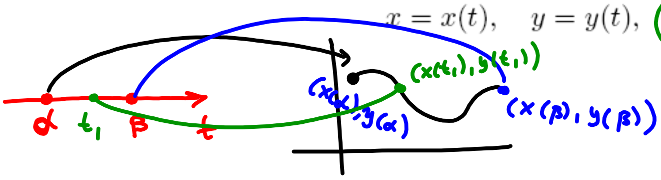
9.3: Arc Length

$\int_a^b dx = b-a$ = length of $[a,b]$

Description of a curve by parametric equations:

parameter domain

$$x = x(t), \quad y = y(t), \quad \alpha \le t \le \beta$$



EXAMPLE 1. *Parameterize the following curves:*

(a) $y = \sqrt{1-x^2}$

Way 1: $x = t$, $y = \sqrt{1-t^2}$, $-1 \le t \le 1$

Note: $x = x$, $y = \sqrt{1-x^2}$, $-1 \le x \le 1$

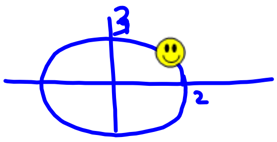
Conclusion: To parameterize $y = f(x)$ we can use x as a parameter.
$x = x$, $y = f(x)$

Way 2: $x = \cos t$, $y = \sqrt{1-\cos^2 t} = \sqrt{\sin^2 t} = |\sin t|$

$\boxed{x = \cos t, \quad y = \sin t, \quad 0 \le t \le \pi}$

(b) $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$



Using $\cos^2 t + \sin^2 t = 1$,
let $\frac{x}{2} = \cos t$ and $\frac{y}{3} = \sin t$.

$\boxed{x = 2\cos t, \quad y = 3\sin t, \quad 0 \le t \le 2\pi}$

Another way: $\boxed{x = 2\cos 5t, \quad y = 3\sin 5t, \quad 0 \le t \le \frac{2\pi}{5}}$

(c) $y = \sin x$, $0 \le x \le \pi$

$x = x$, $y = \sin x$, $0 \le x \le \pi$

(d) $y^4 = x$

$x = t^4$, $y = t$

OR

$x = y^4$, $y = y$

In general, if the curve is given by $x = g(y)$, we can use y as a parameter.

(e) $y = f(x)$, $a \le x \le b$

$x = x$, $y = f(x)$, $a \le x \le b$

(f) $x = g(y)$, $c \le y \le d$

$x = g(y)$, $y = y$, $c \le y \le d$.

DEFINITION 2. *A curve $C : x = x(t),\quad y = y(t),\quad \alpha \le t \le \beta$, is called* **smooth** *if the derivatives $x'(t)$ and $y'(t)$ are continuous and not simultaneously zero on $[\alpha, \beta]$.*

The length of a smooth curve

$$C : x = x(t), \quad y = y(t), \quad \alpha \le t \le \beta,$$

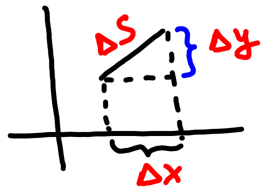
assuming that C is traversed exactly once as t increases from α to β is

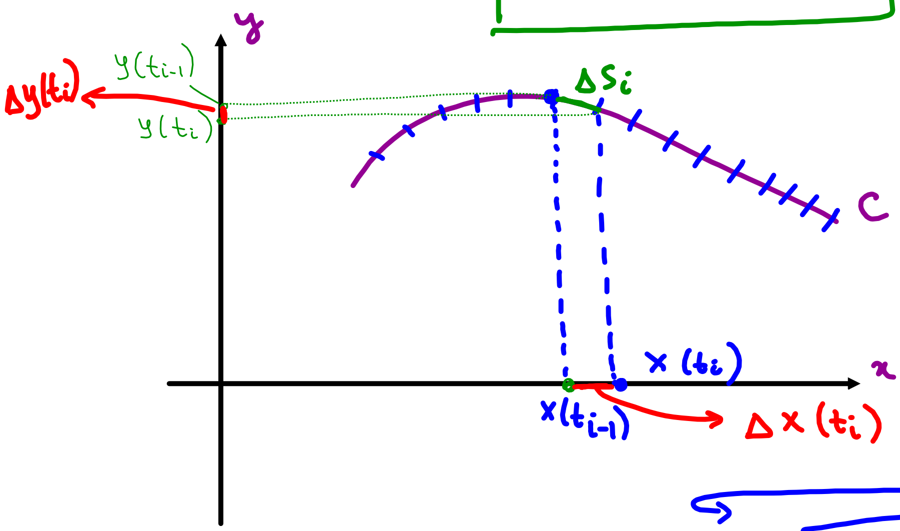
$$\int_\alpha^\beta \underbrace{\sqrt{[x'(t)]^2 + [y'(t)]^2}\,\mathrm{d}t}_{} =: \underbrace{\int_C \mathrm{d}s}_{},$$

