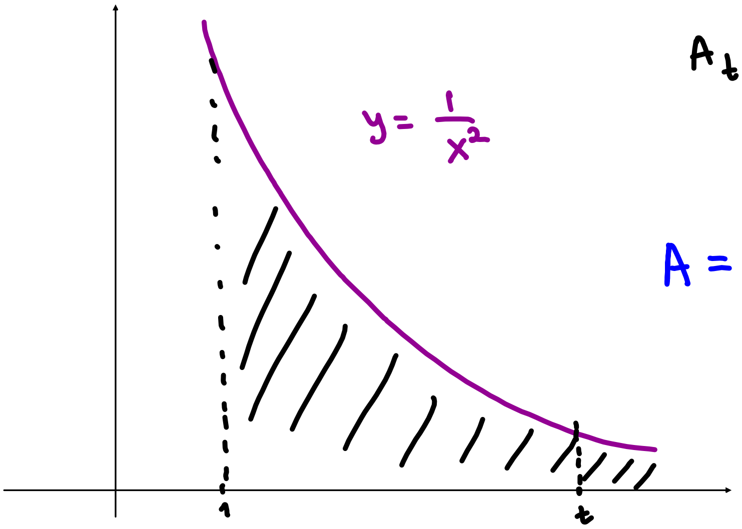
8.9: Improper Integrals

TYPE I: Infinite Interval and Continuous Integrand

EXAMPLE 1. *Evaluate* $\int_1^\infty \frac{1}{x^2}\,dx$ is convergent.

- What is the area, A, under the curve $y = \frac{1}{x^2}$ on $[1, \infty)$ is?
- What is the area, A_t, under the curve $y = \frac{1}{x^2}$ on $[1, t)$, $t > 1$, is?



$$A_t = \int_1^t \frac{dx}{x^2} = -\frac{1}{x}\Big|_1^t = -\left(\frac{1}{t} - 1\right) = 1 - \frac{1}{t}$$

$$A = \lim_{t\to\infty} A_t = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1$$

REMARK 2. Not all areas on an unbounded interval will yield finite areas.

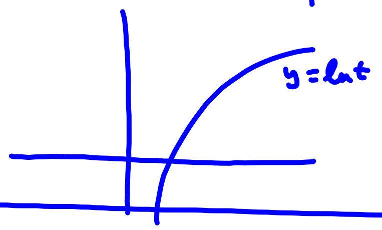
DEFINITION 3. *An improper integral is called* **convergent** *if the associated limit exists and is a finite number. An improper integral is called* **divergent** *if the associated limit does not exist or is $-\infty$, or ∞.*

EXAMPLE 4. *Evaluate* $\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$

$$= \lim_{t\to\infty} \ln|x| \Big|_1^t = \lim_{t\to\infty} \left(\ln|t| - \cancel{\ln 1}^{\,0}\right) = \lim_{t\to\infty} \ln|t|$$

$$= \lim_{t\to\infty} \ln t = \infty$$

So, $\int_1^\infty \frac{1}{x}\,dx$ is divergent.



FACT: If $a > 0$ then $\int_a^\infty \frac{1}{x^p}\,dx$ is convergent as $p > 1$ and divergent as $p \le 1$.

$$\int_a^\infty \frac{dx}{x^p} = \begin{cases} \int_a^\infty \frac{dx}{x}, & p=1 \\ \int_a^\infty \frac{dx}{x^p}, & p \neq 1 \end{cases} = \begin{cases} \lim_{t\to\infty} \int_a^t \frac{dx}{x} & (p=1) \\ \lim_{t\to\infty} \int_a^t x^{-p}\,dx & (p\neq 1) \end{cases} =$$

$$= \begin{cases} \lim_{t\to\infty} \ln|x| \Big|_a^t = \lim_{t\to\infty} \ln t - \ln|a| = \infty \quad (p=1) \text{ divergent} \\ p\neq 1 \quad \lim_{t\to\infty} \frac{x^{-p+1}}{-p+1}\Big|_a^t = \lim_{t\to\infty} \frac{1}{x^{(p-1)}(1-p)}\Big|_a^t = \frac{1}{1-p}\left[\lim_{t\to\infty} \frac{1}{t^{p-1}} - \frac{1}{a^{p-1}}\right] \end{cases}$$

$p > 1$

$\lim_{t\to\infty} \frac{1}{t^{p-1}} = 0$

the integral is convergent.

$p < 1$

$\lim_{t\to\infty} \frac{1}{t^{p-1}}$ DNE

the integral is divergent.

How to deal with Type I Improper Integrals:

- If $\int_a^t f(x)\,\mathrm{d}x$ exists for every $t \geq a$ then

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{t\to\infty} \int_a^t f(x)\,\mathrm{d}x$$

  provided the limit exists and finite.

