8.2: Trigonometric Integrals

Recall:

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$\cot^2 x + 1 = \csc^2 x$$

$$\tan^2 x = \sec^2 x - 1$$

half-angle identities:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\boxed{\sin x \cos x = \frac{1}{2}\sin 2x}$$

EXAMPLE 1. *Evaluate* $I = \int \cos x \sin^{2011} x \, dx$

$$I = \int \underbrace{(\sin x)^{2011}}_{u^{2011}} \underbrace{\cos x \, dx}_{du}$$

$d(\sin x) = \cos x \, dx$

$u = \sin x \qquad du = \cos x \, dx$

$$I = \int u^{2011} du = \frac{u^{2012}}{2012} + C = \frac{\sin^{2012} x}{2012} + C$$

EXAMPLE 2. *Evaluate* $I = \int \sin^5 x \, dx = \int \sin^4 x \underbrace{\sin x \, dx}_{-d(\cos x)}$

use $\sin^2 x + \cos^2 x = 1$

$$I = \int (\sin^2 x)^2 \sin x \, dx = \int (1 - \cos^2 x)^2 \sin x \, dx$$

$u = \cos x$ $\qquad du = -\sin x \, dx$

$$I = -\int (1-u^2)^2 \, du = -\int 1 + u^4 - 2u^2 \, du$$

$$I = -u - \frac{u^5}{5} + \frac{2}{3} u^3 + C$$

$$\boxed{I = -\cos x - \frac{\cos^5 x}{5} + \frac{2}{3} \cos^3 x + C}$$

EXAMPLE 3. *Evaluate* $I = \int \sin^{2010} x \cos^3 x \, \mathrm{d}x$

$$I = \int (\sin x)^{2010} \underbrace{\cos^2 x}_{1-\sin^2 x} \, \cos x \, dx$$

$$\cos x \, dx = d(\sin x)$$

$u = \sin x \qquad du = \cos x \, dx$

$$I = \int u^{2010} (1-u^2) \, du = \int u^{2010} - u^{2012} \, du$$

$$I = \frac{u^{2011}}{2011} - \frac{u^{2013}}{2013} + C$$

$$I = \frac{\sin^{2011} x}{2011} - \frac{\sin^{2013} x}{2013} + C$$

RULE: How to evaluate $\int \sin^n x \cos^m x \, dx$

1. *If n is odd use substitution $u = \cos x$ (Strip out one sine and convert the rest to cosine.)*
2. *If m is odd use substitution $u = \sin x$ (Strip out one cosine and convert the rest to sine.)*
3. *If both n and m are odd use 1 or 2.*
4. *If both n and m are even, use the* ~~*half*~~ double*-angle identities:*

EXAMPLE 4. *Evaluate* $I = \int_0^{\pi/2} \sin^2 x \, dx$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$I = \frac{1}{2}\int_0^{\pi/2} 1 - \cos(2x)\, dx = \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right)\Big|_0^{\pi/2} =$$

$$= \frac{1}{2}\Big(\frac{\pi}{2} \underbrace{- \frac{\sin \pi}{2} - \Big(0 - \frac{\sin 0}{2}\Big)}_{=0}\Big) = \frac{\pi}{4}$$

Note that it immediately implies that

$$\int_0^{\pi/2} \cos^2 x \, dx = \frac{\pi}{4}$$

EXAMPLE 5. *Evaluate* $I = \int \sin^2 x \cos^2 x \, dx$

$$2 \sin x \cos x = \sin 2x$$

$$\boxed{\sin x \cos x = \frac{1}{2} \sin 2x}$$

$$I = \int (\sin x \cos x)^2 \, dx = \int \left(\frac{1}{2} \sin 2x\right)^2 dx$$

$$I = \frac{1}{4} \int \sin^2 2x \, dx =$$

use $\sin^2 2x = \frac{1}{2}(1 - \cos 4x)$

$$I = \frac{1}{4} \int \frac{1}{2}(1 - \cos 4x) \, dx = \frac{1}{8} \int 1 - \cos 4x \, dx$$

$$I = \frac{1}{8}\left(x - \frac{\sin 4x}{4}\right) + C$$

2nd way

$$I = \int \sin^2 x \cos^2 x \, dx = \int \sin^2 x (1 - \sin^2 x) \, dx$$

$$I = \int (\sin^2 x - \sin^4 x) \, dx = \int \frac{1}{2}(1 - \cos 2x) - \left(\frac{1}{2}(1 - \cos 2x)\right)^2 dx$$

$$I = \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) - \frac{1}{4} \int (1 - 2\cos 2x + \cos^2 2x) \, dx$$

$$I = \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) - \frac{1}{4}\left[x - \sin 2x + \int \frac{1 + \cos 2x}{2} \, dx\right]$$

