GALOIS GROUPS OF SCHUBERT PROBLEMS

A Dissertation

by

ABRAHAM MARTIN DEL CAMPO SANCHEZ

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2012

Major Subject: Mathematics

GALOIS GROUPS OF SCHUBERT PROBLEMS

A Dissertation

by

ABRAHAM MARTIN DEL CAMPO SANCHEZ

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Approved by:

Chair of Committee,	Frank Sottile
Committee Members,	Laura Matusevich
	J. Maurice Rojas
	Gabriel Dos Reis
Head of Department,	Emil Straube

August 2012

Major Subject: Mathematics

ABSTRACT

Galois Groups of Schubert Problems. (August 2012)

Abraham Martin del Campo Sanchez, B.S.,Universidad Nacional Autónoma de México;

M.S., Universidad Nacional Autónoma de México

Chair of Advisory Committee: Dr. Frank Sottile

The Galois group of a Schubert problem is a subtle invariant that encodes intrinsic structure of its set of solutions. These geometric invariants are difficult to determine in general. However, based on a special position argument due to Schubert and a combinatorial criterion due to Vakil, we show that the Galois group of any Schubert problem involving lines in projective space contains the alternating group.

The result follows from a particular inequality of Schubert intersection numbers which are Kostka numbers of two-rowed tableaux. In most cases, the inequality follows from a combinatorial injection. For the remaining cases, we use that these Kostka numbers appear in the tensor product decomposition of $\mathfrak{sl}_2\mathbb{C}$-modules. Interpreting the tensor product as the action of certain Toeplitz matrices and using spectral analysis, the inequality can be rewritten as an integral. We establish the inequality by estimating this integral using only elementary Calculus.

ACKNOWLEDGMENTS

I extend my sincerest thanks to my committee for the support and advice. I am especially grateful to my adviser, Frank Sottile, who challenged to set the highest of goals and provided the unconditional support that gave me the ability to reach them. His wisdom helped me to learn not only from success but from the trials that pave the way to inspiration and discovery. I believe that as long as I carry with me the insight and the passion for mathematics instilled in me by my adviser, my next steps are sure to be in the right direction.

I also wish to express my deepest thanks to my friends and family who have remained by my side with constancy and love. When times were tough, they worked with me; when the work was finished, we celebrated together. They are truly partners in my achievements, and they will always occupy a place of honor in my life and in my heart.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER I

INTRODUCTION

Galois (monodromy) groups of problems from enumerative geometry were first treated by Jordan in 1870 [8], who studied several classical problems with intrinsic structure, showing that their Galois group was not the full symmetric group. The modern theory began with Harris, who showed that the algebraic Galois group is equal to a geometric monodromy group [6] and that many problems had Galois group the full symmetric group. In general, we expect that the Galois group of an enumerative problem is the full symmetric group and when it is not, then the geometric problem possesses some intrinsic structure. For instance, the Cayley-Salmon theorem [4, 15] states that a smooth cubic surface over an algebraic closed field contains 27 lines. These lines are not general, as they satisfy certain incidence relations (e.g. there are nine triplets of lines that meet in one point, there are six skew lines, and each line meets ten other lines), which prevents the corresponding Galois group from being the full symmetric group. In fact, Jordan [8] computed the Galois group of this problem and showed that it was contained in the Weyl group $W(E_6)$, a subgroup of $\mathcal{S}_{27}$. The equality of the Galois group with $W(E_6)$ was proven by Harris [6, §III.2].

The Schubert calculus of enumerative geometry [10] is a method to compute the number of solutions to Schubert problems, which are a class of geometric problems involving linear subspaces. The algorithms of Schubert calculus reduce the enumeration to combinatorics. For example, the number of solutions to a Schubert problem involving lines is a Kostka number, which counts the number of tableaux for a rectangular partition with two parts.

The journal model is SIAM Journal on Discrete Mathematics.

The prototypical Schubert problem is the classical problem of four lines, which asks for the number of lines in space that meet four given lines. To answer this, note that three general lines ℓ_1, ℓ_2, and ℓ_3 lie on a unique doubly-ruled hyperboloid, shown in Figure 1. These three lines lie in one ruling, while the second ruling consists of

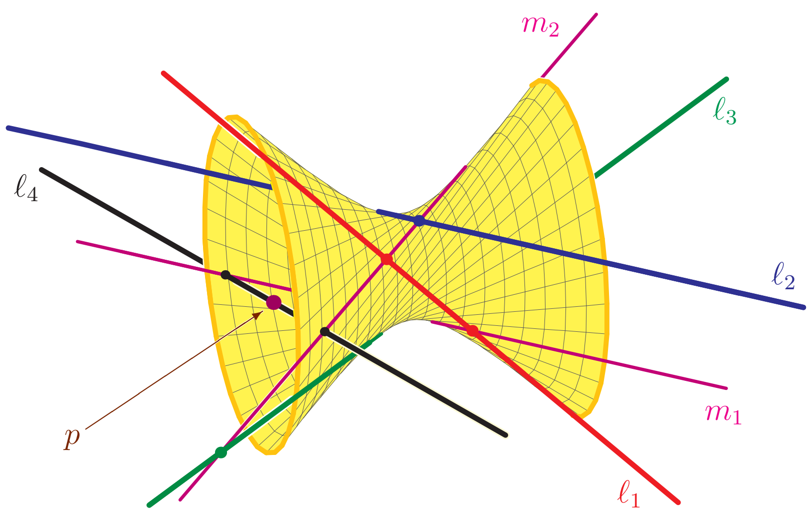

Fig. 1. The two lines meeting four lines in space.

the lines meeting the given three lines. The fourth line ℓ_4 meets the hyperboloid in two points. Each point determines a line in the second ruling, giving two lines m_1 and m_2 which meet our four given lines. In terms of Kostka numbers, the problem of four lines reduces to counting the number of tableaux of shape $\lambda = (2, 2)$ with

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$$

Fig. 2. The two tableaux corresponding to the problem of four lines.

content $(1, 1, 1, 1)$. There exist only two such tableaux as illustrated in Figure 2.

When the field is the complex numbers, Harris' result gives one approach to studying the Galois group—by directly computing monodromy. For instance, the Galois group of the problem of four lines is the group of permutations which are obtained by following the solutions over loops in the space of lines $\ell_1, \ell_2, \ell_3, \ell_4$. Rotating the line ℓ_4 180° about the point p (shown in Figure 1) gives a loop which interchanges the two solution lines m_1 and m_2, showing that the Galois group is the full symmetric group on two letters.

Leykin and Sottile [11] followed this approach, using numerical homotopy continuation [17] to compute monodromy for many *simple* Schubert problems, showing that in each case the Galois group was the full symmetric group. (The problem of four lines is simple.) Billey and Vakil [1] gave an algebraic approach based on elimination theory to compute lower bounds on Galois groups. When the ground field is algebraically closed, Vakil [20] gave a combinatorial criterion, based on classical special position arguments and group theory, which can be used recursively to show that a Galois group contains the alternating group. With this criterion and his geometric Littlewood-Richardson rule [19], he showed that the Galois group of every Schubert problem involving lines in projective space $\mathbb{P}^n$ for $n < 16$ contains the alternating group. We tested Vakil's criterion using his own methods, to find that if $n < 40$, then every Schubert problem involving lines in projective space $\mathbb{P}^n$ has at least alternating Galois group. Inspired by this evidence, we show that for any n, the Galois group of any Schubert problem involving lines in $\mathbb{P}^n$ contains the alternating group. This is Theorem 3.5 in § III.B, and it is the main result of this thesis. This result suggests the absence of intrinsic structure in Schubert problems involving lines.

By Vakil's criterion and a special position argument of Schubert, Theorem 3.5 reduces to a certain inequality among Kostka numbers of two-rowed tableaux. For most cases, the inequality follows from a combinatorial injection of Young tableaux. For the remaining cases, we work in the representation ring of $\mathfrak{sl}_2\mathbb{C}$, where these Kostka numbers also occur. We interpret the tensor product by irreducible $\mathfrak{sl}_2\mathbb{C}$-modules in terms of commuting Toeplitz matrices. Using the eigenvector decomposition of the Toeplitz matrices, we express these Kostka numbers as certain trigonometric integrals. In this way, the inequalities of Kostka numbers become inequalities of integrals, which we establish by estimation.

In contrast with Theorem 3.5, we remark that Galois groups of Schubert problems may not necessarily be full symmetric or alternating in general, in which case we say that the Galois group is deficient. Derksen gave a Schubert problem in the Grassmannian of 4-planes in 8-dimensional space whose Galois group is deficient [20, §3.12]. In [14] a Schubert problem with a deficient Galois group was found in the manifold of flags in 6-dimensional space. Both examples generalize to infinite families of Schubert problems with deficient Galois groups. A more intringuing example is given by Billey and Vakil [1, §7], where they conjecture that a Schubert problem, again in the Grassmannian of 4-planes in 8-dimensional space, has the dihedral group D_4 of order 8 as its Galois group.

CHAPTER II

BACKGROUND

We give an introduction to Schubert calculus based on [5, 7, 12], focusing on those Schubert problems involving lines. We formally introduce their Galois groups, and we explain Vakil's combinatorial criterion to determine if these Galois groups contain the alternating group.

A. Schubert Calculus

The Schubert calculus of enumerative geometry [16, 10] is a method to compute the number of solutions to Schubert problems, which are a class of geometric problems involving linear subspaces of a vector space that have specific positions with respect to other fixed linear spaces. For instance, what are the 3-planes in $\mathbb{C}^7$ meeting 12 (general) fixed 4-planes non-trivially? (There are 462 [16]). The solutions to a Schubert problem are points in the set of k-dimensional linear spaces in $\mathbb{C}^n$.

Definition 2.1. Let $\mathbb{K}$ be a field. The *Grassmannian* $Gr(k, V)$ is the set of all k-dimensional linear subspaces of an n-dimensional $\mathbb{K}$-vector space V. Equivalently, a k-dimensional subspace of V is the same thing as a $(k-1)$-plane in the corresponding projective space $\mathbb{P}V$; in this case we write $\mathbb{G}(k-1, \mathbb{P}V)$. When V is the vector space $\mathbb{K}^n$ we just write $Gr(k, n)$ and $\mathbb{G}(k-1, n-1)$.

For a vector space V with basis $\{v_1, \ldots, v_n\}$ and any positive integer k, we denote by $\bigwedge^k V$ the k^{th} *exterior power* of V, which is generated by $\langle v_{i_1} \wedge \cdots \wedge v_{i_k} \rangle$, and satisfies

$$v_{i_1} \wedge \cdots \wedge v_{i_j} \wedge \cdots \wedge v_{i_l} \wedge \cdots \wedge v_{i_k} = -v_{i_1} \wedge \cdots \wedge v_{i_l} \wedge \cdots \wedge v_{i_j} \wedge \cdots \wedge v_{i_k}.$$

In this way, we realize the Grassmannian as a projective variety as follows. Let $W \subset V$

be a linear subspace of dimension k with basis $\{w_1, \dots, w_k\}$, we define the *Plücker embedding* $Gr(k, V) \to \mathbb{P}(\bigwedge^k V)$, by sending W to the point $\mathbb{K} \cdot (w_1 \wedge \cdots \wedge w_k)$. This map is well-defined for if $\{u_1, \dots, u_k\}$ is another basis for W, then $w_1 \wedge \cdots \wedge w_k = \det(A) \cdot u_1 \wedge \cdots \wedge u_k$, where A is the matrix of change of basis.

Proposition 2.2. *The Plücker map $Gr(k, V) \to \mathbb{P}(\bigwedge^k V)$ is an embedding.*

Proof. For $w = w_1 \wedge \cdots \wedge w_k$ define a map $\psi_w : V \longrightarrow \bigwedge^{k+1} V$ given by $\psi_w(v) = v \wedge w$. Notice that the map ψ_w is linear and $v \wedge w = 0$ if and only if $v \in W$; it follows that $W = \ker(\psi_w)$. From this, we see the Plücker map is an embedding as follows. For $U \in Gr(k, V)$ with basis $\{u_1, \dots, u_k\}$, let $u = u_1 \wedge \cdots \wedge u_k$. If $\mathbb{K} \cdot u = \mathbb{K} \cdot w$, then $\ker(\psi_u) = \ker(\psi_w)$, which implies $U = W$. $\square$

Moreover, $Gr(k, V)$ is a closed subset of $\mathbb{P}(\bigwedge^k V)$, as $\mathbb{K} \cdot w \in \mathbb{P}(\bigwedge^k V)$ lies in $Gr(k, V)$ if and only if $\operatorname{rank}(\psi_w) \leq n-k$, which is a polynomial condition in the coordinates in $\mathbb{P}(\bigwedge^k V)$ of w. Thus, the Grassmannian $Gr(k, V)$ is an algebraic variety.

