9.3: Arc Length

Description of a curve by parametric equations:

$$x = x(t), \quad y = y(t), \quad \underbrace{\alpha \le t \le \beta}_{\text{parameter domain}}$$

EXAMPLE 1. *Parameterize the following curves:*

(a) $y = \sqrt{1 - x^2}$

$x = \cos t, \quad y = \sqrt{1 - \cos^2 t} = \sin t, \quad 0 \le t \le \pi$

Another parametrization:

$x = x, \quad y = \sqrt{1 - x^2}, \quad -1 \le x \le 1$

(b) $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

$\Big(\underbrace{\tfrac{x}{2}}_{\cos t}\Big)^2 + \Big(\underbrace{\tfrac{y}{3}}_{\sin t}\Big)^2 = 1$

$\frac{x}{2} = \cos t, \quad \frac{y}{3} = \sin t$

$x = 2\cos t, \quad y = 3\sin t, \quad 0 \le t \le 2\pi$

Another way

$x = 2\cos(\pi t), \quad y = 3\sin(\pi t)$

$0 \le t \le 2$

(c) $y = \sin x$, $0 \le x \le \pi$

$x = x, \quad y = \sin x, \quad 0 \le x \le \pi$

OR $x = t, \quad y = \sin t, \quad 0 \le t \le \pi$

(d) $y^4 = x$

$x = y^4, \quad y = y$

(e) $y = f(x)$, $a \le x \le b$

$x = x, \quad y = f(x), \quad a \le x \le b$

(f) $x = g(y)$, $c \le y \le d$

$x = g(y), \quad y = y, \quad c \le y \le d$

DEFINITION 2. *A curve* $C: x = x(t), \quad y = y(t), \quad \alpha \le t \le \beta$, *is called* **smooth** *if the derivatives* $x'(t)$ *and* $y'(t)$ *are continuous and not simultaneously zero on* $[\alpha, \beta]$.

The length of a smooth curve = arc length

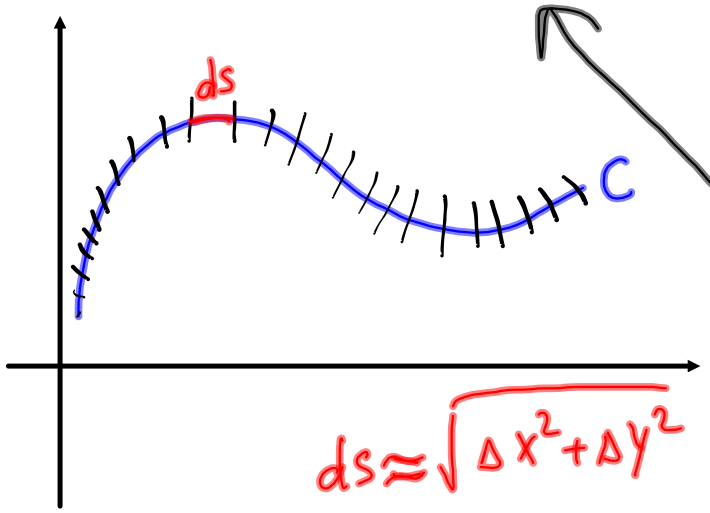
$$C: x = x(t), \quad y = y(t), \quad \alpha \le t \le \beta,$$

assuming that C is traversed exactly once as t increases from α to β is

$$\boxed{L = \int_\alpha^\beta \sqrt{[x'(t)]^2 + [y'(t)]^2}\,\mathrm{d}t =: \int_C \mathrm{d}s,}$$

where $\mathrm{d}s$ is called *differential of the arc length function* and

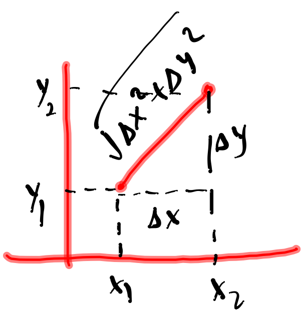
$$\mathrm{d}s = \sqrt{[x'(t)]^2 + [y'(t)]^2}\mathrm{d}t$$ = differential of arc length function.



$$ds \approx \sqrt{\Delta x^2 + \Delta y^2}$$

$$\Delta x = dx = x'(t)\,dt$$

$$\Delta y = dy = y'(t)\,dt$$



EXAMPLE 3. *Find the length of the curve*

$$C: \; x = 3t - t^3, \quad y = 3t^2, \quad 0 \le t \le 2.$$

$$L = \int_C ds$$

$$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

$$ds = \sqrt{(3 - 3t^2)^2 + (6t)^2}\, dt$$

$$ds = \sqrt{9 - 18t^2 + 9t^4 + 36t^2}\, dt = \sqrt{9t^4 + 18t^2 + 9}\, dt$$

$$ds = \sqrt{9\,(t^4 + 2t^2 + 1)}\, dt = \sqrt{9\,(t^2+1)^2}\, dt = 3\,(t^2+1)\, dt$$

$$L = \int_C ds = \int_0^2 3(t^2+1)\, dt = 14$$

Particular cases for the differential of the arc length function:

$x = x(t), \ y = y(t)$

$$\mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

- C is given by $y = f(x)$, $a \leq x \leq b$

$x = x, \quad y = f(x)$

$$ds = \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(f'(x)\right)^2}\, dx \quad \Rightarrow \quad ds = \sqrt{1 + \left(f'(x)\right)^2}\, dx$$

- C is given by $x = g(y)$, $c \leq y \leq d$

$x = g(y), \quad y = y$

$$\Rightarrow ds = \sqrt{\left(g'(y)\right)^2 + 1}\, dy$$

EXAMPLE 4. *Find the arc length of the graph of* $y = \ln \cos x$ *from* $x = 0$ *to* $x = \pi/4$.

