8.3: Trigonometric Substitution

$u = x^2-4 \Rightarrow du = 2x\,dx$

- $\int x\sqrt{x^2-4}\,dx = \int \frac{\sqrt{u}}{2}\,du = \dots$
- $\int \frac{x}{\sqrt{x^2-4}}\,dx = \int \frac{du}{2\sqrt{u}} = \dots$

EXAMPLE 1. *Evaluate* $I = \int \frac{\sqrt{x^2-4}}{x}\,dx$

$I = \int \frac{\sqrt{x^2-4}\;x\,dx}{x\cdot x} = \int \frac{\sqrt{u}\;du/2}{u+4} \rightarrow ?$

$\sqrt{x^2-4} = \sqrt{4\left(\frac{x^2}{4}-1\right)} = 2\sqrt{\left(\frac{x}{2}\right)^2-1} = 2\sqrt{☺^2-1}$

$= 2\sqrt{\sec^2\theta-1} = 2\tan\theta$

$☺ = \boxed{\frac{x}{2} = \sec\theta}$

$x = 2\sec\theta$ (inverse substitution)

$dx = 2\sec\theta\tan\theta\,d\theta$

$I = \int\frac{\sqrt{x^2-4}}{x}\,dx = \int \frac{\cancel{2}\tan\theta}{\cancel{2}\cancel{\sec\theta}}\cdot 2\cancel{\sec}\theta\tan\theta\,d\theta$

$I = 2\int \tan^2\theta\,d\theta = 2\int(\sec^2\theta-1)\,d\theta$

$I = 2[\tan\theta - \theta] + C$ Back to x !!!

Before we had: $\sqrt{x^2-4} = 2\tan\theta \Rightarrow \boxed{\tan\theta = \frac{\sqrt{x^2-4}}{2}}$

$x = 2\sec\theta \Rightarrow \frac{x}{2} = \sec\theta = \frac{1}{\cos\theta}$

$\cos\theta = \frac{2}{x}$

$\boxed{\theta = \arccos\frac{2}{x}}$

$$\boxed{I = \sqrt{x^2-4} - 2\arccos\frac{2}{x} + C}$$

How to get rid of $\sqrt{}$

$$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta$$

$x = \sin\theta$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$\sqrt{x^2+1} = \sqrt{\tan^2\theta + 1} = \sqrt{\sec^2\theta} = |\sec\theta| = \sec\theta$$

$x = \tan\theta$

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = |\tan\theta| = \tan\theta$$

$x = \sec\theta$

$0 \le \theta < \frac{\pi}{2}$

OR

$\pi \le \theta < \frac{3\pi}{2}$

EXAMPLE 2. *Evaluate* $I = \int_2^4 \frac{\sqrt{x^2-4}}{x}\,\mathrm{d}x$

No need back to x here.

Just use $\frac{\sqrt{x^2-4}}{2} = \tan\theta$, $\underline{0 \le \theta < \frac{\pi}{2}}$

$\theta = \arccos\frac{2}{x}$ (see Ex. 1)

By Example 1

$I = 2\tan\theta - 2\theta \Big|_{\boxed{0}}^{\boxed{\pi/3}}$

$x = 2 \Rightarrow \theta = \arccos 1 = 0$

$x = 4 \Rightarrow \theta = \arccos\frac{1}{2} = \frac{\pi}{3}$

$I = 2\tan\frac{\pi}{3} - \frac{2\pi}{3} - 0 = \boxed{2\sqrt{3} - \frac{2\pi}{3}}$

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Table of Inverse Trigonometric Substitutions

$a\cos\theta =$	$\sqrt{a^2 - b^2x^2}$	$x = \frac{a}{b}\sin\theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$a\sec\theta =$	$\sqrt{a^2 + b^2x^2}$	$x = \frac{a}{b}\tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = 1\sec^2\theta$
$a\tan\theta =$	$\sqrt{b^2x^2 - a^2}$	$x = \frac{a}{b}\sec\theta, \quad 0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

for definite integral

EXAMPLE 3. *Evaluate* $I = \int \frac{1}{x^4\sqrt{4-25x^2}}\,dx$

$\sqrt{4-25x^2} = 2\cos\theta$

$x = \frac{2}{5}\sin\theta \Rightarrow dx = \frac{2}{5}\cos\theta\, d\theta$

$$I = \int \frac{1}{\left(\frac{2}{5}\right)^{\cancel{4}\,3}\sin^4\theta \cdot \cancel{2}\cancel{\cos\theta}} \cdot \cancel{\frac{2}{5}}\cancel{\cos\theta}\, d\theta$$

$$I = \frac{5^3}{2^4}\int \frac{d\theta}{\sin^4\theta} = \frac{5^3}{2^4}\int \csc^4\theta\, d\theta$$

$$I = \frac{5^3}{2^4}\int \underbrace{\csc^2\theta}_{1+\cot^2\theta}\ \underbrace{\csc^2 d\theta}_{-d(\cot\theta)}$$