We now consider a cover of $Gr(k, V)$ by Zariski open sets each isomorphic to the affine space $\mathbb{K}^{k(n-k)}$. For this, let $U \subset V$ be a fixed subspace of dimension $n-k$, and set

$$\mathcal{U}_U = \{W \in Gr(k, V) \mid U \cap W = \{0\}\}. \tag{2.1}$$

If $W \in Gr(k, V)$ has basis $\{w_1, \dots, w_k\}$, we can extend this to a basis $\{w_1, \dots, w_n\}$ of V. Let $U = \langle w_{k+1}, \dots, w_n \rangle$, and notice that $W \in \mathcal{U}_U$. Therefore, we see that $Gr(k, V)$ is covered by the union $\bigcup_U \mathcal{U}_U$ for all $U \in Gr(n-k, V)$. We would like to show that this union is an affine open cover.

Proposition 2.3. *Let $U \subset V$ be a fixed subspace of dimension $n-k$. Then, the set $\mathcal{U}_U$ defined in* (2.1) *is open in $Gr(k, V)$ and $\mathcal{U}_U \cong \mathbb{K}^{k(n-k)}$.*

Proof. Suppose that $\{u_1, \dots, u_{n-k}\}$ is a basis for U and let $u = u_1 \wedge \cdots \wedge u_{n-k}$. We can view u as a linear form on $\mathbb{P}(\bigwedge^k V)$ as follows. For $w \in \bigwedge^k V$ define $u(w) := u \wedge w \in \bigwedge^n V \cong \mathbb{K}$. If $W \subset Gr(k, V)$ corresponds to $w \in \bigwedge^k V$, then $u \wedge w \neq 0$ if and only if $U \cap W = \{0\}$, if and only if $U \oplus W = V$. In other words, $\mathcal{U}_U$ is the complement of the zero set of u in $Gr(k, V)$, so $\mathcal{U}_U$ is an open subset of $Gr(k, V)$.

To see that $\mathcal{U}_U$ is affine, let W_0 be a fixed complement to U, then we will show that $\mathcal{U}_U \cong \operatorname{Hom}(W_0, U) \cong \mathbb{K}^{k(n-k)}$. For $f \in \operatorname{Hom}(W_0, U)$ we associate to it its graph $V_f := \{(x, f(x)) \mid x \in W_0\}$, which is a subset of $W_0 \oplus U = V$, and thus, a subspace of V. Notice that $V_f \cap U = 0$ and if $\{w_1, \ldots, w_k\}$ is a basis for W_0, then $\{(w_1, f(w_1)), \ldots, (w_k, f(w_k))\}$ is an independent set in V_f, showing that $\dim V_f = k$. Also, given $W \in Gr(k, V)$ such that $W \oplus U = V$, we can see that W arises as a graph of some $f \in \operatorname{Hom}(W_0, U)$. For $w \in W_0$, write $w = x + y$, with $x \in W$ and $y \in U$. Then, we define $f(w) = -y$, so $x = w + f(w)$. Thus, we can identify the set $\mathcal{U}_U$ with $\operatorname{Hom}(W_0, U)$. Moreover, the identification $\mathcal{U}_U \cong \operatorname{Hom}(W_0, U) \cong \mathbb{K}^{k(n-k)}$ respects the Zariski topology; therefore, $\mathcal{U}_U \cong \operatorname{Hom}(W_0, U) \cong \mathbb{K}^{k(n-k)}$. $\square$

Since $Gr(k, V)$ can be covered by dense open sets, each isomorphic to $\mathbb{K}^{k(n-k)}$, an immediate consequence of Proposition 2.3 is the following.

Corollary 2.4. *The dimension of the Grassmannian $Gr(k, V)$ is $k(n-k)$.*

In the Schubert Calculus, we are interested in describing the conditions for a k-dimensional subspace in $\mathbb{K}^n$ to intersect a sequence of linear subspaces in a prescribed way. The specified positions of the k-planes are in reference to flags in $\mathbb{K}^n$.

Definition 2.5. A (complete) *flag* $F_\bullet$ is a sequence of linear subspaces

$$F_\bullet : F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = \mathbb{K}^n, \text{ where } \dim F_i = i.$$

The possible positions in which a k-plane may sit with respect to a flag $F_\bullet$ are encoded by partitions.

Definition 2.6. A *partition* λ is a weakly decreasing sequence of integers $\lambda : (n-k) \geq \lambda_1 \geq \cdots \geq \lambda_k \geq 0$. To a partition λ we associate a *Young diagram*, which consists of a set of boxes, arranged in left-justified rows, having λ_i boxes in the i-th row, and we define $|\lambda| := \sum_{i=1}^{k} \lambda_i$.

Example 2.7. For $n = 6$ and $k = 3$, the sequence $\lambda = (3, 2, 1)$ is a partition with $|\lambda| = 3+2+1 = 6$. Its corresponding Young diagram is presented in Figure 3.

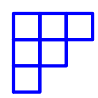

Fig. 3. Young diagram for the partition $\lambda = (3, 2, 1)$.

Definition 2.8. Given a partition λ and a flag $F_\bullet$, we define the *Schubert variety* $\Omega_\lambda F_\bullet$ of $Gr(k, n)$, as

$$\Omega_\lambda F_\bullet := \{H \in Gr(k, n) \mid \dim H \cap F_{n-k+i-\lambda_i} \geq i,\ i = 1, \ldots, k\}. \tag{2.2}$$

A flag $F_\bullet$ in V defines a flag $E_\bullet$ in $\mathbb{P}V$ by letting $E_i = \mathbb{P}F_{i+1}$ for $i = 1, \ldots, n-1$. When the Schubert variety Ω_λ is considered as a subvariety of $\mathbb{G}(k-1, n-1)$, then it consists of the $(k-1)$-planes $\tilde{H}$ in $\mathbb{P}V$ satisfying $\dim \tilde{H} \cap E_{n-k+1+i-\lambda_i} \geq i$ for $i = 1, \ldots, k-1$. This is completely equivalent to (2.2).

Example 2.9. For the partition $\lambda = (1, 0, \ldots, 0) = \square$, the corresponding Schubet variety is $\Omega_\square F_\bullet = \{H \in Gr(k, n) \mid \dim H \cap F_{n-k}\}$. In the space $\mathbb{G}(1, 3)$, the Schubert variety $\Omega_\square F_\bullet$ consists of the lines meeting a fixed line.

Schubert varieties contain an important family of affine open subsets, which we describe next.

Definition 2.10. Given a partition λ and a flag $F_\bullet$, we define the *Schubert cell* $\Omega^\circ_\lambda F_\bullet$ of $Gr(k,n)$, as

$$\Omega^\circ_\lambda F_\bullet := \{H \in Gr(k,n) \mid \dim H \cap F_j = i, \text{ if } n-k+i-\lambda_i \leq j \leq n-k+i-\lambda_{i+1}\}. \tag{2.3}$$

Let $\{e_1, \dots, e_n\}$ be a basis for $\mathbb{K}^n$, so that the flag $F_\bullet$ is defined by letting F_i be the span of $e_{n+1-i}, e_{n+2-i}, \dots, e_{n-1}, e_n$. Then, the Schubert cell $\Omega^\circ_\lambda F_\bullet$ consists of k-planes that are the row span of a reduced row echelon matrix

$$\begin{bmatrix} 0\cdots 0 & 1 & *\cdots * & 0 & *\cdots * & 0 & *\cdots * \\ 0\cdots 0 & 0 & 0\cdots 0 & 1 & *\cdots * & 0 & *\cdots * \\ & \vdots & & \vdots & \ddots & \vdots & *\cdots * \\ 0\cdots 0 & 0 & 0\cdots 0 & 0 & 0\cdots 0 & 1 & *\cdots * \end{bmatrix} \tag{2.4}$$

with a 1 in the ith row at column $i+\lambda_{k-i+1}$, where $*$ represents any number. Note that if we let H to be the row span of the matrix (2.4) when $|\lambda| = 0$ (this is, when $\lambda_i = 0$, for all $i = 1, \dots, k$), then the stars in (2.4) give local coordinates of the point $H \in Gr(k,n)$. By changing bases, these give coordinate charts on the Grassmannian, giving it a manifold structure.

The Schubert variety $\Omega_\lambda F_\bullet$ is the closure of the Schubert cell $\Omega^\circ_\lambda F_\bullet$, thus Schubert cells are dense open subsets.

Proposition 2.11. *For a partition λ and a flag $F_\bullet$, the Schubert variety $\Omega_\lambda F_\bullet$ is an algebraic subset of $Gr(k,n)$ of codimension $|\lambda|$.*

Proof. The condition of $H \cap F_{n-k+i-\lambda_i}$ having dimension at least i can be expressed, in terms of local coordinates, as the vanishing of the minors of order $n+1-\lambda_i$ of a matrix representation of the linear span $\langle H, F_{n-k+i-\lambda_i}\rangle$. Since $\Omega_\lambda F_\bullet$ is defined by such incidence conditions, it is therefore an algebraic subvariety of $Gr(k,n)$.

To see that the codimension of $\Omega_\lambda F_\bullet$ is $|\lambda|$, we consider the Schubert cell $\Omega^\circ_\lambda F_\bullet$, which is a dense open set in $\Omega_\lambda F_\bullet$. Since in (2.4), there are $k^2+|\lambda|$ specified entries and the rest are completely free, we have a homeomorphism $\Omega^\circ_\lambda F_\bullet \cong \mathbb{K}^{k(n-k)-|\lambda|}$. $\square$

Definition 2.12. A *Schubert problem* is a list $(\lambda^1,\ldots,\lambda^m)$ of partitions such that $|\lambda^1|+\cdots+|\lambda^m| = k(n-k)$.

For a Schubert problem $(\lambda^1,\ldots,\lambda^m)$, let $F^1_\bullet,\ldots,F^m_\bullet$ be fixed flags in general position and consider the intersection

$$\Omega_{\lambda^1}F^1_\bullet \cap \Omega_{\lambda^2}F^2_\bullet \cap \cdots \cap \Omega_{\lambda^m}F^m_\bullet \ . \tag{2.5}$$

For fields of characteristic zero, Kleiman [9] showed that the intersection (2.5) is transverse. For fields with positive characteristic, the transversality is due to Sottile [18] when $k=2$, and to Vakil [19] in general. Since the codimension of the intersection (2.5) is precisely the dimension of the Grassmannian (the ambient space), the transversality implies that the intersection (2.5) consists of finitely many k-planes (points in $Gr(k,n)$), as it is a zero-dimensional variety. The number of points in the intersection (2.5) does not depend upon the choice of general $F^1_\bullet,\ldots,F^m_\bullet$. We call this number the *Schubert intersection number* $d(\lambda^1,\ldots,\lambda^m)$, and we say it is the number of solutions to the Schubert problem.

Example 2.13. From Example 2.9, the Schubert problem $(\square,\square,\square,\square)$ in $\mathbb{G}(1,3)$, consists of the lines in $\mathbb{P}^3$ that meet four fixed lines. Therefore, $d(\square,\square,\square,\square)=2$ as explained in the introduction and illustrated in Figure 1.

Example 2.14. The Schubert problem $(\boxplus,\boxplus,\boxplus,\boxplus)$ in $Gr(4,8)$ asks for the 4-dimensional subspaces of $\mathbb{C}^8$ that meet four general 4-planes in a 2-dimensional subspace; in other words, if K_1, K_2, K_3, K_4 are four fixed 4-dimensional subspaces of $\mathbb{C}^4$, how many $H \in Gr(4,8)$ satisfy $\dim H \cap K_i = 2$ for $i = 1, \ldots, 4$? This problem has six solutions, so $d(\boxplus,\boxplus,\boxplus,\boxplus) = 6$. This is the first Schubert problem known where the Galois group is not the full symmetric group $\mathcal{S}_6$, and it is due to Derksen, who showed that the Galois group for this problem is in fact $\mathcal{S}_4$.

B. Schubert problems of lines in projective space

In this work, we are mainly interested in Schubert problems that involve lines in the projective space $\mathbb{P}^n$ meeting other fixed linear subspaces.

From the discussion of the previous section, the Grassmannian $\mathbb{G}(1,n)$ of lines in $\mathbb{P}^n$ is an algebraic manifold of dimension $2n-2$. For this case, given a partition $\lambda = (\lambda_1, \lambda_2)$ and a flag $F_\bullet$, a Schubert variety is

$$\Omega_\lambda F_\bullet = \{\ell \in \mathbb{G}(1,n) \mid \ell \cap F_{n-1-\lambda_1} \neq \varnothing \quad \text{and} \quad \ell \subset F_{n-\lambda_2}\}.$$

We simplify our notation by letting $L := F_{n-1-\lambda_1}$ and $\Lambda := F_{n-\lambda_2}$; these are the relevant parts of the flag $F_\bullet$ that meet a line ℓ in the Schubert variety Ω_λ. Thus we denote a flag by $L \subset \Lambda \subseteq \mathbb{P}^n$. In this setting, we denote Schubert varieties of $\mathbb{G}(1,n)$ by

$$\Omega(L{\subset}\Lambda) \ := \ \Omega_\lambda F_\bullet \ = \ \{\ell \in \mathbb{G}(1,n) \mid \ell \cap L \neq \varnothing \quad \text{and} \quad \ell \subset \Lambda\}\,. \tag{2.6}$$

A Schubert problem on $\mathbb{G}(1,n)$ asks for the lines that meet a fixed, but general collection of flags $L_1{\subset}\Lambda_1, \ldots, L_m{\subset}\Lambda_m$. This set of lines is described by the intersection of Schubert varieties

$$\Omega(L_1{\subset}\Lambda_1) \,\cap\, \Omega(L_2{\subset}\Lambda_2) \,\cap\, \cdots \,\cap\, \Omega(L_m{\subset}\Lambda_m)\,. \tag{2.7}$$

Schubert [16] gave a recursion for determining the number of solutions to a Schubert problem in $\mathbb{G}(1,n)$, when there are finitely many solutions. The geometry behind his recursion is central to our proof, and we will present it in Chapter III.

