9.2 First-Order Linear Differential Equations

A first order ODE is called **linear** if it is expressible in the form

$$y' + p(t)y = g(t) \tag{1}$$

$y = y(t)$

where $p(t)$ and $g(t)$ are given functions.
The method to solve (1) for arbitrary $p(t)$ and $q(t)$ is called

The Method of Integrating Factors

$(fg)' = f'g + fg'$

$I(t)$ a magic function

Multiply both sides of (1) by $I(t)$:

$$y'I + \underbrace{pI}_{= I'}y = gI$$

$$pI = I'$$

$$\frac{dI(t)}{dt} = p(t)\,I(t)$$

$$\int \frac{dI(t)}{I(t)} = \int p(t)\,dt$$

$$\ln|I(t)| = \int p(t)\,dt$$

$$I(t) = e^{\int p(t)dt}$$

$$y'I + I'y = gI$$

$$(yI)' = gI$$

$$\frac{d(y(t)\,I(t))}{dt} = g(t)\,I(t)$$

$$y(t)\,I(t) = \int g(t)\,I(t)\,dt + C$$

$$y(t) = \frac{\int g(t)\,I(t)\,dt + C}{I(t)}$$

The Method of Integrating Factors

Step 1 Put ODE in the form (1).

Step 2 Find the integrating factor

$$I(t) = \mu(t) = e^{\int p(t)\mathrm{d}t}$$

Note: Any μ will suffice here, thus take the constant of integration $C = 0$.

Step 3 Multiply both sides of (1) by μ and use the Product Rule for the left side to express the result as

$$(\mu(t)y(t))' = \mu g(x) \tag{2}$$

Step 4 Integrate both sides of (2). Note: Be sure to include the constant of integration in this step!

Step 5 Solve for the solution $y(t)$.

EXAMPLE 1. *Solve the following DE:* $y' - 3xy = -xe^{x^2}$.

$$y' + p(t)\,y = g(t)$$
$$y' + p(x)\,y = g(x)$$

$$p(x) = -3x$$
$$g(x) = -xe^{x^2}$$

Find an integrating factor:
$$I(x) = e^{\int p(x)\,dx} = e^{-\int 3x\,dx} = e^{-\frac{3x^2}{2}}$$

$$\left(I(x)\,y(x)\right)' = I(x)\left(-xe^{x^2}\right)$$

$$\left(e^{-\frac{3x^2}{2}}y(x)\right)' = e^{-\frac{3x^2}{2}}\cdot e^{x^2}(-x)$$

$$\left(e^{-\frac{3x^2}{2}}y(x)\right)' = -xe^{-\frac{x^2}{2}}$$

$$e^{-\frac{3x^2}{2}}y(x) = -\int xe^{-\frac{x^2}{2}}\,dx$$

$$\boxed{\begin{aligned} -\frac{x^2}{2} &= u \\ du &= -x\,dx \\ -\int xe^{-\frac{x^2}{2}}dx &= \int e^u du \\ &= e^u + C \end{aligned}}$$

$$e^{-\frac{3x^2}{2}}y(x) = e^{-\frac{x^2}{2}} + C \quad \left(\times\, e^{\frac{3x^2}{2}}\right)$$

$$\underbrace{e^{-\frac{3x^2}{2}}\cdot e^{\frac{3x^2}{2}}}_{e^0 = 1}\,y(x) = \underbrace{e^{-\frac{x^2}{2}}\cdot e^{\frac{3x^2}{2}}}_{e^{x^2}} + Ce^{\frac{3x^2}{2}}$$

$$\boxed{y(x) = e^{x^2} + Ce^{\frac{3x^2}{2}}}$$

EXAMPLE 2. *Consider $y' - 2y = \cos(3t)$.*

First order linear DE

Apply Method of Integrating Factor

1. *Find the general solution.*

Multiply the DE by $I(t)$

$$y' I \underbrace{- 2I}_{I'} y = I\cos(3t)$$

$$\underbrace{y' I + I' y}_{(Iy)'} = I\cos(3t)$$

$$(Iy)' = I\cos(3t)$$

Find

$$I' = -2I$$

$$\frac{dI}{dt} = -2I(t)$$

$$\int \frac{dI}{I} = -\int 2\,dt$$

$$\ln I = -2t$$

$$I = e^{-2t}$$

Plug in $I(t) = e^{-2t}$

$$\left(e^{-2t} y\right)' = e^{-2t}\cos(3t)$$

$$e^{-2t} y = \int \underbrace{e^{-2t}\cos(3t)\,dt}_{\text{Loop Integration}}$$

$u = e^{-2t}$ $\quad \cos(3t)\,dt = dv$

$du = -2e^{-2t}\,dt$ $\quad v = \frac{\sin(3t)}{3}$

$$\int e^{-2t}\cos(3t)\,dt = \boxed{M = \frac{1}{3}e^{-2t}\sin 3t + \frac{2}{3}\int e^{-2t}\sin 3t\,dt}$$

$u = e^{-2t}$ $\quad \sin 3t\,dt = dv$

$du = -2e^{-2t}\,dt$ $\quad v = -\frac{\cos 3t}{3}$

$$M = \frac{1}{3}e^{-2t}\sin 3t + \frac{2}{3}\left(-\frac{1}{3}e^{-2t}\cos 3t - \frac{2}{3}\underbrace{\int e^{-2t}\cos 3t\,dt}_{M}\right)$$

$$M = \frac{1}{3}e^{-2t}\sin 3t - \frac{2}{9}e^{-2t}\cos 3t - \frac{4}{9}M$$

$$\frac{13}{9}M = e^{-2t}\left(\frac{1}{3}\sin(3t) - \frac{2}{9}\cos(3t)\right).$$

$$M = \frac{9}{13}e^{-2t}\left(\frac{1}{3}\sin(3t) - \frac{2}{9}\cos(3t)\right) + C$$

$$e^{-2t} y = M + C \Rightarrow y = e^{2t}(M + C) = e^{2t}\cdot\frac{9}{13}e^{-2t}\left(\frac{1}{3}\sin(3t) - \frac{2}{9}\cos(3t)\right) + Ce^{2t}$$

$$y = \frac{9}{13}\left(\frac{1}{3}\sin(3t) - \frac{2}{9}\cos(3t)\right) + Ce^{2t}$$

2. *Find the solution satisfying the initial condition $y(0) = -2/13$.*

$$y(0) = \underbrace{\frac{9}{13}\left(\frac{1}{3}\cdot 0 - \frac{2}{9}\right)}_{} + Ce^{0} = -\frac{2}{13}$$

$$-\frac{2}{13} + C = -\frac{2}{13} \Rightarrow C = 0.$$

$$y = \frac{9}{13}\left(\frac{1}{3}\sin(3t) - \frac{2}{9}\cos(3t)\right)$$