Math 131 Week in Review
Sections 4.2, 4.3, 4.6
4/4/10

1. Sketch a graph of a function $f$ that is continuous on $[-2, 3]$, has an absolute minimum at -1, an absolute maximum at 3, and a local minimum at 0.
A critical number of a function $f$ is a number $c$ in the domain of $f$ such that either $f'(c) = 0$ or $f'(c)$ does not exist. Sketch the graph of a function that has 2 local minima, 1 absolute maximum, and 8 critical numbers.
3. Find the critical numbers of \( g(x) = 2 \cos x - \sin^2 x, 0 \leq x < \pi. \)

\[
g'(x) = -2 \sin x \cdot 2 \sin x \cos x - 2 \sin x \\
-2 \sin x - 2 \sin x \cos x = 0 \\
-2 \sin x \cdot (1 + \cos x) = 0 \\
-2 \sin x = 0 \quad 1 + \cos x = 0 \quad \sin x = 0 \\
\sin x = 0 \quad \cos x = -1 \\
\text{not in domain}
\]

4. Find the critical numbers of \( h(x) = x^3 - 3x^2 + 4x. \)

\[
h'(x) = 3x^2 - 6x + 4 \\
3x^2 - 6x - 4 = 0 \\
\]

\[
x = \frac{6 \pm \sqrt{36 - 4(-4)}}{2(3)} \\
= \frac{6 \pm \sqrt{40}}{6} \\
= \frac{6 \pm 2\sqrt{10}}{6} \\
\]

5. Find the critical numbers of \( F(x) = x^2 e^{-4x}. \)

\[
F'(x) = x^2 e^{-4x}(-4) + 2xe^{-4x} \\
= -4x^2 e^{-4x} + 2xe^{-4x} \\
= -2xe^{-4x}(2x - 1) \\
\]

\[
\frac{4(3 \pm \sqrt{21})}{63} \\
\frac{3 \pm \sqrt{21}}{3}
\]
6. Find the critical numbers of \( G(x) = \frac{2}{3} (x-3)^3 \).

\[
G^{-1}(x) = \frac{4}{3} x^3 \frac{2}{3} (x-3) + \frac{2}{3} x - \frac{2}{3} (x-3)^2
= x - \frac{2}{3} (x-3) \left[ 2x + \frac{2}{3} (x-3) \right]
= x - \frac{2}{3} (x-3) \left( \frac{8}{3} x - 2 \right)
\]

7. Find the absolute maximum and absolute minimum value of \( f(x) = x^4 - 3x^2 + 2 \) on the interval \([-2, 3]\).

\[
f'(x) = 4x^3 - 6x
4x^3 - 6x = 0
2x(2x^2 - 3) = 0
x = 0 \quad 2x^2 - 3 = 0
x = \pm \sqrt{\frac{3}{2}}
\]

\[
\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}\]

\[
x = 0 \quad x = 3 \quad x = \pm \sqrt{\frac{3}{2}}
\]

\[
f(-2) = (-2)^4 - 3(-2)^2 + 2 = 6
f(0) = 0 - 0 + 2 = 2
f\left( \pm \sqrt{\frac{3}{2}} \right) = \frac{9}{4} - 3\left( \frac{3}{4} \right) + 2 = -\frac{1}{4}
\]

\[
f(-2) = 6 \quad f(0) = 2 \quad f\left( \pm \sqrt{\frac{3}{2}} \right) = -\frac{1}{4}
\]

\[
f(3) = 81 - 3(9) + 2 = 56
\]
8. Find the absolute maximum and absolute minimum value of \( G(x) = \frac{x^2 - 9}{x^2 + 9} \) on the interval \([-5, 5]\).

\[
G'(x) = \frac{(x^2 + 9)(2x) - (x^2 - 9)(2x)}{(x^2 + 9)^2}
\]

df everywhere on \([-5, 5]\)

\[
G(-5) = \frac{25 - 9}{25 + 9} = \frac{16}{34} = \frac{8}{17}
\]

\[
G(0) = \frac{-9}{9} = -1
\]

\[
G(5) = \frac{8}{17}
\]

abs min = -1
abs max = \(\frac{8}{17}\)

9. What is the smallest perimeter possible for a rectangle whose area is 16 in\(^2\), and what are its dimensions?


\[
\text{minimize } P = 2w + 2l
\]

\[
A = wl
\]

\[
P(w) = 2w + 2\left(\frac{16}{w}\right)
\]

\[
P'(w) = 2 - 32w^{-2}
\]

\[
2 - \frac{32}{w^2} = 0
\]

\[
w^2 - 16 = 0
\]

\[
w = 4 \text{ in}
\]

\[
l = \frac{16}{4} = 4 \text{ in}
\]