Definition 2.15. When $\Lambda = \mathbb{P}^n$, we may omit Λ and write $\Omega_L := \Omega(L{\subset}\mathbb{P}^n)$, which is a *special Schubert variety*.

Remark 2.16. Note that $\Omega(L{\subset}\Lambda) = \Omega_L$, the latter considered as a subvariety of $\mathbb{G}(1, \Lambda)$. Given $L{\subset}\Lambda$ and $L'{\subset}\Lambda'$, if we set $M := L \cap \Lambda'$ and $M' := L' \cap \Lambda$, then a line $\ell \in \Omega(L{\subset}\Lambda) \cap \Omega(L'{\subset}\Lambda')$ is contained in $\Lambda \cap \Lambda'$ and it meets both M and M'; thus,

$$\Omega(L{\subset}\Lambda) \cap \Omega(L'{\subset}\Lambda') \;=\; \Omega_M \cap \Omega_{M'}\,,$$

the latter intersection taking place in $\mathbb{G}(1, \Lambda \cap \Lambda')$.

Given a Schubert problem (2.7), if we set $\Lambda := \Lambda_1 \cap \cdots \cap \Lambda_m$ and $L'_i := L_i \cap \Lambda$, for each $i = 1, \ldots, m$, then we may rewrite (2.7) as

$$\Omega_{L'_1} \cap \Omega_{L'_2} \cap \cdots \cap \Omega_{L'_m}\,,$$

inside $\mathbb{G}(1, \Lambda)$. Thus, it will suffice to study intersections of special Schubert varieties.

$\square$

Suppose that $\dim L = n{-}1{-}a$, for some positive integer a. Then Ω_L has codimension a in $\mathbb{G}(1, n)$. If $a_\bullet := (a_1, \ldots, a_m)$ is a list of positive integers such that $a_1 + \cdots + a_m = 2n{-}2 = \dim \mathbb{G}(1, n)$, then $a_\bullet$ is a Schubert problem, so if we consider general linear subspaces $L_1, \ldots, L_m$ of $\mathbb{P}^n$ where $\dim L_i = n{-}1{-}a_i$ for $i = 1, \ldots, m$, then the intersection

$$\Omega_{L_1} \cap \Omega_{L_2} \cap \cdots \cap \Omega_{L_m} \tag{2.8}$$

is transverse and therefore zero-dimensional. We call $a_\bullet$ the *type* of the Schubert intersection (2.8).

Remark 2.17. Given positive integers $a_\bullet = (a_1, \ldots, a_m)$ whose sum is even, set $n(a_\bullet) := \frac{1}{2}(a_1 + \cdots + a_m + 2)$. Thus, we do not need to specify n. Henceforth, a

Schubert problem in $\mathbb{G}(1,n)$ will be a list $a_\bullet$ of positive integers with even sum. Since we require that $\dim L_i \geq 0$ for all $i = 1, \ldots, m$, we must have $a_i \leq n(a_\bullet){-}1$.

Definition 2.18. A Schubert problem $a_\bullet$ in $\mathbb{G}(1,n)$ is *valid* if $a_i \leq n(a_\bullet){-}1$ for all $i = 1, \ldots, m$.

For a Schubert problem $a_\bullet$ in $\mathbb{G}(1,n)$, its intersection number $d(a_1, \ldots, a_m)$ is a *Kostka number*, which is the number of Young tableaux of shape $(n(a_\bullet){-}1, n(a_\bullet){-}1)$ and content $(a_1, \ldots, a_m)$ [5, p.25]. These are arrays consisting of two rows of integers, each of length $n(a_\bullet){-}1$ such that the integers increase weakly across each row and strictly down each column, and there are a_i occurrences of i for each $i = 1, \ldots, m$. Let $\mathcal{K}(a_\bullet)$ be the set of such tableaux.

Example 2.19. As observed in Example 2.13, we have $d(\square, \square, \square, \square) = 2$, as it corresponds to the problem of four lines. The set $\mathcal{K}(\square, \square, \square, \square)$ consists of the two tableaux illustrated in Figure 2.

Example 2.20. Figure 4 displays the Young tableaux in $\mathcal{K}(2,2,1,2,3)$, showing that $d(2,2,1,2,3) = 5$.

$$\begin{array}{|c|c|c|c|c|}\hline 1&1&2&2&3\\\hline 4&4&5&5&5\\\hline\end{array}\quad\begin{array}{|c|c|c|c|c|}\hline 1&1&2&2&4\\\hline 3&4&5&5&5\\\hline\end{array}\quad\begin{array}{|c|c|c|c|c|}\hline 1&1&2&3&4\\\hline 2&3&5&5&5\\\hline\end{array}\quad\begin{array}{|c|c|c|c|c|}\hline 1&1&2&4&4\\\hline 2&3&5&5&5\\\hline\end{array}\quad\begin{array}{|c|c|c|c|c|}\hline 1&1&3&4&4\\\hline 2&2&5&5&5\\\hline\end{array}$$

Fig. 4. The five Young tableaux in $\mathcal{K}(2,2,1,2,3)$.

C. Reduced Schubert problems

A simple geometric argument shows that it suffices to consider only certain types of Schubert problems.

Definition 2.21. A Schubert problem $a_\bullet$ is *reduced* if $a_i{+}a_j < n(a_\bullet)$ for any $i<j$.

We may assume that Schubert problems in $\mathbb{G}(1,n)$ are reduced.

Lemma 2.22. *Every Schubert problem in* $\mathbb{G}(1,n)$ *may be recast as an equivalent reduced Schubert problem.*

Proof. Let $a_\bullet = (a_1,\ldots,a_m)$ be a Schubert problem in $\mathbb{G}(1,n)$. If $a_\bullet$ is not reduced, we assume without loss of generality that $a_1{+}a_2 \geq n(a_\bullet)$ and set $n := n(a_\bullet)$. Suppose that $L_1,\ldots,L_m \subset \mathbb{P}^n$ are linear subspaces in general position with $\dim L_i = n{-}1{-}a_i$ for $i=1,\ldots,m$. Since $a_1+a_2 > n{-}1$, then

$$\dim L_1 + \dim L_2 = 2(n{-}1) - (a_1{+}a_2) < 2(n{-}1){-}(n{-}1) = n{-}1;$$

as the subspaces L_1 and L_2 are in general position, they are disjoint and do not span $\mathbb{P}^n$. Every line ℓ in

$$\Omega_{L_1}\cap\Omega_{L_2} \;=\; \{\ell\in\mathbb{G}(1,n)\mid \ell\cap L_i\neq\varnothing \text{ for } i=1,2\}$$

is spanned by its intersections with L_1 and L_2. Thus ℓ lies in the linear span $\langle L_1,L_2\rangle$, which is a proper linear subspace of $\mathbb{P}^n$. Let Λ be a general hyperplane containing $\langle L_1,L_2\rangle$. If we set $L_i' := L_i\cap\Lambda$ for $i=1,\ldots,m$, then a line ℓ that meets each L_i must lie in Λ, thus it must meet L_i' for $i=1,\ldots,m$. Therefore, we have

$$\Omega_{L_1}\cap\Omega_{L_2}\cap\cdots\cap\Omega_{L_m} \;=\; \Omega_{L_1'}\cap\Omega_{L_2'}\cap\cdots\cap\Omega_{L_m'}\,, \tag{2.9}$$

the latter intersection in $\mathbb{G}(1,\Lambda) \cong \mathbb{G}(1,n-1)$. For $i=1,2$, we have $L'_i = L_i$ and so

$$\dim L'_i \;=\; n-1-a_i \;=\; (n-1)-1-(a_i-1) \;=\; \dim\Lambda-1-(a_i-1)\,,$$

and if $i>2$, then

$$\dim L'_i \;=\; n-1-a_i-1 \;=\; (n-1)-1-a_i \;=\; \dim\Lambda-1-a_i\,. \tag{2.10}$$

Thus the righthand side of (2.9) is a Schubert problem of type '$a'_\bullet := (a_1-1, a_2-1, a_3,\ldots,a_m)$, and so we have

$$d(a_1,\ldots,a_m) \;=\; d(a_1-1,a_2-1,a_3,\ldots,a_m)\,.$$

Notice that $a'_1+a'_2 = a_1+a_2-2$ and $n(a'_\bullet) = n(a_\bullet)-1$, so the difference $a'_1+a'_2-n(a'_\bullet)$ is strictly smaller than $a_1+a_2-n(a_\bullet)$. The lemma follows by applying recursively this procedure to $a'_\bullet$ until we obtain a reduced Schubert problem. $\square$

We may also understand Lemma 2.22 combinatorially: the condition $a_1+a_2 \geq n(a_\bullet)$ implies that the first column of every tableaux in $\mathcal{K}(a_\bullet)$ consists of a 1 on top of a 2. Removing this column gives a tableaux in $\mathcal{K}(a'_\bullet)$, and this defines a bijection between these two sets of tableaux. This is illustrated in the following example.

Example 2.23. For $a_\bullet = (3,3,2,2)$ let us consider $\mathcal{K}(3,3,2,2)$, which is depicted in Figure 5. Notice that $n(3,3,2,2)=6$, and that the first two entries satisfy $a_1+a_2 > n(a_\bullet)-1$. Consider $a'_\bullet = (a_1-1,a_2-1,a_3,a_4) = (2,2,2,2)$. The set $\mathcal{K}(2,2,2,2)$ is also presented in Figure 5.

Since the first column of each tableaux in $\mathcal{K}(3,3,2,2)$ consists of a 1 on top of a 2, we give a bijection between $\mathcal{K}(3,3,2,2)$ and $\mathcal{K}(2,2,2,2)$, by erasing the first column in each tableaux of $\mathcal{K}(3,3,2,2)$.

$$\begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&2\\\hline 2&3&3&4&4\\\hline\end{array}\quad\begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&3\\\hline 2&2&3&4&4\\\hline\end{array}\quad\begin{array}{|c|c|c|c|c|}\hline 1&1&1&3&3\\\hline 2&2&2&4&4\\\hline\end{array}\qquad\begin{array}{|c|c|c|c|}\hline 1&1&2&2\\\hline 3&3&4&4\\\hline\end{array}\quad\begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 2&3&4&4\\\hline\end{array}\quad\begin{array}{|c|c|c|c|}\hline 1&1&3&3\\\hline 2&2&4&4\\\hline\end{array}$$

Fig. 5. The tableaux in $\mathcal{K}(3,3,2,2)$ and $\mathcal{K}(2,2,2,2)$.

D. Galois groups

Associated to a Schubert problem is its Galois group, which is a geometric invariant that encodes intrinsic structure of the problem. Not much is known about these geometric invariants for enumerative problems. However, based on his geometric Littlewood-Richardson rule [19] and a simple group theoretical argument, Ravi Vakil [20] deduced a combinatorial criterion to determine when these Galois groups contain the alternating group. We summarize Vakil's presentation in [20, § 5.3]. We start by recalling some algebraic geometry.

Given an irreducible variety X and an open set $U \subset X$, let $\mathcal{O}_X(U)$ be the ring of regular functions on U. Then $\mathcal{O}_X(U)$ is an integral domain. A *rational function* on X is an equivalence class of pairs (U, f), where U is a non-empty open set in X and $f \in \mathcal{O}_X(U)$. Two pairs (U, f) and (V, g) are equivalent if $f = g$ in $U \cap V$. The function field $\mathbb{K}(X)$ is the field of rational functions. We can realize the field $\mathbb{K}(X)$ also as the field of fractions of $\mathcal{O}_X(U)$ for every open U.

Definition 2.24. A morphism $f : W \to X$ of irreducible algebraic varieties is called *dominant* if the image of f is dense in X.

A dominant morphism $f : W \to X$ induces a morphism of function fields $\mathbb{K}(X) \to \mathbb{K}(W)$, because for every $g \in K(X)$, there is a neighborhood $U \in X$ such that $g \in \mathcal{O}_X(U)$. Since f is dense, the image of W is dense, thus the pre-image $f^{-1}(U)$ is a non-empty open set in W. Therefore, define $\mathbb{K}(X) \to \mathbb{K}(W)$ by $f \mapsto f^*g$, where $f^*g := g \circ f \in \mathcal{O}_W(f^{-1}(U)) \subset \mathbb{K}(W)$. This map is well defined as the pullback f^* sends different representatives of an element in $\mathbb{K}(X)$ to representatives of the same element in $\mathbb{K}(W)$.

