8.4: Integration Of Rational Functions By Partial Fractions

EXAMPLE 1. *Evaluate the following integrals:*

(a) $\displaystyle\int \frac{2x-5}{x^2-5x+4}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|x^2-5x+4| + C$

$u = x^2-5x+4$

$du = (2x-5)\,dx$

(b) $\displaystyle\int \frac{x-6}{x^2-5x+4}\,dx$

We will use so called Partial Fraction Decomposition

$$\frac{1}{ax^2+bx+C} = \frac{1}{a(x-x_1)(x-x_2)} = \frac{A}{x-x_1} + \frac{B}{x-x_2}$$

$$\frac{x-6}{x^2-5x+4} = \frac{x-6}{(x-4)(x-1)} = \frac{A}{x-4} + \frac{B}{x-1}$$

$$\frac{x-6}{(x-4)(x-1)} = \frac{A(x-1)+B(x-4)}{(x-4)(x-1)}$$

$x-6 = A(x-1) + B(x-4)$ for all x.

$x=1$: $1-6 = A(1-1) + B(1-4)$

$-5 = -3B \Rightarrow \boxed{B = \frac{5}{3}}$

$x=4$: $4-6 = A(4-1) + 0$

$-2 = 3A \Rightarrow \boxed{A = -\frac{2}{3}}$

$$I = \int\left(\frac{A}{x-4} + \frac{B}{x-1}\right)dx = \int -\frac{2}{3}\cdot\frac{dx}{x-4} + \int \frac{5}{3}\frac{dx}{x-1}$$

$$= -\frac{2}{3}\int\frac{dx}{x-4} + \frac{5}{3}\int\frac{dx}{x-1}$$

$$= -\frac{2}{3}\ln|x-4| + \frac{5}{3}\ln|x-1| + C$$

Partial Fraction Decomposition Process

Rational function: $f(x) = \dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.

Step 0. $\deg P(x) < \deg Q(x)$.

Important to remember: Partial fractions can only be done if the degree of the numerator is **strictly less** than the degree of denominator. (Otherwise, you must first do **long division**.)

Step 1. Factor the denominator as much as possible.

Step 2. For each **linear factor** in the denominator use the following table:

	Factor in denominator	Term in partial fraction decomposition
linear factor	$ax+b$	$\dfrac{A}{ax+b}$
repeated linear factor	$(ax+b)^2$	$\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2}$
	$(ax+b)^k$	$\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \cdots + \dfrac{A_k}{(ax+b)^k}$

For each **prime (irreducible quadratic factor)** in the denominator use the following table:

	Factor in denominator	Term in partial fraction decomposition
prime factor	ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
repeated prime factor	$(ax^2+bx+c)^2$	$\dfrac{A_1x+B_1}{ax^2+bx+c} + \dfrac{A_2x+B_2}{(ax^2+bx+c)^2}$
	$(ax^2+bx+c)^k$	$\dfrac{A_1x+B_1}{ax^2+bx+c} + \dfrac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \dfrac{A_kx+B_k}{(ax^2+bx+c)^k}$

$$ax^2+bx+c = a(x-x_1)(x-x_2)$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Discriminant

$D = b^2-4ac$

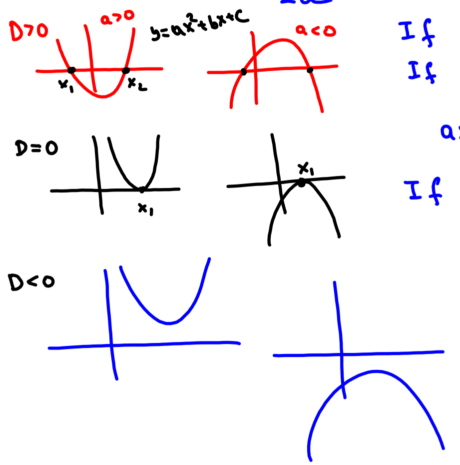
If $D>0$, then x_1, x_2 are distinct roots

If $D=0$, then $x_1 = x_2$ (one root or a repeated root)

$ax^2+bx+c = a(x-x_1)^2$.

If $D<0$, then no real roots.

$\underbrace{ax^2+bx+c}$ (prime factor)



EXAMPLE 2. *Write out the form of the partial fraction decomposition of the following rational functions. (Do not try to solve)*

(a) $$\frac{3x}{\underbrace{(x-1)}_{\text{L.F.}}\underbrace{(3x+12)}_{\text{L.F.}}} = \frac{A}{x-1} + \frac{B}{3x+12}$$

(b) $$\frac{5x^2}{\underbrace{(x-1)^2}_{\text{R.L.F.}}\underbrace{(x^2-1)}} = \frac{5x^2}{(x-1)^2(x-1)(x+1)} = \frac{5x^2}{(x-1)^3(x+1)}$$

$$= \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{A_3}{(x-1)^3} + \frac{B}{x+1}$$

