8.8: Approximate Integration

Not all integrals can be computed. There are two such situations:

- When it is difficult, or even impossible to find antiderivative of integrand in order to apply the Fundamental Theorem of Calculus. For example,

$$\int_0^1 e^{x^2}\,\mathrm{d}x, \quad \int_{-1}^1 \sqrt{1+x^3}\,\mathrm{d}x.$$

- When the integrand is determined from a scientific experiment through instrument reading (table).

In the above cases we can think of the integral as an area problem and using known shapes to estimate the area under the curve (in other words, to find approximate values of definite integrals). For that we need to use the definition of definite integral as a limit of Riemann sums. So, any Riemann sum could be used as an approximation. In particular, taking a partition of $[a,b]$ into n subintervals of the equal length, we get

$$\int_a^b f(x)\,\mathrm{d}x \approx \underbrace{\sum_{i=1}^n f(x_i^*)\Delta x}_{Riemann\quad Sum},$$

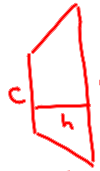
where $\Delta x = (b-a)/n$ and x_i^* is an arbitrary point of the i-th subinterval $[x_{i-1}, x_i]$ of the partition.

Midpoint Riemann Sum Take x_i^* as the midpoint $\bar{x}_i$ of $[x_{i-1}, x_i]$:

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\Delta x = M_n \qquad \left(\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)\right)$$

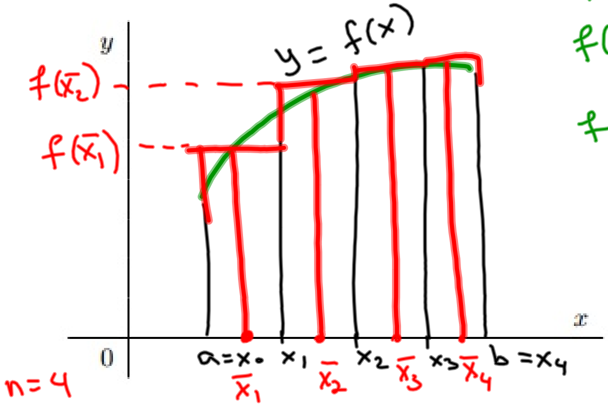
Trapezoid Riemann Sum Take the sum of areas of trapezoids that lie above the i-th subinterval:

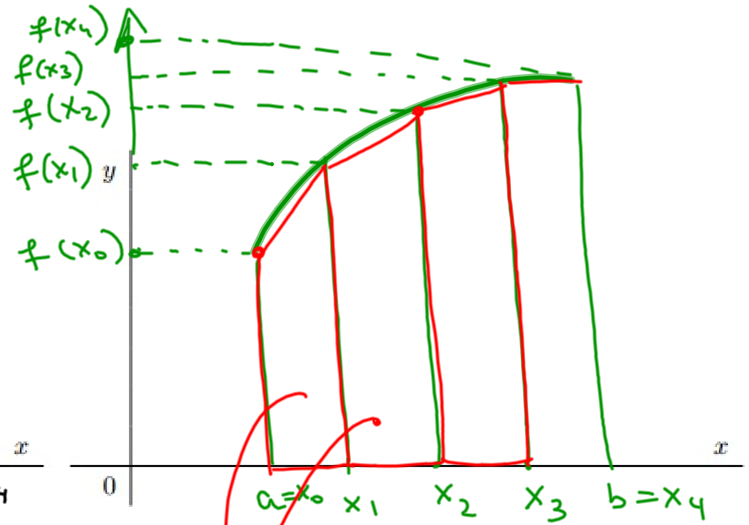
$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} \underbrace{\frac{f(x_{i-1} + f(x_i)}{2}}_{\text{area of the corresponding trapezoid}}\Delta x = T_n$$



$A = \frac{c+d}{2} \cdot h$

$h = \Delta x$





$n = 4$

$\Delta x = \frac{b-a}{n}$

$\bar{x}_1 = a + \frac{\Delta x}{2} = \frac{x_0 + x_1}{2}$

$\bar{x}_2 = a + \frac{3\Delta x}{2}$

$\bar{x}_3 = a + \frac{5\Delta x}{2}$

$\bar{x}_4 = a + \frac{7\Delta x}{2}$

$\frac{f(x_0) + f(x_1)}{2} \cdot \Delta x$

$\frac{f(x_1) + f(x_2)}{2} \Delta x$

Trapezoid Riemann Sum Take the sum of areas of trapezoids that lie above the i-th subinterval:

$$\int_a^b f(x)\,\mathrm{d}x \approx \sum_{i=1}^{n} \frac{f(x_{i-1}) + f(x_i)}{2}\Delta x = T_n$$

$$= \Big(f(x_0)+f(x_1) + f(x_1)+f(x_2) + f(x_2)+f(x_3) + \ldots + f(x_{n-2})+f(x_{n-1}) + f(x_{n-1})+f(x_n)\Big)\frac{\Delta x}{2}$$

Trapezoidal Rule

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n)\right),$$

where $x_i = a + i\Delta x$.