Definition 2.25. A $f : W \to X$ is a dominant morphism is of *degree* d if the induced morphism $\mathbb{K}(X) \to \mathbb{K}(W)$ is a finite degree d field extension.

Suppose that $\pi\colon W \to X$ is a dominant morphism of degree d between irreducible algebraic varieties of the same dimension defined over an algebraically closed field $\mathbb{K}$. We will assume here and throughout that π is generically separable in that the corresponding extension $\pi^*(\mathbb{K}(X)) \hookrightarrow \mathbb{K}(W)$ of function fields is separable. In this case, define the *Galois group* $\mathrm{Gal}_{W\to X}$ of this map to be the Galois group of the Galois closure of the field extension $\mathbb{K}(W)/\pi^*(\mathbb{K}(X))$. This is a subgroup of the symmetric group $\mathcal{S}_d$ on d letters. We say that $\mathrm{Gal}_{W\to X}$ is *at least alternating* if it is $\mathcal{S}_d$ or its alternating subgroup. Vakil's criterion addresses how $\mathrm{Gal}_{W\to X}$ is affected by the Galois group of a restriction of $\pi\colon W \to X$ to a subvariety $Z \subset X$.

Suppose that $\mathbb{K}$ is the field complex numbers $\mathbb{C}$ and x is a regular value of π. Replacing W and X by Zariski open subsets if necessary, we can realize the map $\pi : W \to X$ as a degree d covering. A loop in X based at x has d lifts to W, one for each point in the fiber $\pi^{-1}(x)$. Associating a point in the fiber $\pi^{-1}(x)$ to the endpoint of the corresponding lift gives a permutation in $\mathcal{S}_d$. This defines the usual permutation action of the fundamental group of X on the fiber $\pi^{-1}(x)$. The *monodromy group* of the map $\pi : W \to X$ is the group of permutations of $\pi^{-1}(X)$

which are obtained by lifting closed paths based at x that lie in the set of regular values of π. Harris [6] showed that $\mathrm{Gal}_{W\to X}$ equals the monodromy group.

A *divisor* is a formal linear combination of irreducible subvarieties of X of codimension one. A divisor Z is *Cartier* if it is locally defined by a single equation (that is locally around each point, each component of Z is the zero locus of a regular function). Given a Cartier divisor Z of X, we say that X is *smooth along* Z if each irreducible component of Z meets the smooth locus of X.

The key point of considering Cartier divisors is that they pullback to Cartier divisors under dominant maps: for a given a dominant morphism $\pi : W \to X$ of degree d, the image of W does not lie on any Cartier divisor Z (as $\pi(W)$ is dense), then we can define a Cartier divisor on W by the pullback of the defining equations of Z.

Suppose that we have a fiber diagram

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d, "\pi"'] & W \arrow[d, "\pi"] \\ Z \arrow[r, hook] & X \end{tikzcd} \tag{2.11}$$

where $Z \hookrightarrow X$ is the closed embedding of a Cartier divisor Z of X, X is smooth in codimension one along Z, and $\pi\colon Y \to Z$ is a generically separable, dominant morphism of degree d.

Theorem 2.26 (Vakil's Criterion). *Suppose that Y is either irreducible or has two components, we have the following.*

(a) *If Y is irreducible and $\mathrm{Gal}_{Y\to Z}$ is at least alternating, then $\mathrm{Gal}_{W\to X}$ is at least alternating.*

(b) *If Y has two components, Y_1 and Y_2, each of which maps dominantly to Z of respective degrees d_1 and d_2. If $\mathrm{Gal}_{Y_1\to Z}$ and $\mathrm{Gal}_{Y_2\to Z}$ are at least alternating,* *and if either* $d_1 \neq d_2$ *or* $d_1 = d_2 = 1$*, then* $\mathrm{Gal}_{W\to X}$ *is at least alternating.*

Vakil's Criterion follows by the following group-theoretic argument based on Goursat's Lemma, which we recall first.

Lemma 2.27 (Goursat's Lemma)**.** *Let* $H \subset G_1 \times G_2$ *be a subgroup such that the projections* $H \to G_i$ $(i = 1, 2)$ *are surjective. Then there are normal subgroups* $N_i \triangleleft G_i$ $(i = 1, 2)$ *and an isomorphism* $\phi : G_1/N_1 \xrightarrow{\sim} G_2/N_2$ *such that* $(g_1, g_2) \in H$ *if and only if* $\phi(g_1N_1) = g_2N_2$*.*

Proposition 2.28 (Proposition 5.7 in [20])**.** *Let* G *be a transitive subgroup of* $\mathcal{S}_{a+b}$*. Suppose there is a subgroup* $H \subset G \cap (\mathcal{S}_a \times \mathcal{S}_b)$ *such that the projection* $H \to \mathcal{S}_i$ *(i=a,b) contains the alternating group* A_i*.*

1. *If* $a \neq b$*, then* G *contains the alternating group* A_{a+b}*.*
2. *If* $a = b = 1$*, then* $G = \mathcal{S}_2$*.*

Sketch of the proof of Vakil's criterion. We only give the proof when $\mathbb{K} = \mathbb{C}$, for the general proof refer to [20, Remark 3.5]. In this case, we use Harris' approach and we realize $\mathrm{Gal}_{W\to X}$ as the monodromy group. In Case (a), Y is irreducible and $\pi : Y \to Z$ is a generically separable, dominant morphism of degree d. We obtain the Galois group $\mathrm{Gal}_{Y\to Z}$ by lifting closed paths in the smooth locus of Z based on a regular point $x \in Z$ to a permutation of $\pi^{-1}(x) \subset Y$. Since X is smooth in codimension one along Z, closed paths in the smooth locus of Z at x are also paths in the smooth locus of X. This provides an inclusion $\mathrm{Gal}_{Y\to Z} \hookrightarrow \mathrm{Gal}_{W\to X}$. In particular, if the first group is at least alternating, so is the second.

In Case (b), Y has two components Y_1 and Y_2 that map dominantly to Z. Lifting closed paths in the smooth locus of Z produces a subgroup H of $\mathrm{Gal}_{Y_1\to Z} \times \mathrm{Gal}_{Y_2\to Z}$, such that the projections $\mathrm{Gal}_{Y_i\to Z}$ (for $i = 1, 2$) are surjective. The group H injects into $\mathrm{Gal}_{W\to X}$ via the induced inclusion $\mathcal{S}_{d_1}\times\mathcal{S}_{d_2}\hookrightarrow\mathcal{S}_d$. Notice that W is connected as it is irreducible. Therefore, $\mathrm{Gal}_{W\to X}$ is transitive (any two points in the fiber of a regular value $x\in X$ are connected by a path, which is the lift of the loop coming from the projection of the path to X). By Proposition 2.28, if $\mathrm{Gal}_{Y_i\to Z}$ is at least alternating (for $i=1,2$), so it is $\mathrm{Gal}_{W\to X}$. $\square$

Remark 2.29. This criterion applies to more general inclusions $Z\hookrightarrow X$ of an irreducible variety into X. All that is needed is that X is generically smooth along Z, for then we may replace X by an affine open set meeting Z and there are subvarieties $Z=Z_0\subset Z_1\subset\cdots\subset Z_m=X$ with each inclusion $Z_{i-1}\subset Z_i$ that of a Cartier divisor where Z_i is smooth in codimension one along Z_{i-1}. $\square$

Given a Schubert problem $a_\bullet$, let $n:=n(a_\bullet)$, and set

$$X \ := \ \{(L_1,\dots,L_m)\mid L_i\subset\mathbb{P}^n \text{ is a linear space of dimension } n{-}1{-}a_i\}\,.$$

Consider the total space of the Schubert problem $a_\bullet$,

$$W \ := \{(\ell,L_1,\dots,L_m)\in\mathbb{G}(1,\mathbb{P}^n)\times X\mid \ell\cap L_i\neq\emptyset\,,\ i=1,\dots,m\}\,.$$

Let $p:W\to\mathbb{G}(1,\mathbb{P}^n)$ be the projection to the first coordinate. The fiber over a point $\ell\in\mathbb{G}(1,\mathbb{P}^n)$ is

$$p^{-1}(\ell)=\Omega_{a_1}F_\bullet\times\Omega_{a_2}F_\bullet\times\cdots\times\Omega_{a_m}F_\bullet\,,$$

where each $\Omega_{a_i}F_\bullet$ is a Schubert variety in $\mathbb{G}(n{-}1{-}a_i)$ with respect to a flag $F_\bullet$ such that $F_1=\ell$. Thus, p realizes W as a fiber bundle of $\mathbb{G}(1,n)$ with irreducible fibers. As $\mathbb{G}(1,n)$ is irreducible, W is irreducible. Let $\pi\colon W\to X$ be the other projection. Its fiber over a point $(L_1,\dots,L_m)\in X$ is

$$\pi^{-1}(L_1,L_2,\dots,L_m)\ =\ \Omega_{L_1}\cap\Omega_{L_2}\cap\cdots\cap\Omega_{L_m}\,. \tag{2.12}$$

In this way, the map $\pi\colon W \to X$ contains all Schubert intersections of type $(a_1, \ldots, a_m)$. As the general Schubert problem is a transverse intersection containing $d(a_1, \ldots, a_m)$ points, π is generically separable, and it is a dominant (in fact surjective) map of degree $d(a_1, \ldots, a_m)$.

Definition 2.30. The Galois group $\mathrm{Gal}(a_\bullet)$ of the Schubert problem of type $a_\bullet$ is the Galois group of $\pi\colon W \to X$, where W and X are as defined above. $\square$

CHAPTER III

SCHUBERT'S DEGENERATION

We present the main result of this thesis which states that the Galois group of Schubert problems involving lines is at least alternating. We explain how a special position argument of Schubert [16] together with Vakil's criterion reduces the proof of the main result to establishing an inequality of Kostka numbers. In many cases, the inequality follows from simple counting. The remaining cases are treated in Chapter IV. We also give two infinite families of Schubert problems whose Galois groups are the full symmetric group.

A. Schubert's degeneration

Suppose that M_1 and M_2 are linear subspaces of $\mathbb{P}^n$ in general position such that for $i=1,2$, we have $\dim M_i = n{-}1{-}b_i$ for some positive integers b_1, b_2. Note that M_1, M_2 correspond to linear subspaces N_1, N_2 of $\mathbb{K}^{n+1}$ of dimension $n{-}b_1$ and $n{-}b_2$ respectively. If we assume that $\dim N_1 + \dim N_2 \geq \dim \mathbb{K}^{n+1}$, then $n{-}b_1 + n{-}b_2 \geq n{+}1$, which is equivalent to assume $b_1{+}b_2 \leq n{-}1$. In this case N_1 and N_2 linearly span $\mathbb{K}^{n+1}$. Equivalently, if $b_1{+}b_2 \leq n{-}1$, then the linear span of M_1 and M_2 is $\mathbb{P}^n$ We begin with a simple observation due to Schubert [16].

Lemma 3.1. *Let b_1, b_2 be positive integers with $b_1 + b_2 \leq n{-}1$, and suppose that $M_1, M_2 \subset \mathbb{P}^n$ are linear subspaces with $\dim M_i = n{-}1{-}b_i$ for $i=1,2$. If M_1 and M_2 are in special position in that their linear span $\langle M_1, M_2\rangle$ is a hyperplane Λ, then*

$$\Omega_{M_1} \cap \Omega_{M_2} \;=\; \Omega_{M_1\cap M_2} \bigcup \Omega(M_1{\subset}\Lambda) \cap \Omega_{M_2'}\,, \tag{3.1}$$

where M_2' is any linear subspace of dimension $n{-}b_2$ of $\mathbb{P}^n$ with $M_2' \cap \Lambda = M_2$. Fur-

thermore, the intersection $\Omega_{M_1}\cap\Omega_{M_2}$ is generically transverse, (3.1) *is its irreducible decomposition, and the second intersection of Schubert varieties is also generically transverse.*

Proof. Let M_1, M_2 be in special position. First notice that $M_1\subset\Lambda$ implies $M_2'\cap M_1 = M_2'\cap(\Lambda\cap M_1) = M_2\cap M_1$. If ℓ meets both M_1 and M_2, then either it meets $M_1\cap M_2$ or it lies in their linear span (as ℓ is spanned by its intersection with M_1 and M_2). Hence $\ell\subset\Lambda$ and ℓ meets $\Lambda\cap M_2'$ showing

$$\Omega_{M_1}\cap\Omega_{M_2}\ \subseteq\ \Omega_{M_1\cap M_2}\ \bigcup\ \Omega(M_1{\subset}\Lambda)\cap\Omega_{M_2'}\,.$$

Let $\ell\in\Omega(M_1\subset\Lambda)\cap\Omega_{M_2'}$. By definition $\Omega(M_1\subset\Lambda)=\Omega_{M_1}\cap Gr(1,\Lambda)$, thus $\ell\in\Omega_{M_1}$ and $\ell\subset\Lambda$. Now suppose also that $\ell\in\Omega_{M_2'}$, then ℓ meets both M_2' and Λ, thus $\ell\in\Omega_{M_2'\cap\Lambda}=\Omega_{M_2}$. Lastly, by definition $\Omega_{M_1\cap M_2}\subseteq\Omega_{M_1}\cap\Omega_{M_2}$. For the proof of the transversality statement we refer the reader to [18, Lemma 2.4]. $\square$

Remark 3.2. Suppose that $a_\bullet$ is a reduced Schubert problem. Set $n:=n(a_\bullet)$. Let $L_1,\dots,L_m$ be linear subspaces with $\dim L_i = n{-}a_i{-}1$ in $\mathbb{P}^n$ such that L_{m-1} and L_m span a hyperplane Λ, but otherwise $L_1,\dots,L_m$ are in general position. By Lemma 3.1 we have

$$\begin{aligned}\Omega_{L_1}\cap\cdots\cap\Omega_{L_m}\ &=\ \Omega_{L_1}\cap\cdots\cap\Omega_{L_{m-2}}\ \cap\ \Omega_{L_{m-1}\cap L_m}\\ &\qquad\bigcup\ \Omega_{L_1}\cap\cdots\cap\Omega_{L_{m-2}}\ \cap\ \Omega(L_{m-1}{\subset}\Lambda)\cap\Omega_{L_m'}\,,\end{aligned}$$

where $L_m'\cap\Lambda=L_m$, and so L_m' has dimension $n{-}a_m$.