$$L = \int_C ds$$

$x = x$

$y = \ln \cos x$

$$L = \int_0^{\pi/4} \sqrt{1 + (f'(x))^2}\, dx$$

$$f'(x) = y'(x) = (\ln \cos x)' = \frac{1}{\cos x}(-\sin x) = -\tan x$$

$$L = \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx = \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx$$

$$= \int_0^{\pi/4} |\sec x|\, dx = \int_0^{\pi/4} \sec x\, dx =$$

$$= \ln|\sec x + \tan x|\Big|_0^{\pi/4}$$

$$= \ln(\sqrt{2} + 1) \simeq 0.881$$

EXAMPLE 5. *Find the arc length of the curve* $(y-1)^3 = x^2$ *from* $x = 0$ *to* $x = 8$.

$$L = \int_C ds = \int_0^8 \sqrt{1 + [y'(x)]^2}\, dx$$

$$y - 1 = \sqrt[3]{x^2} = x^{\frac{2}{3}}$$

$$y = 1 + x^{2/3}$$

$$y'(x) = \frac{d}{dx}\left(1 + x^{\frac{2}{3}}\right) = \frac{2}{3} x^{-\frac{1}{3}} = \frac{2}{3x^{1/3}}$$

$$ds = \sqrt{1 + \left(\frac{2}{3x^{1/3}}\right)^2}\, dx = \sqrt{1 + \frac{4}{9x^{2/3}}}\, \cancel{dx} = \sqrt{\frac{9x^{2/3} + 4}{9x^{2/3}}}\, dx$$

$$= \frac{\sqrt{9x^{2/3} + 4}\, dx}{3x^{1/3}}$$

$$L = \int_0^8 \frac{\sqrt{9x^{2/3} + 4}\, dx}{3x^{1/3}}$$

$$\frac{du}{6}$$

$$u = 9x^{\frac{2}{3}} + 4$$

$$du = 9 \cdot \frac{2}{3} x^{-\frac{1}{3}} dx$$

$$du = \frac{6\,dx}{x^{1/3}}$$

$x = 0 \Rightarrow u = 4$

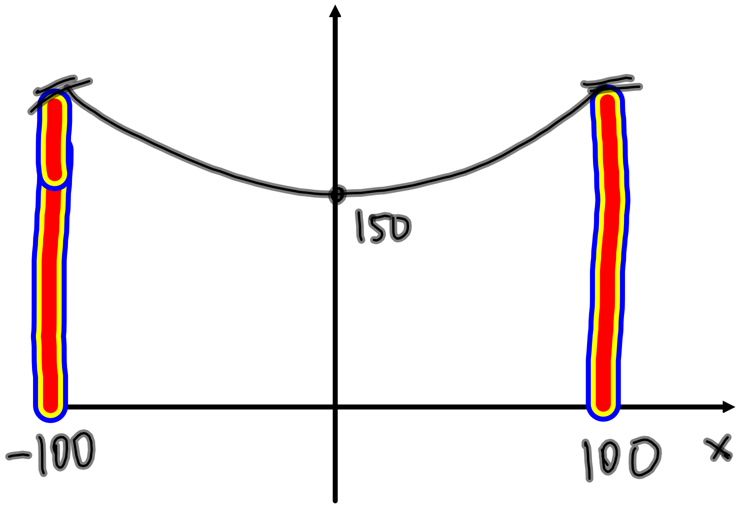
$x = 8 \Rightarrow u = 9 \cdot 4 + 4 = 40$

$$L = \frac{1}{3}\int_4^{40} \sqrt{u}\, \frac{du}{6} = \ldots =$$

EXAMPLE 6. *A telephone wire hanging between two poles (at $x = -100$ and $x = 100$) takes the shape of a catenary with equation*

$$y = 75(e^{x/150} + e^{-x/150}).$$

Find the length of the wire.



$$L = \int_C ds = \int_{-100}^{100} \sqrt{1+(y')^2}\, dx$$

$$y' = 75\left(\frac{1}{150} e^{\frac{x}{150}} - \frac{1}{150} e^{-\frac{x}{150}}\right)$$

$$y' = \frac{1}{2}\left(e^{\frac{x}{150}} - e^{-\frac{x}{150}}\right)$$

$$1 + (y')^2 = 1 + \left(\frac{1}{2}\left(e^{\frac{x}{150}} - e^{-\frac{x}{150}}\right)\right)^2 = 1 + \frac{1}{4}\left(e^{\frac{x}{75}} - 2 + e^{-\frac{x}{75}}\right)$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$\boxed{\begin{gathered}\left(e^{\frac{x}{150}}\right)^2 = e^{\frac{2x}{150}} = e^{\frac{x}{75}} \\ e^{a} e^{-a} = e^{a-a} = e^{0} = 1\end{gathered}}$$

$$1 + (y')^2 = \frac{1}{4}\left(4 + e^{\frac{x}{75}} - 2 + e^{-\frac{x}{75}}\right)$$

$$\sqrt{1+(y')^2} = \sqrt{\frac{1}{4}\left(e^{\frac{x}{75}} + 2 + e^{-\frac{x}{75}}\right)} = \sqrt{\frac{1}{4}\left(e^{\frac{x}{150}} + e^{-\frac{x}{150}}\right)^2}$$

$$L = \int_{-100}^{100} \frac{1}{2}\left(e^{\frac{x}{150}} + e^{-\frac{x}{150}}\right) dx = \frac{1}{2} \cdot 150\left(e^{\frac{x}{150}} - e^{-\frac{x}{150}}\right)\Big|_{-100}^{100}$$

$$L = 75\left(e^{\frac{2}{3}} - e^{-\frac{2}{3}}\right) \simeq 215$$