Midpoint Riemann Sum Take x_i^* as the midpoint $\bar{x}_i$ of $[x_{i-1}, x_i]$:

$$\int_a^b f(x)\,\mathrm{d}x \approx \sum_{i=1}^{n} f(\bar{x}_i)\Delta x = M_n \qquad \left(\bar{x}_i = \frac{1}{2}(x_{i-1}+x_i)\right)$$

EXAMPLE 1. *Using $n = 4$ and the Midpoint and Trapezoid Rules rules approximate the value of*

$$\int_1^2 e^{\frac{1}{x}}\,\mathrm{d}x$$

Midpoint Rule

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = 0.25$$

$$\int_1^2 e^{\frac{1}{x}}\,dx \approx M_4,$$

$$\frac{\Delta x}{2} = 0.125$$

where

$$M_n = \left(f(\bar{x}_1) + f(\bar{x}_2) + f(\bar{x}_3) + f(\bar{x}_4)\right)\Delta x$$

$$= \left(f(1.125) + f(1.375) + f(1.625) + f(1.875)\right)\cdot 0.25$$

$$= \left(e^{\frac{1}{1.125}} + e^{\frac{1}{1.375}} + e^{\frac{1}{1.625}} + e^{\frac{1}{1.875}}\right)\cdot 0.25$$

$$\approx \boxed{2.014207} = M_4$$

Trapezoidal Rule

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n)\right),$$

where $x_i = a + i\Delta x$.

By Trapezoid Rule

$$\int_1^2 e^{\frac{1}{x}}\,dx \approx T_4 = \frac{0.25}{2}\left(e^1 + 2e^{\frac{1}{1.25}} + 2e^{\frac{1}{1.5}} + 2e^{\frac{1}{1.75}} + e^{\frac{1}{2}}\right)$$

$$\approx 2.031893$$

DEFINITION 2. *The* **error** *in approximating* $\int_a^b f(x)\,dx$ *by* M_n *and* T_n *is defined as*

$$E_M = \underbrace{\int_a^b f(x)\,dx}_{} - M_n \quad \text{and} \quad E_T = \underbrace{\int_a^b f(x)\,dx}_{} - T_n,$$

respectively.

EXAMPLE 3. *Find the exact error in using* T_5 *to approximate* $\displaystyle\int_1^2 \frac{dx}{x}$.

$f(x) = \frac{1}{x}$

$$\int_1^2 \frac{dx}{x} \approx T_5$$

$\Delta x = \frac{2-1}{5} = \frac{1}{5}$

$x_0 = 1, \quad x_1 = 1 + \frac{1}{5} = \frac{6}{5}$

$x_2 = x_1 + \frac{1}{5} = \frac{7}{5}$

$x_3 = \frac{8}{5}$

$x_4 = \frac{9}{5}$

$x_{10} = 2$

$$E_T = \int_1^2 \frac{dx}{x} - T_5 = \underbrace{\ln|x|\Big|_1^2}_{} - \frac{\Delta x}{2}\Big(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5)\Big)$$

$$= \ln 2 - \frac{1}{10}\left(1 + 2\cdot\frac{5}{6} + 2\cdot\frac{5}{7} + 2\cdot\frac{5}{8} + 2\cdot\frac{5}{9} + \frac{1}{2}\right)$$

$$\approx -0.695635 + 0.693147 = -0.002488$$

REMARK 4. In both methods we get more accurate approximations when we increase the value of n.

Error Bounds

Theorem

Suppose $|f''(x)| \leq K$ for $a \leq x \leq b$. Then

$$|E_M| \leq \frac{K(b-a)^3}{24n^2} \quad \text{and} \quad |E_T| \leq \frac{K(b-a)^3}{12n^2}.$$

EXAMPLE 5. *Suppose we used T_4 to approximate $\int_1^3 \ln x \, dx$. Find an upper bound on the error.*

$|E_T| \leq \frac{K(3-1)^3}{12 \cdot 4^2}$,

$a=1, \ b=3$

$n=4$

where K is an upper bound for $|f''(x)|$ on $[1,3]$.

To find K; $f(x) = \ln x$

$f'(x) = \frac{1}{x}$

$f''(x) = -\frac{1}{x^2} \Rightarrow |f''(x)| = \frac{1}{x^2}$

Since $\frac{1}{x^2}$ is decreasing on $[1,3]$,

$\max |f''(x)| = |f''(1)| = 1 = K.$

Finally, $|E_T| \leq \frac{1 \cdot 2^3}{12 \cdot 16} \Rightarrow |E_T| \leq \frac{1}{24}$

EXAMPLE 6. *How large should we choose n so that M_n approximates $\int_1^3 \ln x \, dx$ within 0.01?*

Find n, so that $|E_M| \le 0.01$

We know $|E_M| \le \frac{K(b-a)^3}{24n^2} \le 0.01$ $\quad a=1$, $b=3$

By Ex. 5 $K=1$

$$\frac{1\cdot(3-1)^3}{24n^2} \le \frac{1}{100}$$

$$\frac{\cancel{2^3}^{\,1}}{\cancel{24}_{\,3}n^2} \le \frac{1}{100}$$

$$3n^2 \ge 100$$

$$n^2 \ge \frac{100}{3}$$

$$n \ge \sqrt{\frac{100}{3}} = \frac{10}{\sqrt{3}} \approx 5\tfrac{2}{3}$$

$$\frac{10}{\sqrt{3}} = \frac{10\sqrt{3}}{3} \approx \frac{10\cdot 1.7}{3} = \frac{17}{3} = 5\tfrac{2}{3}$$

$$\boxed{n \ge 6}$$