The first intersection has type $(a_1,\dots,a_{m-2},a_{m-1}{+}a_m)$ and the second, once we apply the reduction of Remark 2.16, has type $(a_1,\dots,a_{m-2},\,a_{m-1}{-}1,a_m{-}1)$. This gives Schubert's recursion for Kostka numbers

$$d(a_1,\ldots,a_m) \;=\; d(a_1,\ldots,a_{m-2},\, a_{m-1}+a_m) \;+\; d(a_1,\ldots,a_{m-2},\, a_{m-1}-1, a_m-1)\,. \tag{3.2}$$

As $a_\bullet$ is reduced, the two Schubert problems obtained are both valid. Observe that this recursion holds even if $a_\bullet$ is not reduced. The only modification in that case is that the first term in (3.2) may be zero, for $(a_1,\ldots,a_{m-2},\, a_{m-1}+a_m)$ may not be valid (in this case, $L_{m-1}\cap L_m = \varnothing$).

We consider this recursion for $d(2,2,1,2,3)$, illustrated in Figure 4. The first tableau in Figure 4 has both 4s in its second row (along with its 5s), while the remaining four tableaux have last column consisting of a 4 on top of a 5. If we replace the 5s by 4s in the first tableau and erase the last column in the remaining four tableaux, we obtain $\mathcal{K}(2,2,1,5)$ and $\mathcal{K}(2,2,1,1,2)$, illustrated in Figure 6.

$$\begin{array}{|c|c|c|c|c|}\hline 1&1&2&2&3\\\hline 4&4&4&4&4\\\hline\end{array}\qquad\qquad \begin{array}{|c|c|c|c|}\hline 1&1&2&2\\\hline 3&4&5&5\\\hline\end{array}\quad \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 2&3&5&5\\\hline\end{array}\quad \begin{array}{|c|c|c|c|}\hline 1&1&2&4\\\hline 2&3&5&5\\\hline\end{array}\quad \begin{array}{|c|c|c|c|}\hline 1&1&3&4\\\hline 2&2&5&5\\\hline\end{array}$$

Fig. 6. The tableaux $\mathcal{K}(2,2,1,5)$ and $\mathcal{K}(2,2,1,1,2)$

This shows that $d(2,2,1,2,3) = d(2,2,1,5) + d(2,2,1,1,2)$.

In Section C, we use this recursion to prove the following lemma.

Lemma 3.3. *Suppose that $a_\bullet$ is a valid Schubert problem. Then $d(a_\bullet) \neq 0$ and $m > 1$.*

If $m = 2$ or $m = 3$, then $d(a_\bullet) = 1$. If $m = 4$, then

$$d(a_\bullet) \;=\; 1 \;+\; \min\{a_i\,,\; n(a_\bullet){-}1{-}a_j \mid i, j = 1, \ldots, 4\}\,. \tag{3.3}$$

There are no reduced Schubert problems with $m < 4$. If $a_\bullet$ is reduced and $m = 4$, then $a_1 = a_2 = a_3 = a_4$.

B. Galois groups are at least alternating

We state the main result of this thesis, and we use Vakil's criterion and Schubert's degeneration to deduce the main result from a key combinatorial lemma. We start by defining a *rearrangement* of a Schubert problem $(a_1, \ldots, a_m)$ simply as a listing of the integers $(a_1, \ldots, a_m)$ in some order.

Lemma 3.4. *Let $a_\bullet$ be a reduced Schubert problem involving $m \geq 4$ integers. Unless $a_\bullet = (1, 1, 1, 1)$, then there is a rearrangement $(a_1, \ldots, a_m)$ such that*

$$d(a_1, \ldots, a_{m-2},\, a_{m-1}{+}a_m) \;\neq\; d(a_1, \ldots, a_{m-2},\, a_{m-1}{-}1, a_m{-}1)\,, \tag{3.4}$$

and both terms are nonzero. When $a_\bullet = (1, 1, 1, 1)$, we have equality in (3.4) with both terms equal to 1.

The proof of Lemma 3.4 will occupy most of the remainder of this thesis. We use it to deduce our main theorem, which we now state.

Theorem 3.5. *Let $a_\bullet$ be a Schubert problem in $\mathbb{G}(1, n)$. Then $\mathrm{Gal}(a_\bullet)$ is at least alternating.*

Proof. We use a double induction on the dimension n of the ambient projective space and the number m of conditions. The initial cases are when one of n or m is less than four, for by Lemma 3.3, $d(a_1, \ldots, a_m) \leq 2$ and the trivial subgroups of these small symmetric groups are alternating. Only in case $a_\bullet = (1,1,1,1)$ with $n=3$ is $d(a_\bullet)=2$.

Given a non-reduced Schubert problem, the associated reduced Schubert problem is in a smaller-dimensional projective space, and so its Galois group is at least alternating, by our induction hypothesis. We may therefore assume that $(a_1,\dots,a_m)$ is a reduced Schubert problem, so that for $1\leq i<j\leq m$, we have $a_i+a_j\leq n{-}1$, where $n := \frac{1}{2}(a_1+\cdots+a_m+2)$. Let $\pi\colon W\to X$ be as in Section II.D, so that the fibers of π are intersections of Schubert problems (2.12). Define $Z\subset X$ by

$$Z \;:=\; \{(L_1,\dots,L_m)\in X \mid L_{m-1},L_m \text{ do not span } \mathbb{P}^n\}\,.$$

This subvariety is proper, for if L_{m-1},L_m are general and $a_{m-1}+a_m\leq n{-}1$, they span $\mathbb{P}^n$. Moreover, X is smooth.

Let Y be the pullback of the map $\pi\colon W\to X$ along the inclusion $Z\hookrightarrow X$. By Remark 3.2, Y has two components Y_1 and Y_2 corresponding to the two components of (3.2). The first component Y_1 is the total space of the Schubert problem $(a_1,\dots,a_{m-2},a_{m-1}{+}a_m)$, and so by induction $\mathrm{Gal}_{Y_1\to Z}$ is at least alternating. For the second component $Y_2\to Z$, first replace Z by its dense open subset in which L_{m-1},L_m span a hyperplane $\Lambda=\langle L_{m-1},L_m\rangle$. Observe that under the map from Z to the space of hyperplanes in $\mathbb{P}^n$ given by

$$(L_1,L_2,\dots,L_m) \;\longmapsto\; \langle L_{m-1},L_m\rangle\,,$$

the fiber of $Y_2\to Z$ over a fixed hyperplane Λ is the total space of the Schubert problem $(a_1,\dots,a_{m-2},a_{m-1}{-}1,a_m{-}1)$ in $\mathbb{G}(1,\Lambda)$. Again, our inductive hypothesis and Case (a) of Vakil's criterion (as elucidated in Remark 2.29) implies that $\mathrm{Gal}_{Y_2\to Z}$ is at least alternating.

We conclude by an application of Vakil's criterion that $\mathrm{Gal}_{W\to X}$ is at least alternating, which proves Theorem 3.5. $\square$

C. Some Schubert intersection numbers

We prove Lemma 3.3 by showing that if $a_\bullet$ is a valid Schubert problem, then $d(a_\bullet) \neq 0$, and we also compute $d(a_\bullet)$ for $m \leq 4$. Observe that there are no valid Schubert problems with $m = 1$.

1. $m = 2$

Valid Schubert problems when $m = 2$ necessarily have the form (a, a) with $n(a_\bullet) = a{+}1$. The corresponding geometric problem asks for the lines meeting two general linear spaces of dimension $n{-}a{-}1 = 0$, that is, the lines meeting two general points. Thus $d(a, a) = 1$.

2. $m = 3$

Let (a, b, c) be a valid Schubert problem; thus, $a \leq \frac{1}{2}(a{+}b{+}c)$, which implies that $a \leq b{+}c$. We may assume that $a < b{+}c$ so that $d(a, b, c) = d(a, b{-}1, c{-}1)$ by (2.10). Iterating this will lead to a Schubert problem with $m = 2$, and so we see that $d(a, b, c) = 1$.

3. $m = 4$

Suppose that (a_1, a_2, a_3, a_4) is a valid Schubert problem, and suppose that $a_1 \leq a_2 \leq a_3 \leq a_4$. If it is reduced, then we have

$$a_3 \ + \ a_4 \ \leq \ \frac{1}{2}(a_1 + a_2 + a_3 + a_4) \ \leq \ a_3 \ + \ a_4\,,$$

which implies that the four numbers are equal, say to a. Write $a_\bullet = (a^4)$ in this case. By (3.2),

$$d(a^4) \;=\; d(a,a,2a) \;+\; d(a,a,a{-}1,a{-}1) \;=\; 1 \;+\; d((a{-}1)^4)\,,$$

as $d(a,a,2a) = 1$ and $d(a,a,a{-}1,a{-}1) = d((a{-}1)^4)$, by (2.10). Since $d(1^4) = 2$, as this is the problem of four lines, we have inductively shown that $d(a^4) = a{+}1$, which proves (3.3) when $a_\bullet$ is reduced.

Now suppose that $a_\bullet$ is not reduced, and set

$$\begin{aligned} \alpha(a_\bullet) &:= \min\{a_i \mid i = 1,\ldots,4\}\,, \quad \text{and} \\ \beta(a_\bullet) &:= \min\{n(a_\bullet){-}1{-}a_i \mid i = 1,\ldots,4\}\,. \end{aligned}$$

Since $a_\bullet$ is not reduced and $a_1 \le a_2 \le a_3 \le a_4$, we have $a_1 + a_2 < n(a_\bullet) < a_3 + a_4$ and (2.10) gives

$$d(a_\bullet) \;=\; d(a_1, a_2, a_3{-}1, a_4{-}1)\,.$$

Set $a'_\bullet := (a_1, a_2, a_3{-}1, a_4{-}1)$. We prove (3.3) by showing that

$$\min\{\alpha(a_\bullet), \beta(a_\bullet)\} \;=\; \min\{\alpha(a'_\bullet), \beta(a'_\bullet)\}\,. \tag{3.5}$$

Note that $n(a'_\bullet) = n(a_\bullet){-}1$. Since $a_1 \le a_3$, we have $\alpha(a'_\bullet) = \alpha(a_\bullet) = a_1$ unless $a_1 = a_3$, in which case $a_\bullet = (a,a,a,a{+}2\gamma)$ for some $\gamma \ge 1$. Thus $a'_\bullet = (a{-}1,a,a,a{+}2\gamma{-}1)$, and so $\alpha(a'_\bullet) = \alpha(a_\bullet){-}1$. But then $\beta(a'_\bullet) = \beta(a_\bullet) = a{-}\gamma \le \alpha(a'_\bullet)$, which proves (3.5) when $\alpha(a'_\bullet) \ne \alpha(a_\bullet)$.

Since $a_2 \le a_4$, we have $\beta(a'_\bullet) = \beta(a_\bullet) = n(a_\bullet){-}1{-}a_4$, unless $a_2 = a_4$, in which case $a_\bullet = (a, a{+}2\gamma, a{+}2\gamma, a{+}2\gamma)$ for some $\gamma \ge 1$. Therefore $a'_\bullet = (a, a{+}2\gamma{-}1, a{+}2\gamma{-}1, a{+}2\gamma)$, and so $\beta(a'_\bullet) = \beta(a_\bullet){-}1$. But then $\alpha(a'_\bullet) = \alpha(a_\bullet) = a$, which proves (3.5) when $\beta(a'_\bullet) \ne \beta(a_\bullet)$.

D. Some Schubert problems with symmetric Galois group

While Theorem 3.5 asserts that all Schubert problems involving lines have at least alternating Galois group, we conjecture that these Galois groups are always the full symmetric group. We present some evidence for this conjecture.