- If $\int_t^b f(x)\,\mathrm{d}x$ exists for every $t \leq b$ then

$$\int_{-\infty}^b f(x)\,\mathrm{d}x = \lim_{t\to-\infty} \int_t^b f(x)\,\mathrm{d}x$$

  provided the limit exists and finite.

- If $\int_{-\infty}^c f(x)\,\mathrm{d}x$ and $\int_c^\infty f(x)\,\mathrm{d}x$ are BOTH convergent then

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \int_{-\infty}^c f(x)\,\mathrm{d}x + \int_c^\infty f(x)\,\mathrm{d}x$$

where c is any number.

EXAMPLE 5. *Evaluate* $I = \int_{-\infty}^{0} \frac{1}{\sqrt{20-x}}\, dx$

$$I = \lim_{t \to -\infty} \int_t^0 \frac{dx}{\sqrt{20-x}}$$

$$= \lim_{t \to -\infty} \left(-2\sqrt{20} + 2\sqrt{20-t}\right)$$

$$= -2\sqrt{20} + 2 \underbrace{\lim_{t \to -\infty} \sqrt{20-t}}_{\infty}$$ divergent

$$\int \frac{du}{\sqrt{u}} = \int u^{-\frac{1}{2}}\, du = \frac{u^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{u}$$

$$\int \frac{dx}{\sqrt{20-x}} = -2\sqrt{20-x}\Big|_t^0 = -2\left(\sqrt{20} - \sqrt{20-t}\right)$$

EXAMPLE 6. *Evaluate* $I = \int_{-\infty}^{\infty} xe^{-x^2}\,\mathrm{d}x$

$f(x) = xe^{-x^2}$ is continuous function on $(-\infty, \infty)$

$$I = \int_{-\infty}^{\infty} xe^{-x^2}\,dx = \underbrace{\int_{-\infty}^{0} xe^{-x^2}\,dx}_{I_1} + \underbrace{\int_{0}^{\infty} xe^{-x^2}\,dx}_{I_2}$$

$u = -x^2$

$du = -2x\,dx$

$x\,dx = -\frac{du}{2}$

$\int xe^{-x^2}\,dx = -\frac{1}{2}\int e^u\,du$

$= -\frac{1}{2}e^u = -\frac{1}{2}e^{-x^2}$

$$I_1 = \lim_{t\to-\infty} \int_t^0 xe^{-x^2}\,dx = \lim_{t\to-\infty}\left(-\frac{1}{2}e^{-x^2}\right)\Big|_t^0$$

$$= -\frac{1}{2}\lim_{t\to-\infty}\left(e^0 - e^{-t^2}\right) = -\frac{1}{2}\left(1 - \lim_{t\to-\infty} e^{-t^2}\right)$$

$$= -\frac{1}{2}(1-0) = \boxed{-\frac{1}{2}}$$

Similarly,

$$I_2 = \lim_{t\to\infty}\int_0^t xe^{-x^2}\,dx = -\frac{1}{2}\lim_{t\to\infty} e^{-x^2}\Big|_0^t$$

$$= -\frac{1}{2}\left(\lim_{t\to\infty} e^{-t^2} - 1\right) = -\frac{1}{2}(0-1) = \boxed{\frac{1}{2}}$$

$$I = I_1 + I_2 = -\frac{1}{2} + \frac{1}{2} = 0.$$

EXAMPLE 7. *Evaluate* $I = \int_{-2}^{\infty} \sin x \, \mathrm{d}x$

$f(x) = \sin x$ is continuous on $[-2, \infty)$

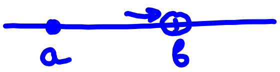
$$I = \lim_{t \to \infty} \int_{-2}^{t} \sin x \, dx = \lim_{t \to \infty} (-\cos x) \Big|_{-2}^{t}$$

$$= -\Big(\underbrace{\lim_{t \to \infty} \cos t}_{\text{DNE}} - \cos(-2) \Big) \text{ DNE}$$

I is divergent.

TYPE II: Discontinuous Integrand and Finite Interval

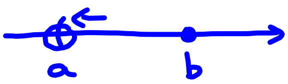
- If $f(x)$ is continuous on $[a, b)$ and not continuous at $x = b$ then



$$\int_a^b f(x)\,\mathrm{d}x = \lim_{t \to b^-} \int_a^t f(x)\,\mathrm{d}x$$

  if the limit exists and finite.

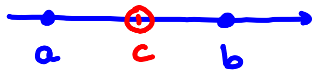
- If $f(x)$ is continuous on $(a, b]$ and not continuous at $x = a$ then



$$\int_a^b f(x)\,\mathrm{d}x = \lim_{t \to a+} \int_t^b f(x)\,\mathrm{d}x$$

  if the limit exists and finite.

