6.5: The substitution rule

The Substitution Rule for indefinite integrals: If $u = g(x)$ is a differentiable function, then

$$\int f(g(x))g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u$$

Note that $\mathrm{d}u = g'(x)\mathrm{d}x$ is differential.

The correct substitution depends on the integral:

$$\int \overset{u^n}{(g(x))^n}\,\overbrace{g'(x)\,\mathrm{d}x}^{du} \qquad \int e^{g(x)}g'(x)\,\mathrm{d}x \qquad \int \frac{g'(x)}{g(x)}\,\mathrm{d}x = \int \frac{du}{u}$$

$$\int \cos(g(x))g'(x)\,\mathrm{d}x \qquad \int \sin(g(x))g'(x)\,\mathrm{d}x \qquad \int \sec^2(g(x))g'(x)\,\mathrm{d}x$$

$$\int \sec(g(x))\tan(g(x))g'(x)\,\mathrm{d}x \qquad \int \csc^2(g(x))g'(x)\,\mathrm{d}x \qquad \int \csc(g(x))\cot(g(x))g'(x)\,\mathrm{d}x$$

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C = \frac{(g(x))^{n+1}}{n+1} + C$$

EXAMPLE 1. *Evaluate each of the following integrals:*

1. $$\int x(x^2+2012)^{2012}\,dx = \int \underbrace{(x^2+2012)^{2012}}_{u^{2012}} \cdot \underbrace{x\,dx}_{\frac{du}{2}} = \int u^{2012} \cdot \frac{du}{2} =$$

$u = x^2 + 2012$

$du = 2x\,dx$

$x\,dx = \frac{du}{2}$

$$= \frac{1}{2}\int u^{2012}\,du =$$

$$= \frac{1}{2}\,\frac{u^{2012+1}}{2012+1} + C =$$

$$= \frac{1}{4026}(x^2+2012)^{2013} + C$$

2. $\int \underbrace{18x^2 \sqrt[4]{6x^3+5}\,dx} = \int \sqrt[4]{u}\,du =$

$\boxed{\begin{array}{l} u = 6x^3+5 \\ du = 18x^2\,dx \end{array}}$

$$= \int u^{\frac{1}{4}}\,du = \frac{u^{\frac{1}{4}+1}}{\frac{1}{4}+1} + C =$$

$$= \frac{4}{5}u^{5/4} + C = \frac{4}{5}(6x^3+5)^{5/4} + C$$

2^* $I = \int 18x^8 \sqrt[4]{6x^3+5}\,dx =$

$$= \int \underbrace{x^6} \; \underbrace{\sqrt[4]{6x^3+5}}_{u^{1/4}} \cdot \underbrace{18x^2\,dx}_{du}$$

Use the same u- sub as in 2,

$$x^6 = (x^3)^2 = \left(\frac{u-5}{6}\right)^2 = \frac{(u-5)^2}{36}$$

$u = 6x^3+5 \Rightarrow x^3 = \frac{u-5}{6}$

$$I = \int \frac{(u-5)^2}{36} \cdot u^{\frac{1}{4}}\,du =$$

$$= \frac{1}{36}\int (u^2 - 10u + 25)\,u^{\frac{1}{4}}\,du =$$

$$= \frac{1}{36}\int u^{2+\frac{1}{4}} - 10u^{1+\frac{1}{4}} + 25u^{\frac{1}{4}}\,du$$

$$= \frac{1}{36}\left(\frac{u^{3+\frac{1}{4}}}{3+\frac{1}{4}} - 10 \cdot \frac{u^{2+\frac{1}{4}}}{2+\frac{1}{4}} + 25\frac{u^{\frac{1}{4}+1}}{\frac{1}{4}+1}\right) + C$$

Simplify and replace u by $6x^3+5$

3. $\int \cos(3x)\sin^{10}(3x)\,dx = \int u^{10}\frac{du}{3} =$

$u = \sin 3x$

$du = 3\cos(3x)\,dx$

$\cos(3x)\,dx = \frac{du}{3}$

$= \frac{1}{3}\cdot\frac{u^{11}}{11} + C$

$= \frac{1}{33}u^{11} + C =$

$= \frac{1}{33}\sin^{11}(3x) + C$

4. $\int (8x-1)e^{4x^2-x}\,dx = \int e^u\,du = e^u + C$

$u = 4x^2 - x$

$du = (8x-1)\,dx$

$= e^{4x^2-x} + C$

5. $\int \sec^2(5y)(5-\tan(5y))^5\,dy = \int \underbrace{(5-\tan(5y))^5}_{u^5}\underbrace{\sec^2(5y)\,dy}_{-\frac{du}{5}}$

