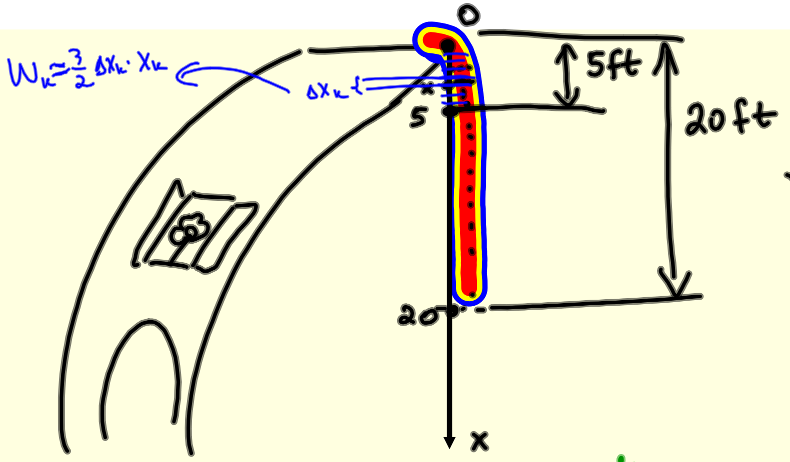
A(y)

$$W = \rho g \frac{\pi}{4} \int_0^8 y^2 (10-y)\,dy$$

$$W = 1000 \cdot 9.8 \cdot \frac{3.14}{4} \int_0^8 (10y^2 - y^3)\,dy$$

$$= 1225\ \text{J}.$$

EXAMPLE 6. *A uniform cable hanging over the edge of a tall building is* $20ft$ *long and weight* $30lb$. *How much work is required to pull* $5ft$ *of the cable to the top?*



$$\text{weight} = \frac{30\ lb}{20\ ft} = \frac{3}{2}\ lb/ft$$

$$W = \underbrace{W_1}_{\substack{\text{work to pull the top 5 ft of the cable} \\ \text{distance} = x\ ft}} + \underbrace{W_2}_{\substack{\text{work to pull the bottom 15 ft of cable} \\ \text{distance} = 5 ft}}$$

weight of a slice.

distance

$$W_1 = \int_0^5 \frac{3}{2}\, x\, dx = \frac{3}{2}\frac{x^2}{2}\Big|_0^5 = \frac{3}{4}\cdot 25 = \frac{75}{4}\ ft\cdot lb$$

distance

$$W_2 = (\underbrace{15}_{20-5} \cdot \frac{3}{2})\, 5 = \frac{225}{2}\ ft\cdot lb$$

weight of 15 ft of cable.

REMARK 7. The exact height of the building doesn't matter.

REMARK 10. The exact height of the building doesn't matter.