(c) $$\frac{7}{x(x^3+x^2+x)} = \frac{7}{x \cdot x\underbrace{(x^2+x+1)}_{\text{P.F.}}} = \frac{7}{\underbrace{x^2}_{\text{R.L.}}\underbrace{(x^2+x+1)}_{\text{P.F.}}}$$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+x+1}$$

(d) $$\frac{x+5}{\underbrace{(x-3)}_{\text{L.F.}}\underbrace{(x^2+25)^2}_{\text{R.P.F.}}} = \frac{A}{x-3} + \frac{Bx+C}{x^2+25} + \frac{Dx+E}{(x^2+25)^2}$$

Note that in the above examples

deg (num.) < deg (denom.)

$$\int \frac{dx}{ax+b} = \frac{1}{a} \ln |ax+b| + C$$

$$\int \frac{dx}{(ax+b)^n} = \frac{1}{a} \cdot \frac{(ax+b)^{1-n}}{1-n} + C ,$$

if $n \neq 1$.

$$\int \frac{dx}{x^n} = \int x^{-n} dx$$

$$= \frac{x^{1-n}}{1-n} + C$$

$$\int \frac{dx}{x^2+1} = \arctan x + C$$

$$\int \frac{dx}{x^2+a^2} = \ldots$$

EXAMPLE 3. *Evaluate* $I = \int \frac{x^2+1}{x^2-x}\,dx$

$\deg(x^2+1) = \deg(x^2-x)$

Note $\frac{x^2+1}{x^2-x} = \frac{(x^2-x)+(x+1)}{x^2-x} = 1 + \frac{x+1}{x^2-x}$

(another way: Do Long Division for polynomials).

$I = \int\left(1 + \frac{x+1}{x^2-x}\right)dx = x + \underbrace{\int \frac{x+1}{x(x-1)}\,dx}_{\text{Apply Part. Fraction Decomp:}} + C$

$\frac{x+1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} = \frac{A(x-1)+Bx}{x(x-1)}$

$x+1 = A(x-1) + Bx$

$x=1$ $\boxed{2 = B}$

$x=0$ $1 = -A$, or $\boxed{A=-1}$

$I = x + \int \frac{A}{x}\,dx + \int \frac{B}{x-1}\,dx + C$

$= x - \int \frac{dx}{x} + 2\int \frac{dx}{x-1} + C = \boxed{x - \ln|x| + 2\ln|x-1| + C}$

$= x - \ln|x| + \ln(x-1)^2 + C$

$= x + \ln\frac{(x-1)^2}{|x|} + C$

EXAMPLE 4. *Evaluate* $I = \int \frac{x^2 - 3x + 7}{(x-1)(x^2+1)}\,dx$

$\deg(x^2 - 3x + 7) = 2 < \deg(x-1)(x^2+1) = 3$

Apply PFD:

$$\frac{x^2 - 3x + 7}{\underbrace{(x-1)}_{\text{L.F.}}\underbrace{(x^2+1)}_{\text{P.Q.F.}}} = \frac{A}{x-1} + \frac{Bx + C}{x^2+1} = \frac{A(x^2+1) + (Bx+C)(x-1)}{(x-1)(x^2+1)}$$

$$x^2 - 3x + 7 = A(x^2+1) + (Bx+C)(x-1)$$

$x = 1$: $\quad 1 - 3 + 7 = A(1^2 + 1)$

$\quad 5 = 2A \Rightarrow \boxed{A = \frac{5}{2}}$

x^2: $\quad 1 = A + B \Rightarrow 1 = \frac{5}{2} + B \Rightarrow \boxed{B = -\frac{3}{2}}$

x: $\quad -3 = C - B \Rightarrow -3 = C + \frac{3}{2} \Rightarrow \boxed{C = -\frac{9}{2}}$

$$I = \int \frac{A}{x-1}\,dx + \int \frac{Bx+C}{x^2+1}\,dx$$

$$= \frac{5}{2}\int \frac{dx}{x-1} + \int \frac{-\frac{3}{2}x - \frac{9}{2}}{x^2+1}\,dx$$

$$= \frac{5}{2}\ln|x-1| - \frac{3}{2}\underbrace{\int \frac{x\,dx}{x^2+1}}_{\substack{u = x^2+1 \\ du = 2x\,dx}} - \frac{9}{2}\underbrace{\int \frac{dx}{x^2+1}}_{\arctan x} + C$$

$$\int \frac{du}{2u} = \frac{1}{2}\ln|u| = \frac{1}{2}\ln(x^2+1)$$

$$= \frac{5}{2}\ln|x-1| - \frac{3}{2}\cdot\frac{1}{2}\ln(x^2+1) - \frac{9}{2}\arctan x + C$$

$$= \frac{5}{2}\ln|x-1| - \frac{3}{4}\ln(x^2+1) - \frac{9}{2}\arctan x + C$$