line integral

where $\mathrm{d}s$ is called *differential of the arc length function* and

$$\boxed{\mathrm{d}s = \sqrt{[x'(t)]^2 + [y'(t)]^2}\mathrm{d}t}$$





$\Delta s = \sqrt{\Delta x^2 + \Delta y^2}$

$\Delta s = ds,$

$ds = \sqrt{(dx)^2 + (dy)^2}$

$\Delta x = dx = x'(t)\,dt$

$\Delta x = dy = y'(t)\,dt$

$ds = \sqrt{(x'(t)\,dt)^2 + (y'(t)\,dt)^2}$

$= \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$

EXAMPLE 3. *Find the length of the curve*

$$C: \ x = 3t - t^3, \quad y = 3t^2, \quad 0 \le t \le 2.$$

$$x'(t) = 3 - 3t^2$$
$$y'(t) = 6t$$

$$ds = \sqrt{(x')^2 + (y')^2}\,dt = \sqrt{(3-3t^2)^2 + (6t)^2}\,dt$$
$$= \sqrt{9 - \underline{18t^2} + 9t^4 + \underline{36t^2}}\,dt \qquad (a+b)^2 = a^2 + 2ab + b^2$$
$$= \sqrt{9t^4 + 18t^2 + 9}\,dt = \sqrt{9(t^4 + 2t^2 + 1)}$$
$$= 3\sqrt{(t^2+1)^2}\,dt = 3\,|t^2+1|\,dt$$
$$= 3(t^2+1)\,dt$$

$$|C| = \int_C ds = \int_0^2 3(t^2+1)\,dt = 3\left(\frac{t^3}{3} + t\right)\Big|_0^2 = 3\left(\frac{8}{3} + 2\right)$$
$$= (8+6) = 14$$

Particular cases for the differential of the arc length function:

$$\mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

- C is given by $y = f(x)$, $a \leq x \leq b$

$t = x$

$x = x$, $y = f(x)$, $a \leq x \leq b$

$$ds = \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + \left(\frac{df}{dx}\right)^2}\,dx$$

$$\boxed{ds = \sqrt{1 + [f'(x)]^2}\,dx}$$

- C is given by $x = g(y)$, $c \leq y \leq d$

$x = g(y)$, $y = y$, $c \leq y \leq d$ $\quad t = y$

$$ds = \sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2}\,dy = \sqrt{\left(\frac{dg}{dy}\right)^2 + 1}\,dy$$

$$\boxed{ds = \sqrt{1 + (g'(y))^2}\,dy}$$

EXAMPLE 4. *Find the arc length of the graph of* $y = \overbrace{\ln \cos x}^{f(x)}$ *from* $x = 0$ *to* $x = \pi/4$.