RULE: How to evaluate $\int \sec^n x \tan^m x \, dx$

1. *If n is even use formula* $\sec^2 x = 1 + \tan^2 x$ *and substitution*

$d(\tan x) = \sec^2 x \, dx$

$$u = \tan x \quad \Rightarrow \quad du = \sec^2 x dx.$$

2. *If m is odd use formula* $\tan^2 x = \sec^2 x - 1$ *and substitution*

$d(\sec x) = \sec x \tan x \, dx$

$$u = \sec x \quad \Rightarrow \quad du = \sec x \tan x dx$$

REMARK 6. Integral $\int \csc^n x \cot^m x \, dx$ can be found by similar methods because of the identity $1 + \cot^2 x = \csc^2 x$

EXAMPLE 7. *Evaluate* $I = \int \sec^{2011} x \tan^5 x \, dx$

$$I = \int (\sec x)^{2010} \underbrace{\tan^4 x}_{(\tan^2 x)^2 = (\sec^2 x - 1)^2} \underbrace{\sec x \tan x \, dx}_{= d(\sec x)}$$

$u = \sec x \qquad du = \sec x \tan x \, dx$

$$I = \int u^{2010} (u^2 - 1)^2 \, du = \int u^{2010} (u^4 - 2u^2 + 1) \, du$$

$$I = \int u^{2014} - 2u^{2012} + u^{2010} \, du$$

$$I = \frac{\sec^{2015} x}{2015} - \frac{2 \sec^{2013} x}{2013} + \frac{\sec^{2011} x}{2011} + C$$

EXAMPLE 8. *Evaluate* $I = \displaystyle\int \frac{\sin^{10} x}{\cos^{14} x}\, \mathrm{d}x$

$$I = \int \frac{\sin^{10} x}{\cos^{10} x} \cdot \frac{1}{\cos^4 x}\, dx = \int (\tan x)^{10} (\sec x)^4\, dx$$

$$I = \int (\tan x)^{10} \underbrace{\sec^2 x}_{1+\tan^2 x} \underbrace{\sec^2 x\, dx}_{d(\tan x)}$$

$$u = \tan x \quad \Rightarrow \quad du = \sec^2 x\, dx$$

$$I = \int u^{10}(1+u^2)\, du = \int u^{10} + u^{12}\, du$$

$$I = \frac{\tan^{11} x}{11} + \frac{\tan^{13} x}{13} + C$$

RULE: How to evaluate $\int \sin(Ax)\cos(Bx)\,\mathrm{d}x$, $\int \sin(Ax)\sin(Bx)\,\mathrm{d}x$, $\int \cos(Ax)\cos(Bx)\,\mathrm{d}x$

Use the following identities:

$$\begin{aligned}\sin A\cos B &= \tfrac{1}{2}\left(\sin(A-B)+\sin(A+B)\right)\\ \sin A\sin B &= \tfrac{1}{2}\left(\cos(A-B)-\cos(A+B)\right)\\ \cos A\cos B &= \tfrac{1}{2}\left(\cos(A-B)+\cos(A+B)\right)\end{aligned}$$

EXAMPLE 9. *Evaluate* $I = \int \cos(\overset{A}{25}x)\cos(\overset{B}{4}x)\,\mathrm{d}x$

$$I = \int \frac{1}{2}\left(\cos((25-4)x) + \cos((25+4)x)\right)dx$$

$$I = \frac{1}{2}\int\left(\cos(21x) + \cos(29x)\right)dx$$

$$I = \frac{1}{2}\left(\frac{\sin(21x)}{21} + \frac{\sin(29x)}{29}\right) + C$$