8.4: Integration Of Rational Functions By Partial Fractions

EXAMPLE 1. *Evaluate the following integrals:*

(a) $\displaystyle\int \frac{2x-5}{x^2-5x+4}\,\mathrm{d}x = \int \frac{du}{u} = \ln|u| + C = \ln|x^2-5x+4| + C$

$u = x^2-5x+4$
$du = (2x-5)dx$

(b) $\displaystyle\int \frac{x-6}{x^2-5x+4}\,\mathrm{d}x$

step 0. $1 = \deg(x-6) < \deg(x^2-5x+4) = 2$

We will use so called Partial Fraction Decomposition

Step 1 Factor the denominator as much as possible

$$\frac{x-6}{x^2-5x+4} = \frac{x-6}{(x-4)(x-1)} = \frac{A}{x-4} + \frac{B}{x-1} \quad \text{for all } x$$

$$\text{common denominator} = \frac{A(x-1)+B(x-4)}{(x-4)(x-1)}$$

$$x-6 = A(x-1) + B(x-4) \quad \text{for all } x$$

$x=4 \Rightarrow$ $4-6 = A(4-1) + B(4-4)$

$-2 = 3A + 0 \Rightarrow \boxed{A = -\frac{2}{3}}$

$x=1 \Rightarrow$ $1-6 = A(1-1) + B(1-4)$

$-5 = -3B \Rightarrow \boxed{B = \frac{5}{3}}$

$$\int \frac{x-6}{x^2-5x+4}\,dx = \int \frac{A\,dx}{x-4} + \int \frac{B\,dx}{x-1}$$

$$= -\frac{2}{3}\int \frac{dx}{x-4} + \frac{5}{3}\int \frac{dx}{x-1}$$

$$= -\frac{2}{3}\ln|x-4| + \frac{5}{3}\ln|x-1| + C$$

Partial Fraction Decomposition Process

Rational function: $f(x) = \dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.

Step 0. $\deg P(x) < \deg Q(x)$.

Important to remember: Partial fractions can only be done if the degree of the numerator is strictly less than the degree of denominator. (Otherwise, you must first do **long division**.)

Step 1. Factor the denominator as much as possible.

Step 2. For each **linear factor** in the denominator use the following table:

Factor in denominator	Term in partial fraction decomposition
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^2$	$\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2}$
$(ax+b)^k$	$\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \cdots + \dfrac{A_k}{(ax+b)^k}$

For each **irreducible quadratic factor** in the denominator use the following table:

Factor in denominator	Term in partial fraction decomposition
ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^2$	$\dfrac{A_1}{ax^2+bx+c} + \dfrac{A_2x+B_2}{(ax^2+bx+c)^2}$
$(ax^2+bx+c)^k$	$\dfrac{A_1x+B_1}{ax^2+bx+c} + \dfrac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \dfrac{A_kx+B_k}{(ax^2+bx+c)^k}$

irreducible quadratic factor

x^2+1 , x^2+x+1

$ax^2+bx+c = a(x-x_1)(x-x_2)$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

we can factor if and only if

$$b^2-4ac \geq 0$$

Note for $x^2+1 \Rightarrow b^2-4ac = 0-4\cdot1\cdot1 = -4 < 0$

$x^2+x+1 \Rightarrow b^2-4ac = 1-4 = -3 < 0$

EXAMPLE 2. *Evaluate* $I = \int \frac{3x+11}{x^2-x-6}\,dx$

$1 = \deg(3x+11) < \deg(x^2-x-6) = 2$

$$\frac{3x+11}{x^2-x-6} = \frac{3x+11}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2} = \frac{A(x+2)+B(x-3)}{(x-3)(x+2)}$$

$$3x+11 = A(x+2) + B(x-3)$$

$x=-2 \Rightarrow 3\cdot(-2)+11 = 0 + B(-2-3)$

$5 = -5B \Rightarrow \boxed{B=-1}$

$x = 3 \Rightarrow 3\cdot 3+11 = A(3+2)+0$

$20 = 5A \Rightarrow \boxed{A=4}$

$$I = A\int \frac{dx}{x-3} + B\int \frac{dx}{x+2}$$

$$= 4\int \frac{dx}{x-3} - \int \frac{dx}{x+2}$$

$$= 4\ln|x-3| - \ln|x+2| + C$$

EXAMPLE 3. *Evaluate* $I = \int \frac{x^2+1}{x^2-x}\,dx$

$2 = \deg(x^2+1) = \deg(x^2-x) = 2$

Use long division OR

$$\begin{array}{r|l} & 1 \\ x^2-x & x^2+1 \\ & -\;x^2-x \\ \hline & 1+x \end{array}$$

$$\frac{x^2+1}{x^2-x} = 1 + \frac{1+x}{x^2-x}$$

$$\frac{x^2+1}{x^2-x} = \frac{x^2-x+x+1}{x^2-x} = \frac{x^2-x}{x^2-x} + \frac{x+1}{x^2-x} = 1 + \frac{x+1}{x^2-x}$$

$$\int \frac{x^2+1}{x^2-x} = \int 1 + \frac{x+1}{x^2-x}\,dx = x + \underbrace{\int \frac{x+1}{x^2-x}\,dx}_{I}$$