The first computation of a Galois group of a Schubert problem that we know of was for the problem $a_\bullet = (1^6)$ in $\mathbb{G}(1,4)$ where $K(a_\bullet) = 5$. Byrnes and Stevens showed that $\mathrm{Gal}(a_\bullet)$ is the full symmetric group [3] and [2, §5.3]. In [11] problems $a_\bullet = (1^{2n-2})$ for $n = 5, \ldots, 9$ were shown to have Galois group the full symmetric group. Both demonstrations used numerical methods.

We describe two infinite families of Schubert problems, each of which has the full symmetric group as Galois group. Both are generalizations of the problem of four lines.

1. Lines that meet four $(a-1)$-planes in $\mathbb{P}^{2a-1}$

In [18, §8.1], the Schubert problem $((a-1)^4)$ in $\mathbb{G}(1, 2a-1)$ was studied and solved. We use its equivalent description in the Grassmannian $Gr(2, 2a)$ of two-dimensional linear subspaces of a $2a$-dimensional space, V (which is identical to $\mathbb{G}(1, 2a-1)$). It involves the 2-planes meeting four general a-dimensional linear subspaces in V. If the a-dimensional subspaces are $H_1, \ldots, H_4$, then any two are in direct sum, as they are in general position. It follows that H_3 and H_4 are the graphs of linear isomorphisms $\varphi_3, \varphi_4 \colon H_1 \to H_2$.

If we set $\psi := \varphi_4^{-1} \circ \varphi_3$, then $\psi \in GL(H_1)$. Note that for any vector $v \in H_1$, the linear span $\langle v, \phi_3(v)\rangle$ is a 2-plane in V that meets H_1, H_2, and H_3. However, if v is an eigenvector of ψ, then $\psi(v) = c \cdot v$ for some $c \in \mathbb{K}$. Thus, $\phi_4^{-1} \circ \phi_3(v) = c \cdot v$, which is equivalent to $\phi_3(v) = \phi_4(c \cdot v) = c \cdot \phi_4(v)$. Therefore, $\varphi_3(v)$ and $\varphi_4(v)$ span the same line in H_2. Thus the span $\langle v, \phi_3(v)\rangle$ equals the span $\langle v, \phi_4(v)\rangle$. Therefore, $\langle v, \phi_3(v)\rangle$ meets H_4. The condition that these four planes are generic is that ψ has distinct eigenvalues and therefore exactly a eigenvectors $v_1, \ldots, v_a \in H_1$, up to a scalar. Then the solutions to the Schubert problem are given by the linear span $\langle v_i, \varphi_3(v_i)\rangle$ for each $i = 1, \ldots, a$. Every element $\psi \in GL(H_1)$ with distinct eigenvalues may occur, which implies that the Galois group is the full symmetric group.

2. Lines that meet a fixed line and n $(n-2)$-planes in $\mathbb{P}^n$

We exhibit another infinite family of Schubert problems with full symmetric Galois group. The solutions of these problems were described in [18, §8.2] in terms of rational normal scrolls, which we recall next. Let $\Lambda_1, \ldots, \Lambda_{n-r+1}$ be $n-r+1$ general $(n-2)$-planes in $\mathbb{P}^n$. For every $i = 1, \ldots, n-t+1$, let $\{\Gamma_i(p)\}_{p\in\mathbb{P}^1}$ be the pencil of hyperplanes that contain Λ_i The $(n-2)$-planes are in general position if for every $p \in \mathbb{P}^1$, the hyperplanes $\Gamma_i(p)$, for $1 \le i \le n-r$, intersect in an r-dimensional plane. The union

$$S_{1,n-2} := \bigcup_{p\in\mathbb{P}^1} \Gamma_i(p) \cap \cdots \cap \Gamma_{n-r}(p). \tag{3.6}$$

is a *rational normal scroll*, which is an irreducible determinantal varieties [7, §9].

We now present another family of Schubert problems with full symmetric Galois group. These are given by the problem $a_\bullet = (1^n, n-2)$ in $\mathbb{G}(1, \mathbb{P}^n)$, which looks for the lines meeting a fixed line ℓ and n planes of dimension $(n-2)$ in $\mathbb{P}^n$. Fixing the line ℓ and all but one $(n-2)$-plane, the lines they meet form a rational normal scroll $S_{1,n-2}$ (taking $r = 1$ and $p \in \ell$ in (3.6)), parametrized by their intersections with ℓ. A general $(n-2)$-plane will meet the scroll in $n-1$ points, each of which gives a solution to the Schubert problem. These points correspond to $n-1$ points of ℓ, and thus to a homogeneous degree $n-1$ form on ℓ. The main consequence of [18, §8.2] is that every such form can arise, which shows this Schubert problem has Galois group the full symmetric group.

E. Inequality of Lemma 3.4 in most cases

We give a combinatorial injection of sets of Young tableaux to establish Lemma 3.4, when we have $a_i \neq a_j$ for some i, j.

Lemma 3.6. *Suppose that $b_1, \ldots, b_m, \alpha, \beta, \gamma$ is a reduced Schubert problem where $\alpha \leq \beta \leq \gamma$ with $\alpha < \gamma$. Then*

$$d(b_1, \ldots, b_m,\ \alpha, \beta + \gamma) \ < \ d(b_1, \ldots, b_m,\ \gamma, \beta + \alpha)\,. \tag{3.7}$$

To see that this implies Lemma 3.4 in the case when $a_i \neq a_j$, for some i, j, we apply Schubert's recursion to to obtain two different expressions for $d(b_1, \ldots, b_m, \alpha, \beta, \gamma)$,

$$\begin{aligned} d(b_1, \ldots, b_m,\ \alpha, \beta{+}\gamma) \ &+ \ d(b_1, \ldots, b_m,\ \alpha, \beta{-}1, \gamma{-}1) \\ &= \ d(b_1, \ldots, b_m,\ \gamma, \beta + \alpha) \ + \ d(b_1, \ldots, b_m,\ \gamma, \beta{-}1, \alpha{-}1)\,. \end{aligned}$$

By the inequality (3.7), at least one of these expressions involves unequal terms. Since all four terms are from valid Schubert problems, none is zero, and so this implies Lemma 3.4 when not all a_i are identical. $\square$

Example 3.7. We illustrate Lemma 3.6 and motivate the ideas behind its proof. Consider the reduced Schubert problem $(2, 2, 2, 1, 2, 3)$. In this case, $\alpha = 1$, $\beta = 2$, and $\gamma = 3$. We verify that $d(2, 2, 2, 1, 5) < d(2, 2, 2, 2, 4)$ by noting that $\mathcal{K}(2, 2, 2, 1, 5)$ contains only three tableaux, as illustrated in Figure 7, whereas $\mathcal{K}(2, 2, 2, 2, 4)$ consists of six tableaux, as shown in Figure 8. As in the proof Lemma 3.6, we give a combinatorial injection $\iota : \mathcal{K}(2, 2, 2, 1, 5) \hookrightarrow \mathcal{K}(2, 2, 2, 2, 4)$. If we replace the first

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&2&3&3\\\hline 4&5&5&5&5&5\\\hline\end{array}\quad\begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&2&3&4\\\hline 3&5&5&5&5&5\\\hline\end{array}\quad\begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&3&3&4\\\hline 2&5&5&5&5&5\\\hline\end{array}$$

Fig. 7. The tableaux $\mathcal{K}(2,2,2,1,5)$

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&2&3&3\\\hline 4&4&5&5&5&5\\\hline\end{array}\quad\begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&2&3&4\\\hline 3&4&5&5&5&5\\\hline\end{array}\quad\begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&3&3&4\\\hline 2&4&5&5&5&5\\\hline\end{array}\quad\begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&2&4&4\\\hline 3&3&5&5&5&5\\\hline\end{array}$$

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&3&4&4\\\hline 2&3&5&5&5&5\\\hline\end{array}\quad\begin{array}{|c|c|c|c|c|c|}\hline 1&1&3&3&4&4\\\hline 2&2&5&5&5&5\\\hline\end{array}$$

Fig. 8. The tableaux $\mathcal{K}(2,2,2,2,4)$

5 (marked in red in Figure 7) in each tableaux of $\mathcal{K}(2,2,2,1,5)$ by a 4, we obtain the first three tableaux of $\mathcal{K}(2,2,2,2,4)$ in Figure 8. Similarly, replacing the last 4 in the second row (marked in red in Figure 8) by a 5 in the first three tableaux of $\mathcal{K}(2,2,2,2,4)$, we obtain all the tableaux of $\mathcal{K}(2,2,2,1,5)$.

We bring the ideas presented in the previous example into a proof of Lemma 3.6.

Proof of Lemma 3.6. We establish the inequality (3.7) via a combinatorial injection

$$\iota \,:\, \mathcal{K}(b_1,\dots,b_m,\,\alpha,\beta+\gamma) \;\hookrightarrow\; \mathcal{K}(b_1,\dots,b_m,\,\gamma,\beta+\alpha)\,,$$

which is not surjective.

Let T be a tableau in $\mathcal{K}(b_1,\dots,b_m,\,\alpha,\beta+\gamma)$ and let A be its sub-tableau consisting of the entries $1,\dots,m$. Then the skew tableau $T\setminus A$ has a bloc of $(m{+}1)$'s of length a at the end of its first row and its second row consists of $\alpha{-}a$ many $(m{+}1)$'s followed by a bloc of $(m{+}2)$'s of length $\beta{+}\gamma$. Form the tableau $\iota(T)$ by changing the last row of $T\setminus A$ to a bloc of $(m{+}1)$'s of length $\gamma{-}a$ followed by $\beta{+}\alpha$ many $(m+2)$'s. This is illustrated in Figure 9. Since $a\leq\alpha<\gamma$, this map is well-defined.

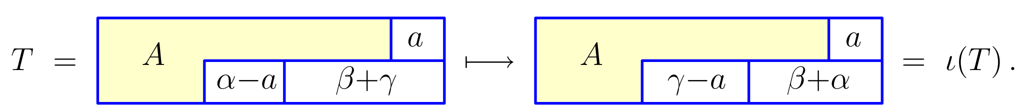

Fig. 9. The map ι

To show that ι is not surjective, set $b_\bullet := (b_1,\dots,b_m,\gamma-\alpha-1,\beta-1)$, which is a valid Schubert problem. Hence $d(b_\bullet)\neq 0$ and $\mathcal{K}(b_\bullet)\neq\varnothing$. For any $T\in\mathcal{K}(b_\bullet)$, we may add $\alpha{+}1$ columns to its end consisting of a $m{+}1$ above a $m{+}2$ to obtain a tableau $T'\in\mathcal{K}(b_1,\dots,b_m,\,\gamma,\beta+\alpha)$. As T' has more than α many $(m{+}1)$'s in its first row, it is not in the image of the injection ι, which completes the proof of the lemma. $\square$

CHAPTER IV

THE INEQUALITY IN THE REMAINING CASES

We prove Lemma 3.4 in the remaining case when $a_1 = \cdots = a_m = a$. Our method will be to first recast Kostka numbers as certain integrals, converting the inequality of Lemma 3.4 into the non-vanishing of an integral, which we establish by induction.

A. Representations of $\mathfrak{sl}_2\mathbb{C}$

Kostka numbers of two-rowed tableaux appear as the coefficients in the decomposition of the tensor products of irreducible $\mathfrak{sl}_2\mathbb{C}$-modules. The *Lie algebra* $\mathfrak{sl}_2\mathbb{C}$ consists of the complex 2×2-matrices whose trace is zero, with the Lie bracket defined by $[x, y] := xy - yx$. The elements

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

generate $\mathfrak{sl}_2\mathbb{C}$ as a vector space. These elements have the following relation with respect to the Lie bracket $[e, f] = h, \quad [h, f] = -2f, \quad [h, e] = 2e$. Similarly, if V is a finite-dimensional vector space over $\mathbb{C}$, the *general Lie algebra* $\mathfrak{gl}(V)$ consists of all the linear maps from V to V endowed with the Lie bracket $[x, y] = x \circ y - y \circ x$.

Definition 4.1. A $\mathfrak{sl}_2\mathbb{C}$-*representation* is a pair (V, ρ) where $\rho : \mathfrak{sl}_2\mathbb{C} \to \mathfrak{gl}(V)$ is a linear map that preserves the Lie bracket.

For any vector space V, the *trivial representation* is the pair (V, ρ) with $\rho = 0$ the zero map. For a finite-dimensional $\mathfrak{sl}_2\mathbb{C}$-representation (V, ρ_V) (not necessarily irreducible), we regard V as a module by setting $x \cdot v := \rho_V(x)v$ for all $x \in \mathfrak{sl}_2\mathbb{C}$.

Consider the basis element h of $\mathfrak{sl}_2\mathbb{C}$. Let v be an eigenvector of $\rho_V(h)$ with eigenvalue a. We define the *weight space*

$$V_a := \{v \in V \mid h \cdot v = av\}.$$

In this case, v is a *weight vector*, and a its *weight*. Moreover, for the other basis elements e, f and any $v \in V_a$ we have

(a) $e \cdot v = 0$ or $e \cdot v$ is an eigenvector of h with eigenvalue $a{+}2$.

