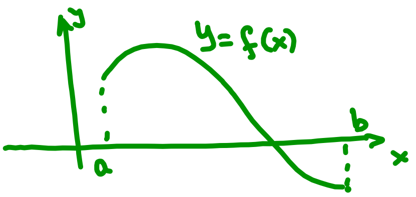
7.5: Average Value of a Function

The average value of finitely many numbers $y_1, y_2, \ldots, y_n$:

$$\bar{y} = \frac{y_1 + y_2 + \ldots + y_n}{n}$$

The average value of a function $y = f(x)$ over the interval $[a, b]$:



$$f_{ave} = \underbrace{\frac{1}{b-a}}_{\text{length of } [a,b]} \int_a^b f(x)\, dx.$$

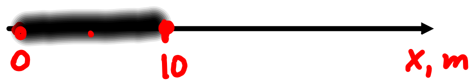
EXAMPLE 1. *Determine the average value of* $f(x) = x^2 - 4x + 7\sin(\pi x)$ *over the interval* $[-\frac{1}{2}; \frac{1}{2}]$.

a b

$$f_{ave} = \frac{1}{\frac{1}{2} - (-\frac{1}{2})} \int_{-\frac{1}{2}}^{\frac{1}{2}} (x^2 - 4x + 7\sin(\pi x))\, dx$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2\, dx - 4 \underbrace{\int_{-\frac{1}{2}}^{\frac{1}{2}} x\, dx}_{=0} + 7 \underbrace{\int_{-\frac{1}{2}}^{1/2} \sin(\pi x)\, dx}_{=0} = 2\int_0^{\frac{1}{2}} x^2\, dx = \frac{2x^3}{3}\Big|_0^{\frac{1}{2}} = \frac{2 \cdot \frac{1}{8}}{3} = \frac{1}{12}$$

EXAMPLE 2. *The temperature of a metal rod, 10 m long, is* $5x$ *(in °C) at a distance* x *meters from one end of the rod. What is the average temperature of the rod?*



$T(x) = 5x$, $0 \le x \le 10$

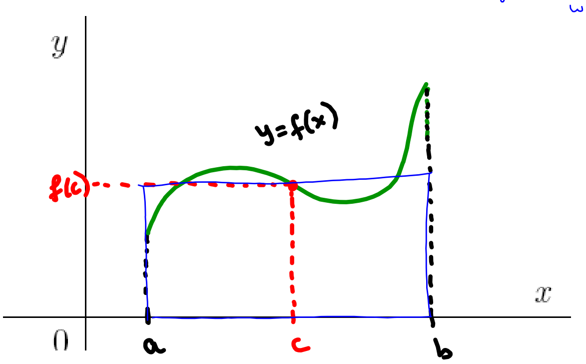
$$T_{ave} = \frac{1}{10-0} \int_0^{10} T(x)\, dx$$

$$= \frac{1}{\cancel{10}_2} \int_0^{10} \cancel{5}x\, dx = \frac{1}{2} \cdot \frac{x^2}{2}\Big|_0^{10}$$

$= 25°C$.

MEAN VALUE THEOREM FOR INTEGRALS: *If f is continuous on $[a,b]$, then there exists a number c on $[a,b]$ s.t.*

$$\int_a^b f(x)\,dx = \underbrace{f(c)}_{\text{height}}\underbrace{(b-a)}_{\text{length of }[a,b]\text{ or width}}.$$



$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$f(c) = f_{ave}$$

The geometric interpretation of the Mean Value Theorem for Integrals: for *positive* functions f, there is a number c s.t. the rectangle with base $[a,b]$ and height $f(c)$ has the same area as the region under the graph of f from a to b.

MEAN VALUE THEOREM FOR INTEGRALS: *If f is continuous on $[a,b]$, then there exists a number c on $[a,b]$ s.t.*

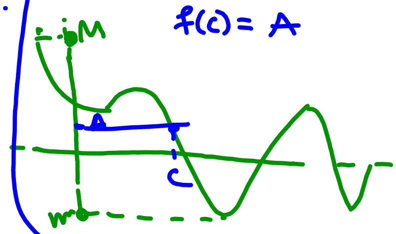
$$\int_a^b f(x)\,dx = f(c)(b-a).$$

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\underbrace{m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M}$$

$$m \le A \le M$$

$$f(c) = A$$



Proof. • Since f is continuous on $[a,b]$, f has absolute max and min on $[a,b]$.

Denote $M = \max_{a\le x\le b} f(x)$, $m = \min_{a\le x\le b} f(x)$.

Denote $A = \frac{1}{b-a}\int_a^b f(x)\,dx$.

• By a known property for definite integrals,

$$m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M, \text{ i.e. } m \le A \le M.$$

Then, it remains to show that there exists a number c on $[a,b]$ such that $f(c) = A$.

Indeed, by the Intermediate Value Theorem (because f is cont. on $[a,b]$) there exist a point c on $[a,b]$ such that $f(c) = A$,

or $$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

or

$$(b-a)f(c) = \int_a^b f(x)\,dx. \quad \square$$

EXAMPLE 3. *If g is continuous and* $\displaystyle\int_{-1}^{7} g(x)\,\mathrm{d}x = 24$ *show that g takes on the value 3 at least once on the interval* $[-1, 7]$.

If $\displaystyle\int_{-1}^{7} g(x)\,dx = 24$, where g is continuous. $\xrightarrow{\text{then.}}$ There is a point $x = c$ on $[-1,7]$ such that $g(c) = 3$.

Proof

By the Mean Value Theorem (since g is continuous on $[-1,7]$) there is a point $x = c$ on $[-1,7]$ such that

$$g(c) = \frac{1}{7-(-1)} \underbrace{\int_{-1}^{7} g(x)\,dx}_{24} = \frac{1}{8} \cdot 24 = 3. \quad \square$$

EXAMPLE 4. *Determine the number c that satisfies the Mean Value Theorem for Integrals for the function $f(x) = x^2 - 2x - 2$ on the interval [1,4]*

Note, $f(x)$ is continuous on $[1,4]$ (as a polynomial).

Hence, by the MVT, there is a number c such that

$$\underbrace{f(c)}_{\downarrow} = \frac{1}{4-1}\int_1^4 (x^2-2x-2)\,dx \text{, where } 1 \le c \le 4.$$

$$c^2 - 2c - 2 = \frac{1}{3}\left(\frac{x^3}{3} - x^2 - 2x\right)\Big|_1^4 = 0$$

$$\frac{1}{3}\left(\frac{64}{3} - 16 - 8 - \frac{1}{3} + 1 + 2\right)$$

$$21 - 24 + 3 = 0$$

$$c^2 - 2c - 2 = 0$$

$$ax^2 + bx + c = 0$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$c_{1,2} = \frac{2 \pm \sqrt{4+8}}{2} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}$$

$$1 - \sqrt{3} < 0 \implies 1 - \sqrt{3} \notin [1,4]$$

$$1 < 1 + \sqrt{3} < 4.$$

Answer: $1 + \sqrt{3}$