8.2: Trigonometric Integrals

RULE: How to evaluate $\int \sin(Ax)\cos(Bx)\,dx$, $\int \sin(Ax)\sin(Bx)\,dx$, $\int \cos(Ax)\cos(Bx)\,dx$ *Use the following identities:*

$$\begin{aligned}
\sin A \cos B &= \tfrac{1}{2}(\sin(A-B)+\sin(A+B)) \\
\sin A \sin B &= \tfrac{1}{2}(\cos(A-B)-\cos(A+B)) \\
\cos A \cos B &= \tfrac{1}{2}(\cos(A-B)+\cos(A+B))
\end{aligned}$$

EXAMPLE 1. *Evaluate* $I = \int \cos(25x)\cos(4x)\,dx = \frac{1}{2}\int(\cos(25x-4x)+\cos(25x+4x))dx$

$= \frac{1}{2}\int(\cos(21x)+\cos(29x))dx = \frac{1}{2}\left[\frac{1}{21}\sin(21x)+\frac{1}{29}\sin(29x)\right]+C$

EXAMPLE 2. *Prove th formula, where m and n are positive integers:*

$$I(m,n) = \int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & \text{if} \quad m \neq n \\ \pi & \text{if} \quad m = n \end{cases}$$

$$I(m,n) = \int_{-\pi}^{\pi} \sin(mx) \sin(nx) \, dx$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \left(\cos(mx - nx) - \cos(mx + nx) \right) dx$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \left(\cos(m-n)x - \cos(m+n)x \right) dx$$

Let $m \neq n$

$$I(m,n) = \frac{1}{2} \left[\frac{1}{m-n} \sin(m-n)x - \frac{1}{m+n} \sin(m+n)x \right] \Bigg|_{-\pi}^{\pi} = 0$$

$\sin k\pi = 0$ for all integer k

But $m-n$ and $m+n$ are integers, because m, n are integers.

Let $m = n$.

$$I(m,m) = \frac{1}{2} \int_{-\pi}^{\pi} \left(\cos(m-m)x - \cos(m+m)x \right) dx$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \left(\underbrace{\cos 0}_{1} - \cos(2mx) \right) dx$$

$$= \frac{1}{2} \left(2\pi - \underbrace{\frac{1}{2m} \sin(2mx) \Big|_{-\pi}^{\pi}}_{0} \right) = \pi .$$

$$\begin{aligned} \sin^2 x &+ \cos^2 x = 1 \\ \tan^2 x &+ 1 = \sec^2 x \end{aligned}$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin x \cos x = \frac{1}{2}\sin 2x$$

$\cos 2x = \cos^2 x - \sin^2 x$

$\cos 2x = 2\cos^2 x - 1$

$\cos 2x = 1 - 2\sin^2 x$

EXAMPLE 3. *Evaluate* $I = \int \cos x \sin^{2016} x \, dx$ $= \int (\sin x)^{2016} \underbrace{\cos x \, dx}$

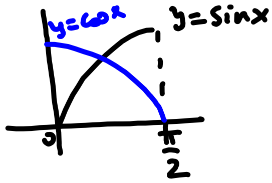
$u = \sin x$

$du = \cos x \, dx$

$$= \int u^{2016} du = \frac{u^{2017}}{2017} + C$$

$$= \frac{\sin^{2017} x}{2017} + C$$

EXAMPLE 4. *Evaluate* $I_1 = \int_0^{\pi/2} \sin^2 x \, \mathrm{d}x$ *and* $I_2 = \int_0^{\pi/2} \cos^2 x \, \mathrm{d}x$



First note that $I_1 = I_2$

$$\sin^2 x = \frac{1}{2}(1 - \cos(2x))$$

$$I_1 = \int_0^{\pi/2} \sin^2 x \, dx = \frac{1}{2} \int_0^{\pi/2} (1 - \cos(2x)) \, dx$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - \overbrace{\frac{1}{2}\sin(2x)\Big|_0^{\pi/2}}^{=0}\right)$$

$$= \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4} = I_2$$

EXAMPLE 5. *Evaluate* $I = \int \sin^5 x \, dx = \int \sin^4 x \underbrace{\sin x \, dx}$

$\cos^2 x + \sin^2 x = 1$

$= \int (\sin^2 x)^2 \sin x \, dx = \int (1 - \cos^2 x)^2 \underbrace{\sin x \, dx}$

$u = \cos x \Rightarrow du = -\sin x dx$

$\sin x \, dx = -du$

$= -\int (1 - u^2)^2 du = -\int (1 - 2u^2 + u^4) du$

$= -\left(u - \frac{2u^3}{3} + \frac{u^5}{5}\right) + C$

$= -\frac{u^5}{5} + \frac{2u^3}{3} - u + C = -\frac{\cos^5 x}{5} + \frac{2\cos^3 x}{3} - \cos x + C$