Use u-sub. $u = \cot\theta \Rightarrow du = -\csc^2\theta\, d\theta$

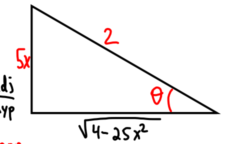
$$I = \frac{125}{16}\int (1+u^2)(-du) = -\frac{125}{16}\int (1+u^2)\,du$$

$$I = -\frac{125}{16}\left(u + \frac{u^3}{3}\right) + C$$

$u \downarrow \theta \downarrow x$

Back to θ: $I = -\frac{125}{16}\left(\cot\theta + \frac{1}{3}\cot^3\theta\right) + C$

Back to x:



$\cos\theta = \frac{\sqrt{4-25x^2}}{2} = \frac{adj}{hyp}$

$\boxed{\sin\theta = \frac{5}{2}x} = \frac{5x}{2} = \frac{opp}{hyp}$

$\boxed{\cot\theta = \frac{adj}{opp} = \frac{\sqrt{4-25x^2}}{5x}}$

Use that or Pythag. Theorem.

$$I = -\frac{125}{16}\left(\cot\theta + \frac{\cot^3\theta}{3}\right) + C$$

$$I = -\frac{125}{16}\left[\frac{\sqrt{4-25x^2}}{5x} + \frac{1}{3}\cdot\frac{\left(\sqrt{4-25x^2}\right)^3}{(5x)^3}\right] + C$$

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EXAMPLE 4. *Evaluate* $I = \int \frac{x^5}{(x^2+36)^{3/2}}\,dx$

$$I = \int \frac{x^5\,dx}{\left(\sqrt{x^2+36}\right)^3}$$

$$\boxed{x = 6\tan\theta \quad \Rightarrow \quad \sqrt{x^2+36} = 6\sec\theta \qquad dx = 6\sec^2\theta\, d\theta}$$

$$I = \int \frac{\cancel{6^5}^{\,2}\tan^5\theta \cdot 6\,\cancel{\sec^2}\theta\, d\theta}{\cancel{6^3}\,\sec^{\cancel{3}}\theta} = 6^3 \int \frac{\tan^5\theta\, d\theta}{\sec\theta}$$

$$I = 6^3 \int \frac{\sin^5\theta}{\cos^5\theta}\cdot\cos\theta\, d\theta = 6^3 \int \frac{\sin^5\theta}{\cos^4\theta}\, d\theta$$

$$I = 6^3 \int \frac{\overset{=(1-\cos^2\theta)^2}{\sin^4\theta}}{\cos^4\theta}\, \underbrace{\sin\theta\, d\theta}_{-d(\cos\theta)}$$

$$\boxed{\begin{array}{l} u = \cos\theta \\ du = -\sin\theta\, d\theta \end{array}}$$

$$I = 6^3 \int \frac{(1-\cos^2\theta)^2}{\cos^4\theta}\, \sin\theta\, d\theta$$

$(a-b)^2 = a^2 - 2ab + b^2$

$$I = -6^3 \int \frac{(1-u^2)^2}{u^4}\, du = -6^3 \int \frac{1-2u^2+u^4}{u^4}\, du$$

$$I = -6^3 \int u^{-4} - 2u^{-2} + 1\, du$$

$$I = -6^3\left(\frac{u^{-3}}{-3} - \frac{2u^{-1}}{-1} + u\right) + C$$

$$I = 6^3\left(\frac{1}{3u^3} - \frac{2}{u} - u\right) + C$$

Back to x:
$u \to \theta \to x$
usually use right triangle

$$I = 6^3\left(\frac{1}{3\cos^3\theta} - \frac{2}{\cos\theta} - \cos\theta\right) + C$$

$$I = 6^3\left(\frac{1}{3}\sec^3\theta - 2\sec\theta - \cos\theta\right) + C$$

From the previous (see above) we have:

$$\sec\theta = \frac{\sqrt{x^2+36}}{6} \quad \Rightarrow \quad \cos\theta = \frac{6}{\sqrt{x^2+36}}$$

$$I = 6^3\left(\frac{1}{3}\frac{\left(\sqrt{x^2+36}\right)^3}{6^3} - \frac{2}{6}\sqrt{x^2+36} - \frac{6}{\sqrt{x^2+36}}\right) + C$$

EXAMPLE 5. *Evaluate* $I = \int \frac{x}{\sqrt{2x^2-4x-7}}\,dx$

To get rid of $\sqrt{\ }$ use complete squares method and then make an appropriate trig. substitution

$(a \pm b)^2 = a^2 \pm 2ab + b^2$

$$\sqrt{2x^2-4x-7} = \sqrt{2\left(x^2-2x-\frac{7}{2}\right)} = \sqrt{2\left(\underbrace{x^2-2x\cdot 1+1^2}_{(x-1)^2}\underbrace{-1^2-\frac{7}{2}}_{-\frac{9}{2}}\right)}$$

$$= \sqrt{2\left((x-1)^2-\frac{9}{2}\right)} = \sqrt{2(x-1)^2-9}$$

[See table (for example)

