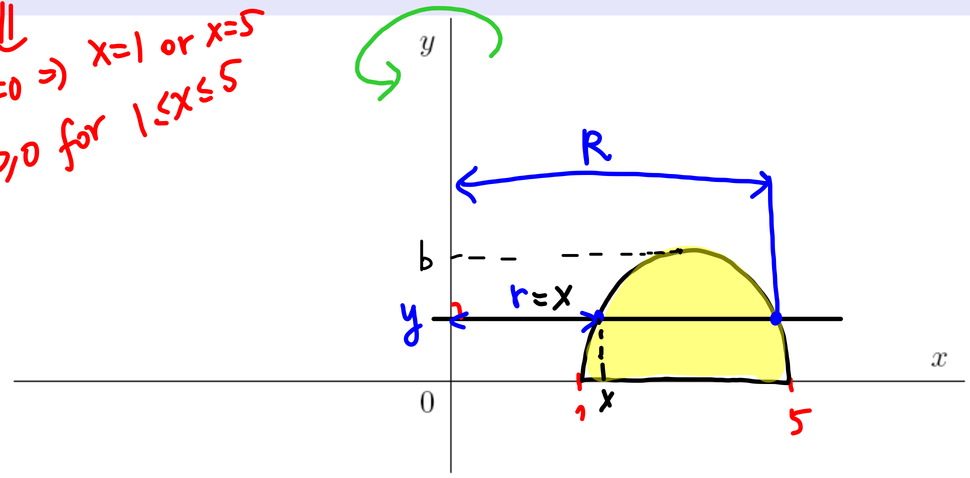
7.3: Volumes by Cylindrical Shells

EXAMPLE 1. *Determine the volume of the solid obtained by rotating the region bounded by* $y = 4(x-1)(x-5)^2$ *and the* x*-axis about the* y*-axis.*

$\Downarrow$

$y=0 \Rightarrow x=1$ or $x=5$

$y \geq 0$ for $1 \leq x \leq 5$



$$V = \pi\int_0^b [R(y)]^2 - [r(y)]^2\,dy$$

Let us list the troubles in application washer method here:

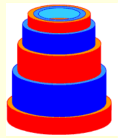
1. It is difficult/impossible to get explicit formula for inner/outer radius in terms of y.

2. To find *b* we need to find absolute maximum of the function defining the curve on the interval [1,5].

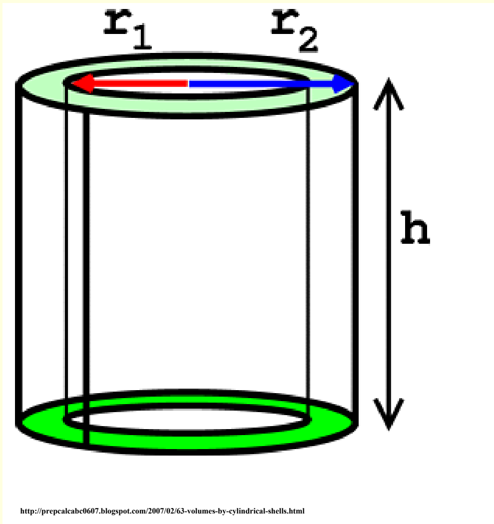
How to avoid these troubles?

Unlike the washer method in shell method the cross sections are always **parallel** to the axis of revolution.

`Think of cutting your solid by a cylindrical cutter centered on the axis of revolution and pushed down into the solid.`



A `cylindrical shell` is a solid bounded by two concentric circular cylinders with the same height. The volume of the cylindrical shell:



$V = 2\pi r h \,\Delta r,$

where $r = \dfrac{r_1 + r_2}{2}$ average radius | $\Delta r = r_2 - r_1$ thickness

$$V = \underbrace{2\pi(\text{radius})}_{\text{circumference}} \cdot (\text{height})(\text{thickness})$$