- If $f(x)$ is continuous on $[a, c)$ and $(c, b]$ and not continuous at $x = c$, and the integrals $\int_a^c f(x)\,\mathrm{d}x$ and $\int_c^b f(x)\,\mathrm{d}x$ are both convergent then



$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$$

EXAMPLE 8. *Evaluate* $I = \int_0^{10} \frac{1}{\sqrt{10-x}}\,dx$

$f(x) = \frac{1}{\sqrt{10-x}}$ is continuous on $[0,10)$ and not continuous at $x=10$.

$$\int \frac{dx}{\sqrt{10-x}} = -2\sqrt{10-x}$$

$$\int \frac{du}{\sqrt{u}} = 2\sqrt{u}$$

$$I = \lim_{t\to 10^-} \int_0^t \frac{dx}{\sqrt{10-x}} = \lim_{t\to 10^-} \left(-2\sqrt{10-x}\right)\Big|_0^t$$

$$= -2\left(\lim_{t\to 10^-} \sqrt{10-t} - \sqrt{10}\right) = -2\left(0-\sqrt{10}\right) = 2\sqrt{10} \quad \text{convergent}$$

EXAMPLE 9. *Evaluate* $I = \int_0^1 \ln x\,dx$

$f(x) = \ln x$ is continuous on $(0,1]$ and not continuous at $x=0$.

$$I = \lim_{t\to 0^+} \int_t^1 \ln x\,dx$$

$$\int \ln x\,dx$$

$u = \ln x$, $dv = dx$

$du = \frac{dx}{x}$ $\quad v = x$

$$\int \ln x\,dx = x\ln x - \int x\frac{dx}{x} = x\ln x - \int dx = x\ln x - x$$

$$= \lim_{t\to 0^+} \left(x\ln x - x\right)\Big|_t^1$$

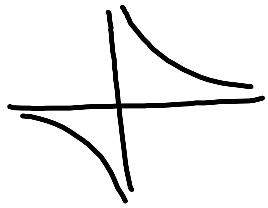
$$= \lim_{t\to 0^+} \left(1\cdot \overset{=0}{\ln 1} - 1 - (t\ln t - t)\right)$$

$$= -1 - \underbrace{\lim_{t\to 0^+} t\ln t}_{=0} - 0$$

$$= -1 \quad I \text{ is convergent.}$$

$$\lim_{t\to 0^+} t\ln t \overset{0\cdot\infty}{=} \lim_{t\to 0^+} \frac{\ln t}{\left(\frac{1}{t}\right)} \overset{L.R.}{=} \lim_{t\to 0^+} \frac{(\ln t)'}{\left(\frac{1}{t}\right)'} = \lim_{t\to 0^+} \frac{\frac{1}{t}}{\left(-\frac{1}{t^2}\right)} = \lim_{t\to 0^+} -t = 0$$

EXAMPLE 10. *Evaluate* $I = \int_{-2}^{3} \frac{1}{x^3}\,dx$



$f(x) = \frac{1}{x^3}$ is not continuous at $x = 0$.

$$I = \int_{-2}^{3} \frac{dx}{x^3} = \int_{-2}^{0} \frac{dx}{x^3} + \int_{0}^{3} \frac{dx}{x^3}$$

$$= \lim_{t \to 0^-} \int_{-2}^{t} \frac{dx}{x^3} + \lim_{t \to 0^+} \int_{t}^{3} \frac{dx}{x^3}$$

$$\int \frac{dx}{x^3} = \int x^{-3}\,dx = \frac{x^{-3+1}}{-3+1} = \frac{x^{-2}}{-2} = -\frac{1}{2x^2}$$

$$= -\frac{1}{2} \lim_{t \to 0^-} \frac{1}{x^2}\Big|_{-2}^{t} - \frac{1}{2} \lim_{t \to 0^+} \frac{1}{x^2}\Big|_{t}^{3}$$

$$= -\frac{1}{2}\Big(\underbrace{\lim_{t \to 0^-} \frac{1}{t^2}}_{\text{DNE}} - \frac{1}{4}\Big) - \frac{1}{2}\Big(\frac{1}{9} - \lim_{t \to 0^+} \frac{1}{t^2}\Big)$$ divergent.

Now we consider an integral involving both of these cases.