$$\left.\begin{aligned} ds &= \sqrt{1 + [y'(x)]^2}\, dx \\ y'(x) &= \frac{d}{dx}\left(\ln(\cos x)\right) = \frac{1}{\cos x}\cdot(\cos x)' = -\frac{\sin x}{\cos x} = -\tan x \end{aligned}\right\} \Rightarrow$$

$$ds = \sqrt{1 + (-\tan x)^2}\, dx = \sqrt{1+\tan^2 x}\, dx$$

$$= \sqrt{\sec^2 x}\, dx = |\sec x|\, dx = \sec x\, dx,$$

because $0 \le x \le \pi/4$

$$|C| = \int_C ds = \int_0^{\pi/4} \sec x\, dx =$$

let $u = \sin x$

$x = 0 \Rightarrow u = \sin 0 = 0$

$x = \pi/4 \Rightarrow u = \sin \pi/4 = \frac{\sqrt{2}}{2}$

$$= \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right|\Bigg|_0^{\sqrt{2}/2} = \frac{1}{2}\left(\ln \frac{1+\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}} - \underbrace{\ln\frac{1+0}{1-0}}_{\overset{\shortparallel}{0}}\right)$$

$$= \frac{1}{2}\ln\frac{2+\sqrt{2}}{2-\sqrt{2}} = \frac{1}{2}\ln\frac{(2+\sqrt{2})(2+\sqrt{2})}{(2-\sqrt{2})(2+\sqrt{2})}$$

$\leftarrow (a+b)^2 = a^2+2ab+b^2$

$\leftarrow (a-b)(a+b) = a^2-b^2$

$$= \frac{1}{2}\ln\frac{4+4\sqrt{2}+2}{4-2} = \frac{1}{2}\ln\frac{6+4\sqrt{2}}{2} = \frac{1}{2}\ln(3+2\sqrt{2})$$

$$= \ln\sqrt{3+2\sqrt{2}}\,.$$

$$\int \sec x \, dx = \int \frac{dx}{\cos x} = \int \frac{\cos x \, dx}{\cos x \, \cos x}$$

$$= \int \frac{\cos x \, dx}{\cos^2 x} = \int \frac{\cos x \, dx}{1 - \sin^2 x} =$$

$u = \sin x$

$du = \cos x \, dx$

$$= \int \frac{du}{1-u^2} = \int \frac{du}{(1-u)(1+u)} \overset{PFD}{=} \frac{1}{2}\left(\int \frac{du}{1-u} + \int \frac{du}{1+u}\right)$$

$$= \frac{1}{2}\left(-\ln|1-u| + \ln|1+u|\right) = \frac{1}{2} \ln \left|\frac{1+u}{1-u}\right|$$

$$= \ln \sqrt{\left|\frac{1+u}{1-u}\right|} = \ln \sqrt{\left|\frac{1+\sin x}{1-\sin x}\right|} + C$$

$$\int \sec(x) \, dx = \ln \sqrt{\left|\frac{1+\sin x}{1-\sin x}\right|} + C$$

EXAMPLE 5. *Find the arc length of the curve $(y-1)^3 = x^2$ from $\overbrace{x = 0 \text{ to } x = 8}^{x \geq 0}$.*

$$(y-1)^3 = x^2 \Rightarrow x = \pm\sqrt{(y-1)^3} = (y-1)^{3/2} = g(y)$$

$$g'(y) = \frac{d}{dy}\left((y-1)^{3/2}\right) = \frac{3}{2}(y-1)^{3/2-1} = \frac{3}{2}(y-1)^{\frac{1}{2}}$$

$$1 + [g'(y)]^2 = 1 + \frac{9}{4}(y-1) = 1 + \frac{9}{4}y - \frac{9}{4} = \frac{9}{4}y - \frac{5}{4}$$

$$= \frac{1}{4}(9y-5)$$

$$ds = \sqrt{1 + (g'(y))^2}\, dy = \sqrt{\frac{1}{4}(9y-5)}\, dy = \frac{1}{2}\sqrt{9y-5}\, dy$$

$$|C| = \int_C ds = \frac{1}{2}\int_1^5 \sqrt{9y-5}\, dy = \frac{1}{\cancel{2}} \cdot \frac{1}{9} \cdot \frac{\cancel{2}}{3}(9y-5)^{3/2}\Big|_1^5 =$$