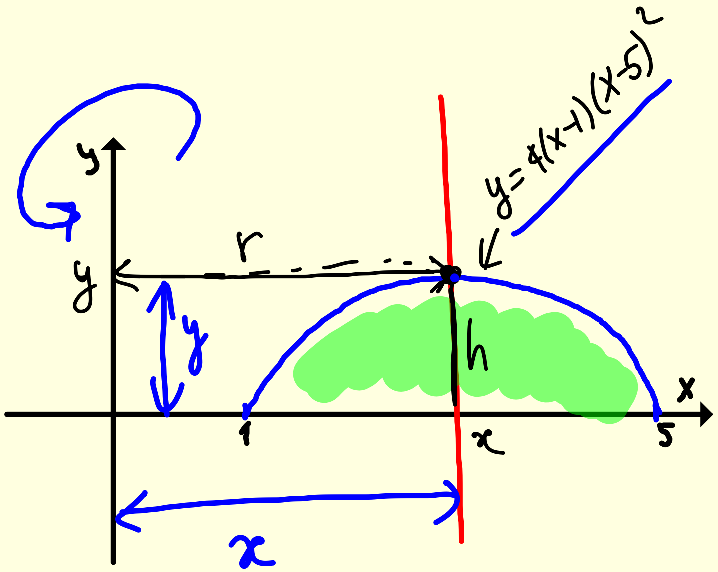
$$V = \int_a^b A(x)\,dx = 2\pi \int_a^b r(x)\,h(x)\,dx$$

$$\boxed{V = 2\pi \int_a^b (\text{radius})(\text{height})\,dx}$$

Solution of Example 1:

Determine the volume of the solid obtained by rotating the region bounded by

$y = 4(x-1)(x-5)^2$ *and the x-axis about the y-axis.*



$$V = \int_1^5 A(x)\,dx$$

$$V = 2\pi \int_1^5 r(x)\,h(x)\,dx$$

$r(x) = x$

$h(x) = y = 4(x-1)(x-5)^2$

$$V = 2\pi \int_1^5 x \cdot 4(x-1)(x-5)^2\,dx = \ldots$$

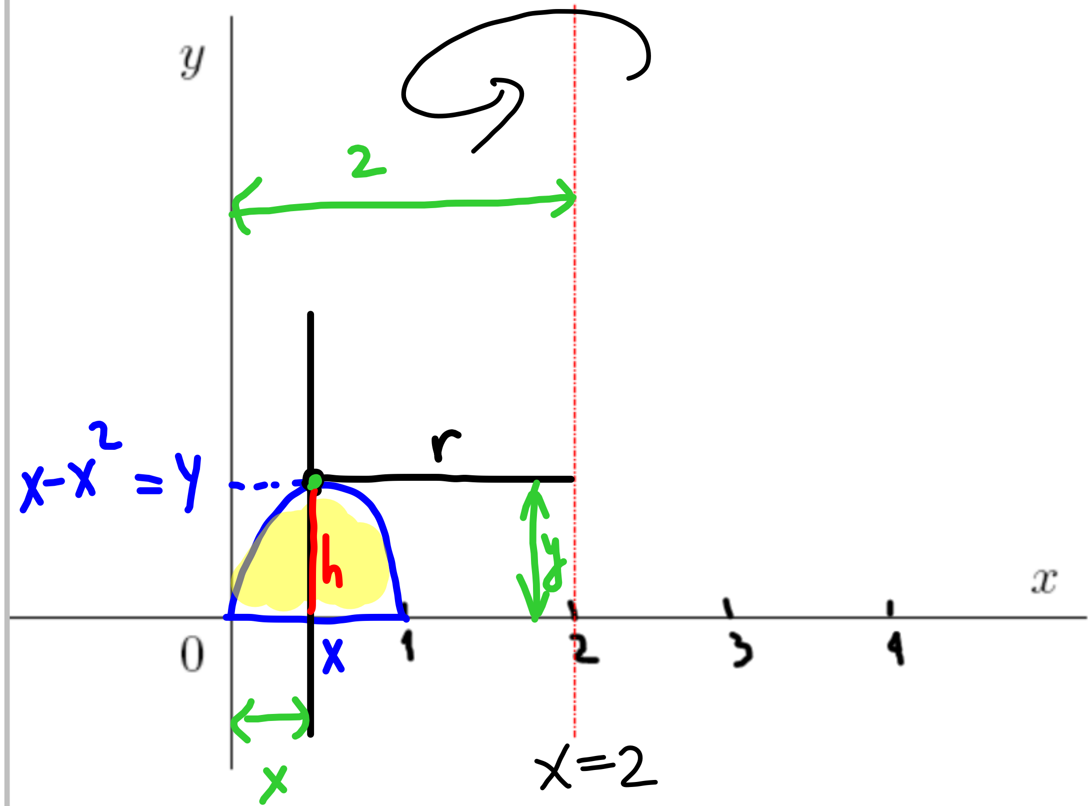
SUMMARY (Method of Cylindrical Shells)

- Area of cross sections: $A(x) = 2\pi(\text{radius})(\text{height})$ =(circumference)(height)
- For rotation about a *vertical* axis we use $V = \displaystyle\int_a^b A(x)\,\mathrm{d}x$.