Use PFD to evaluate I

$1 = \deg(x+1) < \deg(x^2-x) = 2$

$$\frac{x+1}{x^2-x} = \frac{x+1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} = \frac{A(x-1)+Bx}{x(x-1)}$$

$x+1 = A(x-1) + Bx$

$x=1 \Rightarrow \boxed{2 = B}$

$x=0 \Rightarrow 1 = -A \Rightarrow \boxed{A=-1}$

$$\int \frac{x^2+1}{x^2-x}\,dx = x + I = x + \int \frac{A}{x}\,dx + \int \frac{B}{x-1}\,dx$$

$$= x - \int \frac{dx}{x} + 2\int \frac{dx}{x-1}$$

$$= x - \ln|x| + 2\ln|x-1| + C$$

$$= x + \ln \frac{(x-1)^2}{|x|} + C$$

EXAMPLE 4. *Evaluate* $I = \int \frac{x^2}{(x-3)(x+2)^2}\,dx$

$2 = \deg(\text{numerator}) < \deg(\text{denominator}) = 3$

$$\frac{x^2}{(x-3)(x+2)^2} = \frac{A}{x-3} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$(x+2)^2 \quad (x-3)(x+2) \quad x-3$

$$1 \cdot x^2 = A(x+2)^2 + B(x-3)(x+2) + C(x-3)$$

$x = 3 \Rightarrow \quad 9 = 25A + 0 + 0 \Rightarrow \boxed{A = \frac{9}{25}}$

$x = -2 \Rightarrow \quad 4 = 0 + 0 + C(-2-3) \Rightarrow \boxed{C = -\frac{4}{5}}$

Equate coefficients of x2: $1 = A + B$

$$B = 1 - A = 1 - \frac{9}{25} = \frac{16}{25}$$

$$I = A\int \frac{dx}{x-3} + B\int \frac{dx}{x+2} + C\int \frac{dx}{(x+2)^2}$$

u-sub.

$$I = \frac{9}{25}\ln|x-3| + \frac{16}{25}\ln|x+2| + \frac{4}{5(x+2)} + C$$

EXAMPLE 5. *Integrate* $I = \int \frac{x^4 - x^3 - 12x^2 + 10}{x^3 - 4x^2}\,dx$

$\deg(\text{num.}) = 4 > \deg(\text{denom.}) = 3$

LONG DIVISION

$$
\begin{array}{r|l}
 & x + 3 \\
\hline
x^3 - 4x^2 & x^4 - x^3 - 12x^2 + 10 \\
 & -\;(x^4 - 4x^3) \\
\hline
 & 3x^3 - 12x^2 \\
 & -\;(3x^3 - 12x^2) \\
\hline
 & 0 + 10
\end{array}
$$

$$I = \int x + 3 + \frac{10}{x^3 - 4x^2}\,dx = \frac{x^2}{2} + 3x + 10\underbrace{\int \frac{dx}{x^3 - 4x^2}}$$

$\deg(\text{num}) = 0 < \deg(\text{denom}) = 3$

$\Downarrow$ use PFD

$$\frac{1}{x^3 - 4x^2} = \frac{1}{x^2(x-4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-4}$$

$$0 \cdot x^2 + 1 = Ax(x-4) + B(x-4) + Cx^2$$

$x = 0 \Rightarrow 1 = 0 + (-4B) + 0 \Rightarrow \boxed{B = -\frac{1}{4}}$

$x = 4 \Rightarrow 1 = 0 + 0 + 16C \Rightarrow \boxed{C = \frac{1}{16}}$

Equate coefficients of x2: $0 = A + C \Rightarrow A = -C = \boxed{-\frac{1}{16}}$

$$I = \frac{x^2}{2} + 3x + 10\left[A\int\frac{dx}{x} + B\int\frac{dx}{x^2} + C\int\frac{dx}{x-4}\right]$$

$$I = \frac{x^2}{2} + 3x + 10\left[-\frac{1}{16}\ln|x| - \frac{1}{4}\cdot\left(-\frac{1}{x}\right) + \frac{1}{16}\ln|x-4|\right] + C$$

$$I = \frac{x^2}{2} + 3x - \frac{5}{8}\ln|x| + \frac{5}{2x} + \frac{5}{8}\ln|x-4| + C$$

EXAMPLE 6. *Write out the form of the partial fraction decomposition of the following function:*

$$f(x) = \frac{x^3 + 5x^2 - 2012}{\underbrace{x}_{\text{linear}}\underbrace{(x+12)^2}\underbrace{(x^2+2x-3)}_{(x+3)(x-1)}\underbrace{(x^2+x+1)(x^2+25)^3}_{\text{irredicible}}}$$

$$\deg(\text{num}) = 3 < \deg(\text{denom.}) = 1+2+2+2+2\cdot 3 = 13$$

$$f(x) = \frac{A}{x} + \frac{B_1}{x+12} + \frac{B_2}{(x+12)^2} + \frac{C}{x+3} + \frac{D}{x-1}$$

$$+ \frac{Ex+F}{x^2+x+1} + \frac{G_1x+H_1}{x^2+25} + \frac{G_2x+H_2}{(x^2+25)^2} + \frac{G_3x+H_3}{(x^2+25)^3}$$

Note: *It would be extremely tedious to work out by hand the numerical values of the coefficients. Mapple command* **convert(*f*,partfrac,*x*)** *can find them.*