$u = 5 - \tan(5y)$

$du = -5\sec^2(5y)\,dy$

$\sec^2(5y)\,dy = -\frac{du}{5}$

$= -\frac{1}{5}\int u^5\,du = -\frac{1}{5}\frac{u^6}{6} + C =$

$= -\frac{1}{30}(5-\tan(5y))^6 + C$

6. $\int \frac{x}{7x^2+12}\,dx = \int \frac{du/14}{u} =$

$u = 7x^2+12 \geq 0$

$du = 14x\,dx$

$x\,dx = \frac{du}{14}$

$= \frac{1}{14}\int \frac{du}{u} = \frac{1}{14}\ln|u| + C =$

$= \frac{1}{14}\ln(7x^2+12) + C$

7. $\int \tan x \, dx = \int \frac{\sin x \, dx}{\cos x} = -\int \frac{du}{u} =$

$u = \cos x$

$du = -\sin x \, dx$

$= -\ln|u| + C =$

$= -\ln|\cos x| + C$

The Substitution Rule for definite integrals: If $u = g(x)$ is a differentiable function, then

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u$$

$u = 5x$

$$\int_0^{\pi/10} \cos(5x)\,dx = \int_{\dots}^{\dots} \cos u \,\frac{du}{5} = \frac{1}{5}\sin u\Big|_{\dots}^{\dots}$$

$u = 5x$

$du = 5\,dx$

$$= \frac{1}{5}\sin 5x\Big|_0^{\pi/10} = \frac{1}{5}\left(\sin\frac{\pi}{2} - \sin 0\right) = \boxed{\frac{1}{5}}$$

New step:

$x = 0 \Rightarrow u = 5\cdot 0 = 0$

$x = \frac{\pi}{10} \Rightarrow u = 5\cdot\frac{\pi}{10} = \frac{\pi}{2}$

$$\int_0^{\pi/2} \cos u\,\frac{du}{5} = \frac{1}{5}\sin u\Big|_0^{\pi/2} = \frac{1}{5}(1-0) = \boxed{\frac{1}{5}}$$

EXAMPLE 2. *Evaluate each of the following integrals:*

1. $\displaystyle\int_{e^2}^{e^6} \frac{(\ln x)^4}{x}\,\mathrm{d}x$ $= du$

New step

$u = \ln x$

$du = \frac{dx}{x}$

$x = e^2 \Rightarrow u = \ln e^2 = 2$

$x = e^6 \Rightarrow u = \ln e^6 = 6$

new bounds

$$\int_2^6 u^4\,du = \frac{u^5}{5}\Big|_2^6 = \frac{1}{5}\left(6^5 - 2^5\right) = \frac{7744}{5}$$

2. $\int_0^1 \frac{1}{(1+3x)^3} - \frac{3}{1+3x}\,\mathrm{d}x$

$u = 1+3x$ | $x=0 \Rightarrow u=1$

$du = 3dx$ | $x=1 \Rightarrow u = 1+3 = 4$

$$\int_1^4 \left(\frac{1}{u^3} - \frac{3}{u}\right)\frac{du}{3} = \frac{1}{3}\int_1^4 u^{-3}du - \int_1^4 \frac{du}{u} =$$

$$= \frac{1}{3}\cdot\frac{u^{-3+1}}{-3+1}\Big|_1^4 - \ln|u|\Big|_1^4 =$$

$= \frac{1}{u^2}$

$$-\frac{1}{6}\left(\frac{1}{16}-1\right) - (\ln 4 - \ln 1) =$$

$$= -\frac{1}{\cancel{6}_2}\cdot\left(-\frac{\cancel{15}^{5}}{16}\right) - \ln 4 = \frac{5}{32} - \ln 4$$

3. $\int_0^{0.5} (\sin(\pi y) - e^y)\,\mathrm{d}y = \int_0^{0.5} \sin \pi y\, dy - \int_0^{0.5} e^y dy =$

$u = \pi y$ | $y=0 \Rightarrow u=0$

$du = \pi dy$ | $y = 0.5 \Rightarrow u = \frac{\pi}{2}$

$$= \int_0^{\pi/2} \sin u \,\frac{du}{\pi} - e^y\Big|_0^{0.5} =$$

$$= -\frac{\cos u}{\pi}\Big|_0^{\pi/2} - (e^{0.5} - e^0) =$$

$$= -\frac{1}{\pi}(0-1) - \sqrt{e} + 1 = \frac{1}{\pi} - \sqrt{e} + 1$$