  For rotation about a *horizontal* axis we use $V = \displaystyle\int_c^d A(y)\,\mathrm{d}y$

  Note: Exactly opposite of washer method.
- For the limits of integration we take the only range of x or y covering one side of the solid (not the complete range).

EXAMPLE 2. *Find the volume of the solid obtained by rotating the region enclosed by the curves $y = x - x^2$ and $y = 0$ about the line $x = 2$.*



$$V = \int_0^1 A(x)\,dx$$

$$V = 2\pi \int_0^1 r(x)h(x)\,dx$$

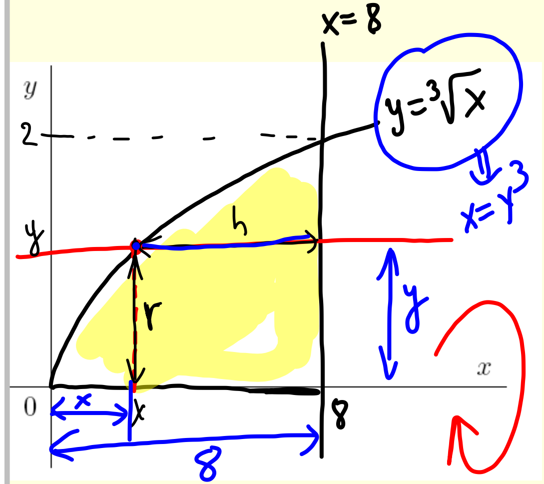
$$r(x) = 2-x$$

$$h(x) = y = x - x^2$$

$$V = 2\pi \int_0^1 (2-x)(x-x^2)\,dx = \ldots = \boxed{\frac{\pi}{2}}$$

EXAMPLE 3. *Set up, but do not evaluate, the volume of the solid obtained by rotating the region enclosed by the curves $y = \sqrt[3]{x}, x = 8, y = 0$ about the x-axis using the method of cylindrical shells and then using the washer method.*

Cylindrical Shells Method Applications:



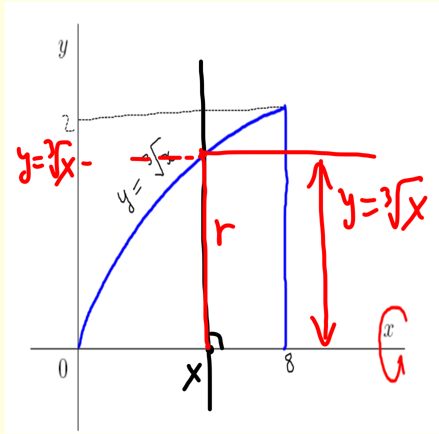
$V = \int_0^2 A(y)\,dy =$

$= 2\pi \int_0^2 r(y)\,h(y)\,dy$

$r(y) = y\,;\ h(y) = 8 - x = 8 - y^3$

$$\boxed{V = 2\pi \int_0^2 y\,(8 - y^3)\,dy}$$

Washer Method Application



$V = \int_0^8 A(x)\,dx$

$V = \pi \int_0^8 [r(x)]^2\,dx$

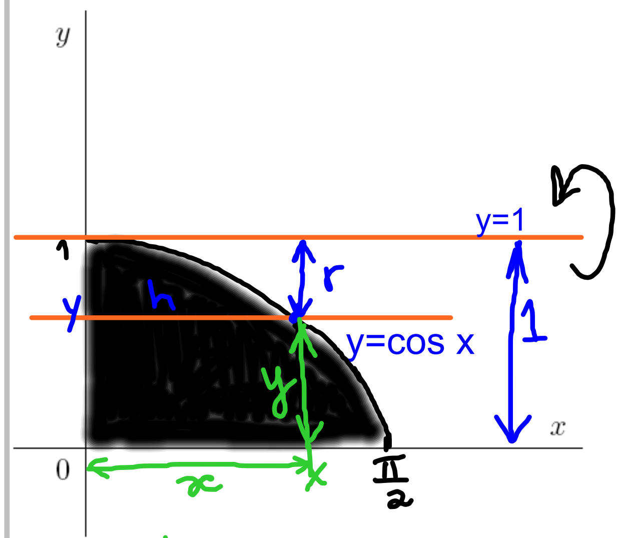
$r(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$

$V = \pi \int_0^8 \left(x^{\frac{1}{3}}\right)^2 dx$

$$\boxed{V = \pi \int_0^8 x^{\frac{2}{3}}\,dx}$$

EXAMPLE 4. *Set up, but do not evaluate, the volume of the solid obtained by rotating the region enclosed by the curves $y = \cos x, x = 0, x = \pi/2$ and $y = 0$ about the line $y = 1$ using the method of cylindrical shells and then using the washer method.*