(b) $f \cdot v = 0$ or $f \cdot v$ is an eigenvector of h with eigenvalue $a{-}2$.

Proposition 4.2. *Let V be a finite-dimensional $\mathfrak{sl}_2\mathbb{C}$-module, then there exists an eigenvector $v \in V$ such that $e \cdot v = 0$.*

Proof. Since $\mathbb{C}$ is algebraically closed, $\rho_V(h)$ has at least one eigenvalue and hence at least one eigenvector w with eigenvalue a. If non-zero, the vector $e^k \cdot w$ (for $k = 1, 2, \ldots$) is an eigenvector for $\rho_V(h)$ with eigenvalue $a{+}2k$. Therefore, the sequence $w, e \cdot w, e^2 \cdot w, \ldots$ is an infinite sequence of linearly independent eigenvectors in V. However V is finite-dimensional, so these cannot all be non-zero, hence there exists a $k \geq 0$ such that $e^k \cdot w \neq 0$ and $e^{k+1} \cdot w = 0$. Set $v = e^k \cdot w$, so from the relation $[h, e] = he - eh = 2e$ we have

$$\begin{aligned}
h \cdot v &= h(e^k \cdot w) = he \cdot e^{k-1} \cdot w = (eh + 2e) \cdot e^{k-1} \cdot w \\
&= e(h + 2) \cdot e^{k-1} \cdot w = e(he + 2e) \cdot e^{k-2} \cdot w \\
&= \cdots = e^k(h + 2k) \cdot w = (a + 2k)e^k \cdot w = (a{+}2k)v.
\end{aligned}$$

Thus v is an eigenvalue satisfying $e \cdot v = 0$. $\square$

Definition 4.3. Let V be a finite-dimensional $\mathfrak{sl}_2\mathbb{C}$-module. A *highest weight* is a weight a for which its weight space satisfies $V_a \neq \{0\}$ but $V_{a+2} = \{0\}$. A *highest weight vector* is a weight vector v with weight the highest weight.

A finite-dimensional irreducible $\mathfrak{sl}_2\mathbb{C}$-module is determined by its highest weight: if v is a maximal vector of an irreducible module V with highest weight a, then $v, f \cdot v, f^2 \cdot v, \cdots, f^a \cdot v$ form a basis for V. Weyl's complete reducibility theorem states that every finite-dimensional $\mathfrak{sl}_2\mathbb{C}$-module can be written as a direct sum of irreducible modules, thus it is parametrized by its highest weights.

Knowing the weight decompositions of given representations tells us the weight decomposition of all their tensor products. The Clebsch-Gordan formula describes the tensor product of irreducible $\mathfrak{sl}_2\mathbb{C}$-modules. If V_a and V_b are irreducible $\mathfrak{sl}_2\mathbb{C}$-modules with highest weights a and b respectively, then the Clebsch-Gordan formula is

$$V_a \otimes V_b = V_{b+a} \oplus V_{b+a-2} \oplus \cdots \oplus V_{|b-a|}. \tag{4.1}$$

B. Kostka numbers as integrals

Let V_a be the irreducible module of $\mathfrak{sl}_2\mathbb{C}$ with highest weight a, for $a = 0, 1, 2, \ldots$. According to Young's rule [5, Corollary 1, p.92] for a Schubert problem $a_\bullet = (a_1, \ldots, a_m)$, the Kostka number $K(a_\bullet)$ appears as the multiplicity of the trivial $\mathfrak{sl}_2\mathbb{C}$-module V_0 in the tensor product $V_{a_1} \otimes \cdots \otimes V_{a_m}$.

The representation ring R of $\mathfrak{sl}_2\mathbb{C}$ is the free abelian group on the isomorphism classes $[V_a]$ of irreducible modules, modulo the relations $[V_a] + [V_b] - [V_a \oplus V_b]$. Setting $[V_a] \cdot [V_b] := [V_a \otimes V_b]$ equips R with the structure of a ring with unit $[V_0]$. Multiplication by $[V_a]$ is a linear operator M_a on R,

$$M_a([V_b]) \ := \ [V_a] \cdot [V_b] \ = \ [V_{b+a}] + [V_{b+a-2}] + \cdots + [V_{|b-a|}]\,, \tag{4.2}$$

by the Clebsch-Gordan formula (4.1). In the basis $\{[V_a]\}$, the operator M_a is represented by an infinite Toeplitz matrix with entries 0 and 1 given by the formula (4.2).

For instance,

$$M_2 \;=\; \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & \\ & & & \vdots & & & & \ddots \end{pmatrix}, \qquad M_3 \;=\; \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \\ & & & \vdots & & & & & \ddots \end{pmatrix}.$$

Since R is a commutative ring, the operators $\{M_a \mid a \geq 0\}$ commute. If we extend scalars to $R_{\mathbb{C}} := R \otimes_Z \mathbb{C}$, then the $\{M_a\}$ are a system of commuting operators on a complex vector space. We describe their system of joint eigenvectors and eigenvalues.

Proposition-Definition 4.4. For each $0 \leq \theta \leq \pi$ and integer $a \geq 0$, set

$$\begin{aligned} \mathbf{v}(\theta) &:= (\sin\theta, \sin 2\theta, \ldots, \sin{(j{+}1)\theta}, \ldots)^{\top} \;=\; \sum_j \sin{(j{+}1)\theta}\,[V_j]\,, \\ \lambda_a(\theta) &:= \frac{\sin{(a{+}1)\theta}}{\sin\theta}\,. \end{aligned}$$

Then $\mathbf{v}(\theta)$ is an eigenvector of M_a with eigenvalue $\lambda_a(\theta)$.

Proof. Let us compute $M_a(\mathbf{v}(\theta))$, which is

$$\Bigl(\sin{(a{+}1)\theta}\,,\ \ldots\ \sum_{i=0}^{j} \sin{(a{+}1{-}j{+}2i)\theta}\,,\ \ldots\,,\ \sum_{i=0}^{a} \sin{(k{+}1{-}a{+}2i)\theta}\,,\ \ldots\Bigr)^{\top},$$

where the sum with upper index j is the coefficient of $[V_j]$ in $M_a(\mathbf{v}(\theta))$ for $0 \leq j \leq a$, and the sum involving $k{+}1{-}a{+}2i$ is the coefficient of $[V_k]$ for $k \geq a$.

Recall that $\sin\alpha\sin\beta = \frac{1}{2}(\cos{(\alpha-\beta)} - \cos{(\alpha+\beta)})$. Thus if $c > 0$ and $b \geq 0$, we have

$$\sin\theta \cdot \sum_{i=0}^{b} \sin(c{+}2i)\theta \;=\; \frac{1}{2}\sum_{i=1}^{b}\big(\cos(c{-}1{+}2i)\theta \,-\, \cos(c{+}1{+}2i)\theta\big)$$
$$\;=\; \frac{1}{2}(\cos(c{-}1)\theta \,-\, \cos(c{+}2b{+}1)\theta) \;=\; \sin(b{+}1)\theta\sin(c{+}b)\theta\,.$$

Using this, we see that $\sin\theta M_a\mathbf{v}(\theta)$ equals

$$(\sin\theta\sin(a{+}1)\theta\,,\,\ldots,\,,\sin(j{+}1)\theta\sin(a{+}1)\theta\,,\,\ldots\,,\,\sin(a{+}1)\theta\sin(k{+}1)\theta\,,\,\ldots)^{\top}\,,$$

which completes the proof of the proposition. $\square$

Recall that for $b, c > 0$ we have

$$\int_0^{\pi} \sin b\theta\ \sin c\theta\ d\theta \;=\; \begin{cases} 0 & \text{if} \quad b \neq c \\ \pi/2 & \text{if} \quad b = c \end{cases}\,.$$

This computation shows that our system of eigenvectors is complete.

Proposition 4.5. *For any $a = 0, 1, 2, \ldots,$ we have*

$$[V_a] \;=\; \frac{2}{\pi}\int_0^{\pi} \sin(a{+}1)\theta\ \mathbf{v}(\theta)\ d\theta\,.$$

We express the Kostka numbers as integrals.

Theorem 4.6. *For any $a \geq 1$, we have*

$$M_a([V_0]) \;=\; \frac{2}{\pi}\int_0^{\pi} \lambda_a(\theta)\ \sin\theta\ \mathbf{v}(\theta)\ d\theta\,.$$

Let $a_\bullet = (a_1, \ldots, a_m)$ be any valid Schubert problem. Then

$$d(a_\bullet) \;=\; \frac{2}{\pi}\int_0^{\pi}\Big(\prod_{i=1}^{m}\lambda_{a_i}(\theta)\Big)\ \sin^2\theta\ d\theta\,. \tag{4.3}$$

Proof. The first part of the theorem follows from Proposition 4.5, as $M_a(\mathbf{v}(\theta)) = \lambda_a(\theta)\cdot\mathbf{v}(\theta)$. Note that $M_a([V_0]) = [V_a]$, thus for the second part we use that the Kostka number $d(a_\bullet)$ appears as the coefficient of $[V_0]$ in the product $[V_{a_1}\otimes V_{a_2}\otimes\cdots\otimes V_{a_m}]$, which is equivalent to find the coefficient of $[V_0]$ in the following composition

$$M_{a_1}\circ M_{a_2}\circ\cdots\circ M_{a_m}([V_0]) \;=\; \frac{2}{\pi}\int_0^\pi \Bigl(\prod_{i=1}^m \lambda_{a_i}(\theta)\Bigr)\,\sin\theta\,\mathbf{v}(\theta)\;d\theta\,.$$

The equality (4.3) follows from Proposition-Definition 4.4, as the coefficient of $[V_0]$ in $\mathbf{v}(\theta)$ is $\sin\theta$. $\square$

Example 4.7. In Example 2.13 we saw that the problem of 4 lines $a_\bullet = (\square,\square,\square,\square)$ has Kostka number $d(\square,\square,\square,\square) = 2$. Theorem 4.6 rewrites this number as

$$\begin{aligned} d(\square,\square,\square,\square) &= \frac{2}{\pi}\int_0^\pi \lambda_1(\theta)^4\sin^2\theta\;d\theta = \frac{2}{\pi}\int_0^\pi \frac{\sin^4 2\theta}{\sin^2\theta}\;d\theta \\ &= \frac{32}{\pi}\int_0^\pi \sin^2\theta\cos^4\theta\;d\theta = \frac{32}{\pi}\int_0^\pi(\cos^4\theta-\cos^6\theta)\;d\theta = 2. \end{aligned}$$

C. Inequality of Lemma 3.4 when $a_\bullet = (a^m)$

We complete the proof of Theorem 3.5 by establishing the inequality of Lemma 3.4 for Schubert problems not covered by Lemma 3.6. For these, every condition is the same, so $a_\bullet = (a,\ldots,a) = (a^m)$.

If $a = 1$, then we may use the hook-length formula [5, §4.3]. If $\mu+b = 2c$ is even, then the Kostka number $d(1^\mu, b)$ is the number of Young tableaux of shape $(c, c-b)$, which is

$$d(1^\mu, b) \;=\; \frac{\mu!(b+1)}{(c-b)!(c+1)!}$$

When $m = 2c$ is even, the inequality of Lemma 3.4 is that $d(1^{2c-2}) \neq K(1^{2c-2}, 2)$. We compute

$$d(1^{2c-2}) \;=\; \frac{(2c-2)!(1)}{c!(c+1)!} \qquad\text{and}\qquad d(1^{2c-2},2) \;=\; \frac{(2c-2)!(3)}{(c-2)!(c+1)!}$$

and so

$$d(1^{2c-2},2)/K(1^{2c-2}) \;=\; 3\frac{c!(c+1)!}{(c-2)!(c+1)!} \;=\; 3\frac{c-1}{c+1} \;\neq\; 1\,, \tag{4.4}$$

when $c > 2$, but when $c = 2$ both Kostka numbers are 1, which proves the inequality of Lemma 3.4, when each $a_i = 1$.

We now suppose that $a_\bullet = (a^{\mu+2})$ where $a > 1$ and $\mu \cdot a$ is even. (We write $m = \mu + 2$ to reduce notational clutter.) The case $a = 2$ is different because in the inequality (3.4), we will show that

$$d(2^\mu, 4) \;-\; d(2^\mu, 1, 1) \;\neq\; 0\,,$$

and the left-hand side is negative for $\mu \leq 13$ and otherwise positive. This is shown in Table I.

Lemma 4.8. *For all $\mu \geq 2$, we have $d(2^\mu, 4) \neq d(2^\mu, 1, 1)$, and both terms are nonzero. If $\mu < 14$ then $d(2^\mu, 4) < d(2^\mu, 1, 1)$ and if $\mu \geq 14$, then $d(2^\mu, 4) > d(2^\mu, 1, 1)$.*

The remaining cases $a \geq 3$ have a uniform behaviour.

