8.3: Trigonometric Substitution

integral with	substitution	identity
$a^2 - x^2$	$x = a\sin\theta, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$a^2 + x^2$	$x = a\tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$x^2 - a^2$	$x = a\sec\theta, \quad 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$
$ax^2 + bx + c$	complete squares and then do the correct substitution	

EXAMPLE 1. *Use a trigonometric substitution to eliminate the radical:*

(a) $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = |\cos\theta|$

$x = \sin\theta \Rightarrow \theta = \arcsin x$ (with an appropriate domain of θ)

(b) $\sqrt{1-3x^2} = \sqrt{1-(\sqrt{3}x)^2} = \sqrt{1-\sin^2\theta} = |\cos\theta|$

$\sqrt{3}x = \sin\theta$

$\theta = \arcsin\sqrt{3}x$

(c) $\sqrt{x^2+16}$ $= \sqrt{(4\tan\theta)^2+16}$ $\quad x = 4\tan\theta \qquad 1+\tan^2\theta = \sec^2\theta$

$= \sqrt{16\tan^2\theta+16} = \sqrt{16(\tan^2\theta+1)}$

$= 4\sqrt{\sec^2\theta} = 4|\sec\theta|$

(d) $\sqrt{49x^2+16}$ $= \sqrt{(7x)^2+16} = \sqrt{16\tan^2\theta+16} = 4|\sec\theta|$

$7x = 4\tan\theta$

$x = \frac{4}{7}\tan\theta$

EXAMPLE 2. *Evaluate the given integral:*

(a) $\displaystyle\int \frac{1}{x^2\sqrt{5-x^2}}\,dx$

Remove radical: $\boxed{x=\sqrt{5}\sin\theta} \Rightarrow \boxed{dx=\sqrt{5}\cos\theta\,d\theta}$

$\sqrt{5-x^2} = \sqrt{(\sqrt{5})^2-x^2}$

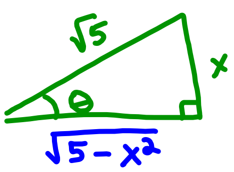
$= \sqrt{5-(\sqrt{5}\sin\theta)^2} = \sqrt{5(1-\sin^2\theta)} = \sqrt{5\cos^2\theta}$

$= \boxed{\sqrt{5}\cos\theta}$ (formally, it is $\sqrt{5}\cdot|\cos\theta|$

$$\int \frac{dx}{x^2\sqrt{5-x^2}} = \int \frac{\cancel{\sqrt{5}}\,\cancel{\cos\theta}\,d\theta}{(\sqrt{5}\sin\theta)^2\,\cancel{\sqrt{5}}\,\cancel{\cos\theta}} = \frac{1}{5}\int \frac{d\theta}{\sin^2\theta} = \frac{1}{5}\int \csc^2\theta$$

$$= -\frac{1}{5}\cot\theta + C = -\frac{\sqrt{5-x^2}}{5x} + C$$

Back to x: $x=\sqrt{5}\sin\theta \Rightarrow \sin\theta = \frac{x}{\sqrt{5}}$



$\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{\sqrt{5-x^2}}{x}$

(b) $\int_{\sqrt{5}/2}^{\sqrt{10}/2} \frac{1}{x^2\sqrt{5-x^2}}\,dx$

Way 1: By Ex 2

(b) $\int_{\sqrt{5}/2}^{\sqrt{10}/2} \frac{1}{x^2\sqrt{5-x^2}}\,dx = -\frac{\sqrt{5-x^2}}{5x}\Bigg|_{\sqrt{5}/2}^{\sqrt{10}/2}$

$= -\frac{1}{5}\left[\frac{\sqrt{5-\frac{10}{4}}}{\frac{\sqrt{10}}{2}} - \frac{\sqrt{5-\frac{5}{4}}}{\frac{\sqrt{5}}{2}}\right]$

$= -\frac{1}{5}\left[\frac{\sqrt{10}\cancel{/2}}{\sqrt{10}\cancel{/2}} - \frac{\sqrt{15}\cancel{/2}}{\sqrt{5}\cancel{/2}}\right]$

$= -\frac{1}{5}[1-\sqrt{3}] = \frac{1}{5}[\sqrt{3}-1].$

Way 2 (don't back to x)

$x = \sqrt{5}\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Find bounds for θ:

$x = \frac{\sqrt{5}}{2} \Rightarrow \frac{\sqrt{5}}{2} = \sqrt{5}\sin\theta \Rightarrow$

$\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

$x = \frac{\sqrt{10}}{2} \Rightarrow \frac{\sqrt{10}}{2} = \sqrt{5}\sin\theta \Rightarrow \sin\theta = \frac{\sqrt{2}}{2}$

$\Rightarrow \theta = \frac{\pi}{4}$

(b) $\int_{\sqrt{5}/2}^{\sqrt{10}/2} \frac{1}{x^2\sqrt{5-x^2}}\,dx \overset{\text{By Ex.2}}{=} -\frac{1}{5}\cot\theta\Bigg|_{\pi/6}^{\pi/4}$

$= -\frac{1}{5}\left(\cot\frac{\pi}{4} - \cot\frac{\pi}{6}\right) = -\frac{1}{5}(1-\sqrt{3})$

$= \frac{1}{5}(\sqrt{3}-1).$

(c) $\displaystyle\int_0^{4/7} \frac{1}{(49x^2+16)^{3/2}}\,dx = I$

By Ex.1(c)

$x = \frac{4}{7}\tan\theta$

$\sqrt{49x^2+16} = 4\sec\theta \quad , \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$dx = \frac{4}{7}\sec^2\theta\, d\theta$

$x = 0 \Rightarrow 0 = \frac{4}{7}\tan\theta \Rightarrow \theta = 0$

$x = \frac{4}{7} \Rightarrow \frac{4}{7} = \frac{4}{7}\tan\theta \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

$$I = \int_0^{4/7} \frac{dx}{\left(\sqrt{49x^2+16}\right)^3} = \int_0^{\pi/4} \frac{\frac{4}{7}\sec^2\theta\, d\theta}{(4\sec\theta)^3} = \frac{4}{7\cdot 4^3}\int \frac{\sec^2\theta\, d\theta}{\sec^3\theta}$$

$$= \frac{1}{7\cdot 16}\int_0^{\pi/4} \frac{d\theta}{\sec\theta} = \frac{1}{112}\int_0^{\pi/4} \cos\theta\, d\theta$$

$$= \frac{1}{112}\sin\theta\Big|_0^{\pi/4} = \frac{1}{112}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{224}.$$