Cylindrical Shells Method Applications:



$$V = \int_0^1 A(y)\,dy$$

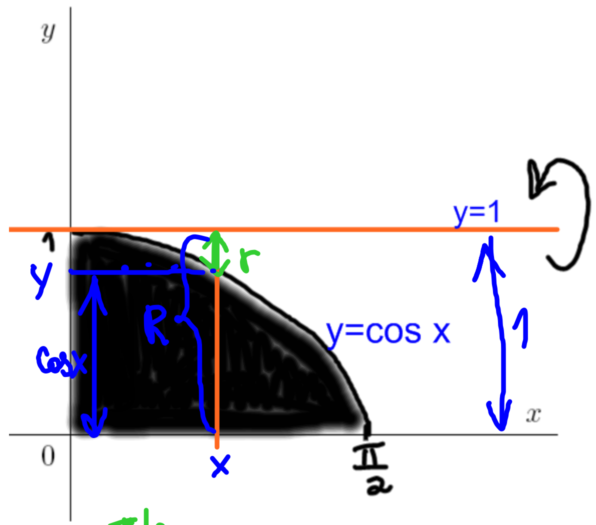
$$V = 2\pi \int_0^1 r(y)\,h(y)\,dy$$

$$r(y) = 1 - y$$

$$h(y) = x = \arccos y$$

$$V = 2\pi \int_0^1 (1-y)\arccos y\,dy$$

Washer Method Application



$$V = \int_0^{\pi/2} A(x)\,dx$$

$$V = \pi \int_0^{\pi/2} [R(x)]^2 - [r(x)]^2\,dx$$

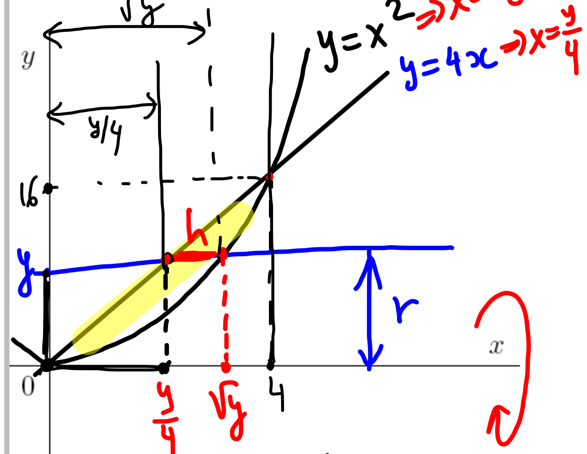
$$R(x) = 1$$

$$r(x) = 1 - y = 1 - \cos x$$

$$V = \pi \int_0^{\pi/2} 1 - (1-\cos x)^2\,dx$$

EXAMPLE 5. *Set up, but do not evaluate, the volume of the solid obtained by rotating the region enclosed by the curves $y = x^2$ and $y = 4x$ about the x-axis using the method of cylindrical shells and then using the washer method.*

Cylindrical Shells Method Applications:



Intersection point

$x^2=4x \Rightarrow x(x-4)=0 \Rightarrow x=0,\ x=4$

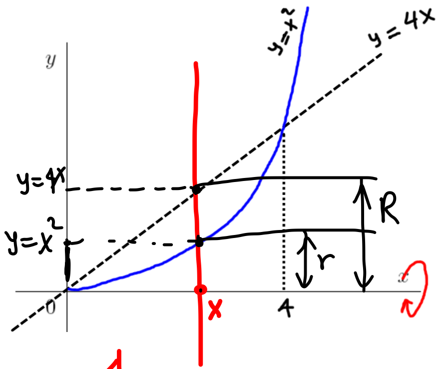
$$V=\int_0^{16} A(y)\,dy$$

$$V=2\pi\int_0^{16} r(y)\cdot h(y)\,dy$$

$r(y)=y$; $h(y)=\sqrt{y}-\frac{y}{4}$

$$\boxed{V=2\pi\int_0^{16} y\left(\sqrt{y}-\frac{y}{4}\right)dy}$$

Washer Method Application



$$V=\int_0^4 A(x)\,dx$$

$$V=\pi\int_0^4 [R(x)]^2-[r(x)]^2\,dx$$

$r(x)=x^2$, $R(x)=4x$

$$V=\pi\int_0^4 (4x)^2-(x^2)^2\,dx$$

$$\boxed{V=\pi\int_0^4 (16x^2-x^4)\,dx}$$