4\(\text{in} \times 4\(\text{in}
\]

\[
P = 2(4) + 2(4) = 16 \text{ in}
\]
10. Two sides of a triangle have lengths $a$ and $b$, and the angle between them is $\theta$. What value of $\theta$ maximize the triangle’s area? [Hint: $A = \frac{1}{2} ab \sin \theta$.]


\[ A = \frac{1}{2} ab \sin \theta \quad 0 \leq \theta \leq 180^\circ \]

\[ A'(\theta) = \frac{1}{2} ab \cos \theta \]

\[ \frac{1}{2} ab \cos \theta = 0 \]

\[ \cos \theta = 0 \]

\[ \theta = 90^\circ, 270^\circ, \ldots \]
11. The height of an object moving vertically is given by \( s = -16t^2 + 96t + 112 \), with \( s \) in ft and \( t \) in sec.


Find

i. the object’s velocity when \( t = 0 \),
   \[
   v(t) = s'(t) = -32t + 96
   \]
   \[
   v(0) = -32(0) + 96 = 96 \text{ ft/sec}
   \]

ii. its maximum height and when it occurs, and
   \[
   v(t) = -32t + 96
   \]
   \[
   -32t + 96 = 0
   \]
   \[
   t = \frac{96}{32} = 3 \text{ sec}
   \]
   \[
   s(t) = -16(t^2) + 96(t) + 112
   \]
   \[
   = -144 + 288 + 112
   \]
   \[
   = 256 \text{ ft}
   \]

iii. its velocity when \( s = 0 \).

\[
-16t^2 + 96t + 112 = 0
\]
\[
-16(t^2 - 6t - 7) = 0
\]
\[
-16(t-7)(t+1) = 0
\]
\[
t = 7 \text{ sec}, t = -1 \text{ not sec}
\]

\[
\begin{align*}
   v(7) &= -32(t) + 96 \\
   &= -224 + 96 \\
   &= -128 \text{ ft/sec}
\end{align*}
\]
12. Jane is 2 mi offshore in a boat and wishes to reach a coastal village 6 mi down a straight shoreline from the point nearest the boat. She can row 2 mph and can walk 5 mph. Where should she land her boat to reach the village in the least amount of time?


\[
D(x) = \frac{\sqrt{x^2+4}}{2} + \frac{6-x}{5}
\]

\[
= \frac{1}{2} \sqrt{x^2+4} + \frac{6}{5} - \frac{1}{5} x
\]

\[
D'(x) = \frac{1}{2} \left( \frac{1}{2} (x+4) - \frac{1}{2} (2x) \right) - \frac{1}{5}
\]

\[
= \frac{x}{2 \sqrt{x^2+4}} - \frac{1}{5}
\]

\[
\frac{\sqrt{2x^2+4}}{2} - \frac{1}{5} = 0
\]

\[
\frac{x}{2 \sqrt{x^2+4}} = \frac{1}{5}
\]

\[
5x = 2 \sqrt{x^2+4}
\]

\[
25x^2 = 4(x^2+4)
\]

\[
25x^2 = 4x^2 + 16
\]

\[
21x^2 = 16
\]

\[
x^2 = \frac{16}{21}
\]

\[
x = \pm \frac{4}{\sqrt{21}}
\]

\[
x = \frac{4}{\sqrt{21}} \approx 0.973 \text{ mi}
\]
14. A rectangle is to be inscribed on the x-axis under the arch of the curve \( y = -x^2 + 3 \). What are the dimensions of the rectangle with **largest** area, and what is the largest area?

\[
A = \omega l
\]

\[
A(x) = 2x\left(-x^2 + 3\right)
\]

\[
= -2x^3 + 6x
\]

**maximize:**

\[
A'(x) = -6x^2 + 6
\]

\[
-6x^2 + 6 = 0
\]

\[
-6(x^2 - 1) = 0
\]

\[
-6(x+1)(x-1) = 0
\]

\[
x = -1, \ x = 1
\]

\[
y = -(1)^2 + 3 = 2
\]

\[
\omega; \ \text{th} = 2x = 2(1) = 2
\]

\[
\text{length} = -x^2 + 3 = 2
\]

\[
A + \text{ca} = 2(2) = 4
\]