$\sqrt{b^2\star^2-a^2} = a\tan\theta$ when $\boxed{\star = \frac{a}{b}\sec\theta}$]

We have $\star = x-1$, $b=\sqrt{2}$, $a=3$

$x-1 = \frac{3}{\sqrt{2}}\sec\theta \Rightarrow x = 1+\frac{3}{\sqrt{2}}\sec\theta$

Then $\sqrt{2x^2-4x-7} = 3\tan\theta$ | $dx = \frac{3}{\sqrt{2}}\sec\theta\tan\theta\,d\theta$

Plug in:

$$I = \int \frac{x\,dx}{\sqrt{2x^2-4x-7}} = \int \frac{1+\frac{3}{\sqrt{2}}\sec\theta}{\cancel{3}\cancel{\tan\theta}} \cdot \frac{\cancel{3}}{\sqrt{2}}\sec\theta\cancel{\tan\theta}\,d\theta$$

$$I = \int \frac{1}{\sqrt{2}}\left(1+\frac{3}{\sqrt{2}}\sec\theta\right)\sec\theta\,d\theta$$

$$I = \underbrace{\int \frac{1}{\sqrt{2}}\sec\theta\,d\theta}_{\text{see Example 7 below}} + \frac{3}{2}\underbrace{\int \sec^2\theta\,d\theta}_{\tan\theta}$$

$$I = \frac{1}{\sqrt{2}}\ln|\sec\theta+\tan\theta| + \frac{3}{2}\tan\theta + C$$

Use $\sec\theta = \frac{\sqrt{2}}{3}(x-1)$, $\tan\theta = \frac{\sqrt{2x^2-4x-7}}{3}$

$$I = \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}}{3}(x-1)+\frac{\sqrt{2x^2-4x-7}}{3}\right| + \frac{1}{2}\sqrt{2x^2-4x-7} + C$$

EXAMPLE 6. *Evaluate* $I = \int \frac{x}{\sqrt{3-2x-x^2}}\,dx$

Complete squares

$$3-2x-x^2 = -(x^2+2x-3)$$

$$= -(\underbrace{x^2+2x+1}-1-3) = -[(x+1)^2-4] = 4-(x+1)^2$$

Thus, $\sqrt{3-2x-x^2} = \sqrt{4-(x+1)^2} = 2\cos\theta$ (1)

where $x+1 = 2\sin\theta \Rightarrow x = 2\sin\theta - 1$ (2)

$$\Downarrow$$

$$dx = 2\cos\theta\, d\theta$$

$$I = \int \frac{2\sin\theta - 1}{\cancel{2\cos\theta}} \cancel{2\cos\theta}\, d\theta = \int (2\sin\theta - 1)\, d\theta$$

$$I = -2\cos\theta - \theta + C$$

(1) $\Rightarrow -2\cos\theta = -\sqrt{3-2x-x^2}$

(2) $\Rightarrow \theta = \arcsin\frac{x+1}{2}$

Finally, $I = -\sqrt{3-2x-x^2} - \arcsin\frac{x+1}{2} + C$

Example 7. *Evaluate* $\int \sec(x)dx$.

$$I = \int \sec x \, dx = \int \frac{dx}{\cos x} = \int \frac{\cos x}{\cos x} \frac{dx}{\cos x}$$

$$I = \int \frac{\cos x \, dx}{\cos^2 x} = \int \frac{\cos x \, dx}{1 - \sin^2 x}$$

$u = \sin x$
$du = \cos x \, dx$

$$I = \int \frac{du}{1-u^2}$$

Use $\frac{1}{1-u^2} = \frac{1}{2}\left[\frac{1}{1+u} + \frac{1}{1-u}\right]$

(Wait until Section 8.4 to learn how to get this)

$$I = \int \frac{1}{2}\left[\frac{1}{1+u} + \frac{1}{1-u}\right] du = \frac{1}{2}\left[\ln|1+u| - \ln|1-u|\right] + C$$

$$I = \frac{1}{2} \ln\left|\frac{1+u}{1-u}\right| + C = \ln\sqrt{\left|\frac{1+u}{1-u}\right|} + C$$

$$I = \boxed{\ln\sqrt{\frac{|1+\sin x|}{|1-\sin x|}} + C}$$

It is possible to simplify it more

$$I = \ln\sqrt{\frac{|1+\sin x|}{|1-\sin x|} \cdot \frac{1+\sin x}{1+\sin x}} = \ln\sqrt{\frac{(1+\sin x)^2}{1-\sin^2 x}} + C$$

$$I = \ln\left|\frac{1+\sin x}{\cos x}\right| + C$$

$$\boxed{\int \sec x \, dx = \ln|\sec x + \tan x| + C}$$