8.1: Integration by Parts

- $\int x^n e^{kx}\,\mathrm{d}x$
- $\int x^n \sin(kx)\,\mathrm{d}x$, $\int x^n \cos(kx)\,\mathrm{d}x$
- $\int x^n \ln x\,\mathrm{d}x$
- $\int e^x \cos x\,\mathrm{d}x$, $\int e^x \sin x\,\mathrm{d}x$
- $\int \sec^n x\,\mathrm{d}x$ and some integrals involving inverse trigonometric functions.
- $\int \arcsin x\,dx$, $\int \arctan x\,dx$ . . .

The **integration by parts formula**:

f(x), g(x) both are integrable.

$$\int fg'\,dx = fg - \int f'g\,dx$$

Proof: We know $(fg)' = f'g + fg'$

Integrate both parts: $\int (fg)'\,dx = \int f'g\,dx + \int f g'\,dx$

By FTC, $fg = \int f'g\,dx + \underbrace{\int fg'\,dx}$,

$\int \underbrace{f}_{u}\,\underbrace{g'dx}_{dv} = \underbrace{fg}_{uv} - \int \overbrace{f'}^{du}\underbrace{g}_{v}\,dx$. □

Rewrite the above formula using the following substitutions:

$$u = f(x), \qquad v = g(x)$$

$du = f'(x)dx$, $dv = g'(x)dx$

$$\boxed{\int u\,dv = uv - \int v\,du} \quad (+C)$$

$\int u\,dv = uv - \int v\,du$ ILPET

LIPET rule to choose u and dTEPIL rule to choose dv:

whichever function comes first in the list below, it should be u.

L Logarithmic functions $\ln x$, $\log x$, $\ln 3x$

I Inverse trigonometric functions $\arcsin x$, $\arctan x$, $\arccos x$ ($\sin^{-1}x$, $\tan^{-1}x$, $\cos^{-1}x$, $\cot^{-1}x$)

P Polynomial functions $1, x, x^2, x^{2016}+5, 3x^2-4, \dots$

E Exponential functions e^x, e^{2x}, ...

du **T** Trigonometric functions $\sin x, \cos x, \sec 3x, \tan 7x, \dots$

EXAMPLE 1. *Evaluate* $I = \int x\cos(5x)\,dx$

$u = x \Rightarrow du = dx$

$\int \cos 5x\,dx = \int dv \Rightarrow v = \frac{1}{5}\sin(5x)$

$$I = uv - \int v\,du = \frac{x}{5}\sin(5x) - \int \frac{1}{5}\sin(5x)\,dx$$

$$= \frac{x}{5}\sin(5x) - \frac{1}{5}\left(\frac{1}{5}(-\cos(5x))\right) + C$$

$$= \frac{x}{5}\sin(5x) + \frac{1}{25}\cos(5x) + C$$

Integration by parts formula for definite integrals:

Way2 :
$$\int_a^b u\,dv = uv\big|_a^b - \int_a^b v\,du,$$

$\int_a^b u\,dv$

way 1: Evaluate $\int u\,dv$, then plug in the bounds.

where

$$uv\big|_a^b = u(b)v(b) - u(a)v(a).$$

EXAMPLE 2. *Evaluate* $I = \int_{-1}^{3} \underbrace{x}_{u} \underbrace{e^{3x}\,dx}_{dv}$

$u = x \Rightarrow du = dx$

$e^{3x}dx = dv \Rightarrow v = \int e^{3x}dx = \frac{1}{3}e^{3x}$

$I = uv\big|_{-1}^{3} - \int_{-1}^{3} v\,du = \frac{x}{3}e^{3x}\Big|_{-1}^{3} - \frac{1}{3}\int_{-1}^{3} e^{3x}dx$

$= \frac{1}{3}\left[3e^{3\cdot 3} - (-1)e^{3\cdot(-1)}\right] - \frac{1}{3}\cdot\frac{1}{3}e^{3x}\Big|_{-1}^{3}$

$= \frac{1}{3}\left[3e^{9} + \frac{1}{e^3}\right] - \frac{1}{9}\left[e^{9} - \frac{1}{e^3}\right]$

$= e^{9} + \frac{1}{3e^3} - \frac{1}{9}e^{9} + \frac{1}{9e^3} = \frac{8}{9}e^{9} + \frac{4}{9e^3}$

$\frac{3/1}{3} + \frac{1}{9}$

EXAMPLE 3. *Evaluate* $I = \int \underbrace{x^2}_{u} \underbrace{\sin(5x)\,dx}_{dv}$

$u = x^2 \Rightarrow du = 2x\,dx$

$dv = \sin(5x)\,dx \Rightarrow v = -\frac{1}{5}\cos(5x)$

$I = uv - \int v\,du = -\frac{x^2}{5}\cos(5x) - \int -\frac{1}{5}\cos(5x)\,2x\,dx$

$= -\frac{x^2}{5}\cos(5x) + \frac{2}{5}\underbrace{\int x\cos(5x)\,dx}$ =

Apply the formula again

$U = x$

$dV = \cos(5x)\,dx$

or use Ex. 2

$= -\frac{x^2}{5}\cos(5x) + \frac{2}{5}\left(\frac{x}{5}\sin(5x) + \frac{1}{25}\cos(5x)\right) + C$

EXAMPLE 4. *Evaluate* $I = \int \underbrace{\ln x}_{u} \underbrace{dx}_{dv}$

$u = \ln x \Rightarrow du = \frac{dx}{x}$

$dv = dx \Rightarrow v = x$

$$I = uv - \int v\,du = x\ln x - \int x \frac{dx}{x} = x\ln x - \int dx$$

$$\int \ln x\,dx = x\ln x - x = x(\ln x - 1)$$

EXAMPLE 5. *Evaluate* $I = \int \arcsin x \, dx$

$u = \arcsin x \longrightarrow du = \frac{dx}{\sqrt{1-x^2}}$

$dv = dx \longrightarrow v = x$

$I = uv - \int v\,du = x \arcsin x - \int \frac{x\,dx}{\sqrt{1-x^2}} = x \arcsin x - \int \frac{-\frac{1}{2}dU}{\sqrt{U}}$

Use U-sub for $\int \frac{x\,dx}{\sqrt{1-x^2}}$

$U = 1 - x^2 \Rightarrow dU = -2x\,dx$

$x\,dx = -\frac{1}{2}dU$

$= x \arcsin x + \frac{1}{2}\int U^{-\frac{1}{2}}dU$

$= x \arcsin x + \frac{1}{2} \frac{U^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$

$I = x \arcsin x + U^{\frac{1}{2}} + C = x \arcsin x + \sqrt{1-x^2} + C.$

EXAMPLE 6. *Evaluate* $I = \int e^x \cos x \, dx$

$u = \cos x \qquad dv = e^x dx$

$du = -\sin x \, dx \qquad v = e^x$

$I = uv - \int v \, du = e^x \cos x + \underbrace{\int e^x \sin x \, dx}$

$U = \sin x \qquad dV = e^x dx$

$dU = \cos x \, dx \qquad V = e^x$

$I = e^x \cos x + UV - \int V \, dU$

$I = \underbrace{e^x \cos x + e^x \sin x} - \underbrace{\int e^x \cos x \, dx}_{I}$

$I = e^x (\cos x + \sin x) - I$

$2I = e^x (\cos x + \sin x)$

$$\boxed{I = \frac{1}{2} e^x (\cos x + \sin x)}$$

Loop integral.