Find bounds of integration (parameter domain):

$$(y-1)^3 = x^2 \Rightarrow y - 1 = x^{2/3}$$

$$y = 1 + x^{2/3}$$

$$x = 0 \Rightarrow y = 1 + 0^{2/3} = 1$$

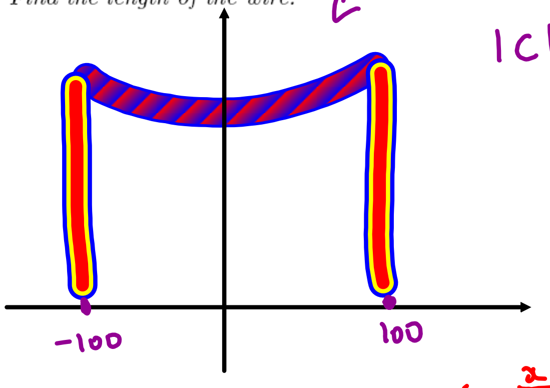
$$x = 8 \Rightarrow y = 1 + 8^{2/3} = 1 + 2^2 = 5$$

$$\frac{1}{27}\left(40^{3/2} - 4^{3/2}\right)$$

EXAMPLE 6. *A telephone wire hanging between two poles (at $x = -100$ and $x = 100$) takes the shape of a catenary with equation*

$$y = 75(e^{x/150} + e^{-x/150}).\qquad -100 \le x \le 100$$

Find the length of the wire.



$$|C| = \int_C ds = \int_{-100}^{100} \sqrt{1 + [y'(x)]^2}\, dx$$

$$y'(x) = \cancel{75}^{1}\left(\frac{1}{\cancel{150}_{2}} e^{x/150} - \frac{1}{\cancel{150}_{2}} e^{-\frac{x}{150}}\right)$$

$$= \frac{1}{2}\left(e^{\frac{x}{150}} - e^{-\frac{x}{150}}\right)$$

$$1 + [y'(x)]^2 = 1 + \frac{1}{4}\left(e^{\frac{2}{150}} - e^{-\frac{x}{150}}\right)^2$$

$$= 1 + \frac{1}{4}\left(\left(e^{\frac{x}{150}}\right)^2 - 2 \cdot e^{\frac{x}{150}} \cdot e^{-\frac{x}{150}} + \left(e^{-\frac{x}{150}}\right)^2\right)$$

$$= 1 + \frac{1}{4}\left(e^{\frac{x}{75}} - 2 + e^{-\frac{x}{75}}\right)$$

$$= \frac{4 + e^{\frac{x}{75}} - 2 + e^{-\frac{x}{75}}}{4} = \frac{e^{\frac{x}{75}} + 2 + e^{-\frac{x}{75}}}{4}$$

$$= \frac{1}{4}\left(\left(e^{\frac{x}{150}}\right)^2 + 2e^{\frac{x}{150}} \cdot e^{-\frac{x}{150}} + \left(e^{-\frac{x}{150}}\right)^2\right)$$

$$= \frac{1}{4}\left(e^{\frac{x}{150}} + e^{-\frac{x}{150}}\right)^2$$

$$|C| = \int_{-100}^{100} \sqrt{1 + [y'(x)]^2}\, dx = \int_{-100}^{100} \sqrt{\frac{1}{4}\left(e^{\frac{x}{150}} + e^{-\frac{x}{150}}\right)^2}\, dx$$

$$= \frac{1}{2} \int_{-100}^{100} \left(e^{\frac{x}{150}} + e^{-\frac{x}{150}}\right) dx$$

$$= \frac{1}{2} \cdot 150 \left(e^{\frac{x}{150}} - e^{-\frac{x}{150}}\right)\Big|_{-100}^{100}$$

$$= 75\left(e^{\frac{100}{150}} - e^{-\frac{100}{150}} - \left(e^{-\frac{100}{150}} - e^{\frac{100}{150}}\right)\right)$$

$$= 75 \cdot 2\left(e^{\frac{2}{3}} - e^{-\frac{2}{3}}\right) = 150\left(e^{\frac{2}{3}} - e^{-\frac{2}{3}}\right).$$