EXAMPLE 11. *Evaluate* $I = \int_0^{\infty} \frac{1}{x^2}\,dx$

(TYPE I and II)
$f(x) = \frac{1}{x^2}$ is not cont. at $x=0$.

$$I = \int_0^{\infty} \frac{dx}{x^2} = \underbrace{\int_0^{1} \frac{dx}{x^2}}_{\text{TYPE 2}} + \underbrace{\int_1^{\infty} \frac{dx}{x^2}}_{\text{TYPE 1}}$$

convergent ($p = 2 > 1$)

TYPE 2, because $f(x) = \frac{1}{x^2}$ is not cont. at $x=0$

TYPE 1, because $f(x) = \frac{1}{x^2}$ is cont. on $[1, \infty)$

divergent, because

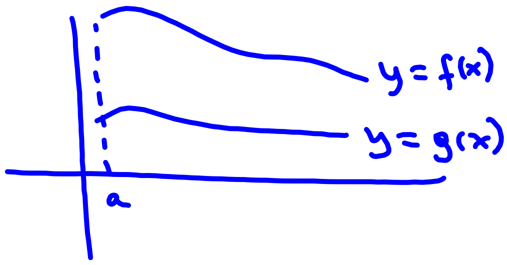
$$\int_0^1 \frac{dx}{x^2} = \lim_{t \to 0^+} \int_t^1 \frac{dx}{x^2} = -\lim_{t\to 0^+} \frac{1}{x}\Big|_t^1$$

$$= -\lim_{t \to 0^+}\left(1 - \frac{1}{t}\right) \text{ DNE}$$

Conclusion: I is divergent.

Comparison Theorem: Suppose $f(x)$ and $g(x)$ are continuous functions s.t $f(x) \geq g(x) \geq 0$ for $x \geq a$. Then

1. if $\displaystyle\int_a^\infty f(x)\,\mathrm{d}x$ is convergent then $\displaystyle\int_a^\infty g(x)\,\mathrm{d}x$ is convergent;
2. if $\displaystyle\int_a^\infty g(x)\,\mathrm{d}x$ is divergent then $\displaystyle\int_a^\infty f(x)\,\mathrm{d}x$ is divergent.



EXAMPLE 12. *Determine whether the following integrals are convergent or divergent.*

(a) $I = \int_1^{\infty} \frac{\sin^2 x}{x^2}\,\mathrm{d}x$

$f(x) = \frac{\sin^2 x}{x^2}$ is continuous on $[1, \infty)$ (TYPE I)

$|\sin x| \leq 1 \Rightarrow \sin^2 x \leq 1$

$$\frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$$

But $\int_1^{\infty} \frac{1}{x^2}\,dx$ is convergent $(p = 2 > 1)$

By Comparison theorems, I is also convergent.

(b) $I = \int_1^{\infty} \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\, \mathrm{d}x$

$f(x) = \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}$ is continuous on $[1,\infty)$ (TYPE I)

$\sqrt{\frac{1+\sqrt{x}}{x}} = \sqrt{\frac{1}{x}+1}$

$\sqrt{\frac{1}{x}} = \frac{1}{x^{\frac{1}{2}}}$

$p = \frac{1}{2}$

$1+\sqrt{x} \geq 1$

$\frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} \geq \frac{1}{\sqrt{x}}$

But $\int_1^{\infty} \frac{dx}{\sqrt{x}}$ is divergent $(p = \frac{1}{2} \leq 1)$

So, by Comparison Theorems I is divergent.

$f(x) = \frac{1}{x+e^{2x}}$ is continuous on $[1,\infty)$

(c) $I = \int_1^\infty \frac{1}{x+e^{2x}}\,dx$

Attempt 1. $x + e^{2x} \geq x$

Since $x > 0$ and $e^{2x} > 0$ on $[1,\infty)$

$$\frac{1}{x+e^{2x}} \leq \frac{1}{x}$$

STOP here, because $\int_1^\infty \frac{dx}{x}$ is divergent and we cannot apply Comparison theorems.

Attempt 2. $x + e^{2x} \geq e^{2x}$

$$\frac{1}{x+e^{2x}} \leq \frac{1}{e^{2x}}$$

(because x and e^{2x} are both positive on $[1,\infty)$)

Let us examine $\int_1^\infty \frac{1}{e^{2x}}\,dx$. We have

$$\int_1^\infty \frac{dx}{e^{2x}} = \int_1^\infty e^{-2x}\,dx = \lim_{t\to\infty} \int_1^t e^{-2x}\,dx = \lim_{t\to\infty} \left.\frac{e^{-2x}}{-2}\right|_1^t$$

$$= -\frac{1}{2}\lim_{t\to\infty}\left(e^{-2t} - e^{-2}\right) = -\frac{1}{2}\left(0 - e^{-2}\right)$$

convergent.

Conclusion: $\frac{1}{x+e^{2x}} \leq \frac{1}{e^{2x}}$ for all $x \in [1,\infty)$

and $\int_1^\infty \frac{dx}{e^{2x}}$ is convergent.

So, by Comparison Theorems, I is also convergent.