Lemma 4.9. *For $a \geq 3$ and for all $\mu \geq 2$ with $a \cdot \mu$ even we have*

$$d(a^\mu,\, 2a) \;<\; d(a^\mu, (a-1)^2)\,. \tag{4.5}$$

We establish Lemma 4.8 in Subsection D and Lemma 4.9 when $\mu \geq 4$ in Subsection E. Now, we compute $d(a^\mu, (a-1)^2)$ for $\mu = 3$ and a is even.

Lemma 4.10. *For $a = 2b$ with $b \geq 1$ we have*

$$d(a^3, (a-1)^2) = \frac{(5b^2 + 3b)}{2}. \tag{4.6}$$

Table I. The inequality (3.4) for the case $a_\bullet = (2^{\mu+2})$

μ	$d(2^\mu, 4)$	$d(2^\mu, 1, 1)$	Difference
2	1	2	-1
3	2	4	-2
4	6	9	-3
5	15	21	-6
6	40	51	-11
7	105	127	-22
8	280	323	-43
9	750	835	-85
10	2025	2188	-163
11	5500	5798	-298
12	15026	15511	-485
13	41262	41835	-573
14	113841	113634	207
15	315420	310572	4848
16	877320	853467	23853
17	2448816	2356779	92037
18	6857307	6536382	320925
19	19259046	18199284	1059762
20	54237210	50852019	3385191

Proof. We apply Schubert's recursion (3.2) to write

$$\begin{aligned} d(a^3,(a{-}1)^2) &= d(a^3,2(a{-}1)) + d(a^3,(a{-}2)^2) \\ &= d(a^3,2(a{-}1)) + d(a^3,2(a{-}2)) + d(a^3,(a{-}3)^2) \\ &= \cdots = \sum_{j=1}^{a} d(a^3,2(a{-}j)). \end{aligned}$$

Note that $n(a^3,2(a{-}j)) = n((2b)^3,2(2b{-}j)) = 5b{-}j{+}1$ for $j = 1,\ldots,a$. We use Lemma 3.3 to compute $d(a^3,2(a{-}j))$ by computing

$$\min\{a,\ 2(a-2j),\ n(a_\bullet){-}1{-}a,\ n(a_\bullet){-}1{-}2(a{-}j)\} \quad \text{for } j = 1,\ldots,a.$$

This is equivalent to compute

$$\min\{2b,\ 2(2b{-}j),\ 3b{-}j,\ b{+}j\} \quad \text{for } j = 1,\ldots,2b. \tag{4.7}$$

When $1 \le j \le b$, then the following inequalities hold $2b \ge b{+}j$, $4b{-}2j \ge b{+}j$, and $3b{-}j \ge b{+}j$; therefore, the minimum in (4.7) is $b{+}j$. When $b \le j \le 2b$, in particular $j \ge b$; which implies $2b \ge 4b{-}2j$, $3b{-}j \ge 4b{-}2j$, and $b{+}j \ge 4b{-}2j$. Thus, the minimum in (4.7) is $4b{-}2j$. Replacing j by $i = j - b$, we get $4b{-}2j = 2(b{-}i)$ for $0 \le i \le b$. Therefore, by Lemma (3.3)

$$\begin{aligned} \sum_{j=1}^{a} d(a^3,2(a{-}j)) &= \sum_{j=1}^{b} d(a^3,2(a{-}j)) + \sum_{j=b+1}^{2b} d(a^3,2(a{-}j)) \\ &= \sum_{j=1}^{b}(b{+}j{+}1) + \sum_{i=1}^{b}(2(b{-}i)+1) \\ &= \left[b(b{+}1) + \frac{b(b{+}1)}{2}\right] + \left[2b^2{-}b(b{+}1)+b\right] \\ &= (b{+}1)\left(\frac{3b}{2}\right) + b^2 = \frac{(5b^2+3b)}{2}. \end{aligned}$$

□

Corollary 4.11. *For a even and $\mu = 3$ we have*

$$d(a^3, 2a) \;<\; d(a^3, (a-1)^2). \tag{4.8}$$

Proof. Let $a = 2b$ for some $b \geq 1$. Notice that for $a_\bullet = (a^3, 2a)$, we have $n(a_\bullet) = 5b+1$, so $d(a^3, 2a) = 1 + b$ by Lemma 3.3. On the other hand, $d(a^3, (a-1)^2) = \frac{1}{2}(5b^2+3b)$ by Lemma 4.10. The inequality (4.8) follows as $b \geq 1$ implies $2(b+1) < (5b^2+3b)$. □

Lemma 4.12. *For a even and $\mu = 2$ we have*

$$d(a^2, 2a) \;<\; d(a^2, (a-1)^2). \tag{4.9}$$

Proof. From Lemma 3.3, we have $d(a^2,\, 2a) = 1$ and $d(a^2, (a-1)^2) = a$. □

Proof of Lemma 3.4 *when $a_\bullet = (a^m)$.* We established the case when $a = 1$ by direct computation in (4.4). Lemma 4.8 covers the case when $a = 2$ as $\mu = m-2$. The case $m \leq 4$ follows from Lemma 3.3, as for this case $d(a^m,\, 2a) = 1$ and either $d(a^m, (a-1)^2) = 1$ or $d(a^m, (a-1)^2) = a$. The remaining cases are covered by Lemma 4.12, Corollary 4.11 and Lemma 4.9. This completes the proof of Lemma 3.4 and of Theorem 3.5. □

D. Proof of Lemma 4.8

By the computations in Table I, we only need to show that $d(2^\mu, 4) - d(2^\mu, 1, 1) > 0$ for $\mu \geq 14$. Using (4.3), we have

$$\begin{aligned} d(2^\mu, 4) - d(2^\mu, 1, 1) \;&=\; \frac{2}{\pi}\int_0^\pi \lambda_2(\theta)^\mu \big(\lambda_4(\theta) \;-\; \lambda_1(\theta)^2\big)\, \sin^2\theta)\; d\theta \\ &=\; \frac{2}{\pi}\int_0^\pi \lambda_2(\theta)^\mu \big(\sin 5\theta\, \sin\theta \;-\; \sin^2 2\theta\big)\; d\theta\;. \end{aligned}$$

The integrand $f(\theta)$ of the last integral is symmetric about $\theta = \pi/2$ in that $f(\theta) = f(\pi - \theta)$. Thus it suffices to prove that if $\mu \geq 14$, then

$$\int_0^{\pi/2} \lambda_2(\theta)^\mu (\sin 5\theta \ \sin\theta \ - \ \sin^2 2\theta)\, d\theta \ > \ 0\,. \tag{4.10}$$

To simplify our notation, set $F_2(\theta) \ := \ \sin 5\theta \ \sin\theta \ - \ \sin^2 2\theta$. We graph the functions $F_2(\theta)$ and $\lambda_2(\theta)$, and the integrand in (4.10) for $\mu = 8$ in Figure 10.

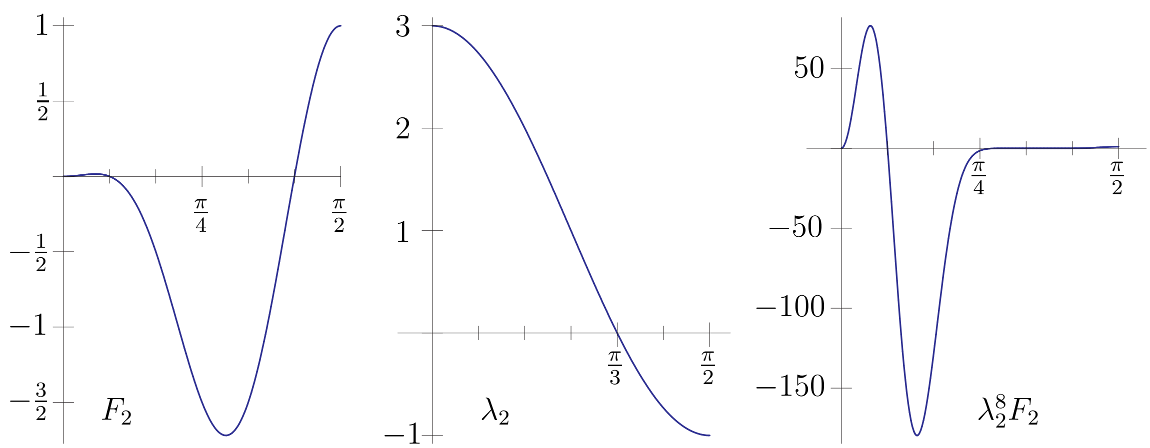

Fig. 10. The functions F_2, λ_2, and $\lambda_2^8 F_2$.

We have

$$\int_0^{\frac{\pi}{2}} \lambda_2^\mu F_2 \ \geq \ \int_0^{\frac{\pi}{3}} \lambda_2^\mu F_2 \ - \ \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left|\lambda_2^\mu F_2\right|.$$

We prove Lemma 4.8 by showing that for $m \geq 14$, we have

$$\int_0^{\frac{\pi}{3}} \lambda_2^\mu F_2 \;>\; \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left|\lambda_2^\mu F_2\right|. \tag{4.11}$$

We estimate the right-hand side. On $[\frac{\pi}{3}, \frac{\pi}{2}]$. the function λ_2 is decreasing and negative, so $|\lambda_2| \leq |\lambda_2(\frac{\pi}{2})| = 1$. Similarly, the function F_2 increases from $-3/2$ at $\frac{\pi}{3}$ to 1 at $\frac{\pi}{2}$. Thus

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left|\lambda_2^\mu F_2\right| \;\leq\; \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{3}{2} \;=\; \frac{\pi}{4}\,.$$

It is therefore enough to show that

$$\int_0^{\frac{\pi}{3}} \lambda_2^\mu F_2 \;>\; \frac{\pi}{4}\,, \tag{4.12}$$

for $\mu \geq 14$. This inequality holds for $\mu = 14$, as

$$\int_0^{\frac{\pi}{3}} \lambda_2^{14} F_2 \;=\; \frac{1062882}{17017}\sqrt{3} + 69\pi\,.$$

Suppose now that the inequality(4.12) holds for some $\mu \geq 14$. As F_2 is positive on $[0, \frac{\pi}{12}]$ and negative on $[\frac{\pi}{12}, \frac{\pi}{3}]$, this is equivalent to

$$\int_0^{\frac{\pi}{12}} \lambda_2^\mu F_2 \;>\; -\int_{\frac{\pi}{12}}^{\frac{\pi}{3}} \lambda_2^\mu F_2 \;+\; \frac{\pi}{4}\,,$$

and both integrals are positive.

For $\theta \in [0, \frac{\pi}{12}]$, $F_2(\theta) \geq 0$ and $\lambda_2(\theta) \geq \lambda_2(\frac{\pi}{12}) = 1 + \sqrt{3}$ as λ_2 is decreasing on $[0, \frac{\pi}{2}]$. Thus

$$\int_0^{\frac{\pi}{12}} \lambda_2^{m+1} F_2 \;\geq\; \int_0^{\frac{\pi}{12}} \left(1+\sqrt{3}\right) \cdot \lambda_2^m F_2\,. \tag{4.13}$$

Similarly, for $\theta \in [\frac{\pi}{12}, \frac{\pi}{3}]$, $F_2(\theta) \leq 0$ and $1+\sqrt{3} \geq \lambda_2(\theta) \geq 0$, so

$$-\int_{\frac{\pi}{12}}^{\frac{\pi}{3}} \left(1+\sqrt{3}\right) \cdot \lambda_2^\mu F_2 \;\geq\; -\int_{\frac{\pi}{12}}^{\frac{\pi}{3}} \lambda_2^m F_2\,. \tag{4.14}$$

From the induction hypothesis and equations (4.13) and (4.14), we have

$$\int_0^{\frac{\pi}{12}} \lambda_2^{m+1} F_2 \;\;\geq\;\; \left(1+\sqrt{3}\right) \cdot \int_0^{\frac{\pi}{12}} \lambda_2^m F_2 \;\;>\;\; (1+\sqrt{3})\left(-\int_{\frac{\pi}{12}}^{\frac{\pi}{3}} \lambda_2^m F_2 \;+\; \frac{\pi}{4}\right)$$
$$> \;\; -\int_{\frac{\pi}{12}}^{\frac{\pi}{3}} \lambda_2^{m+1} F_2 \;+\; \frac{\pi}{4}\,.$$

This completes the proof of Lemma 4.8. □

E. Proof of Lemma 4.9

We must show that $d(a^\mu, (a-1)^2) - d(a^\mu, 2a) > 0$ when $a\mu$ is even and $\mu \geq 4$. By the integral formula for Kostka numbers (4.3), this is equivalent to

$$\frac{2}{\pi}\int_0^{\pi} \lambda_a(\theta)^\mu \left(\sin^2 a\theta - \sin{(2a+1)\theta}\, \sin\theta\right) d\theta \;>\; 0\,. \qquad (4.15)$$

Write

$$F_a(\theta) \;:=\; 2(\sin^2 a\theta - \sin{(2a+1)\theta}\, \sin\theta) \;=\; 1 - 2\cos 2a\theta + \cos{(2a+2)\theta}\,.$$

These functions have symmetry about $\theta = \frac{\pi}{2}$,

$$F_a(\theta) \;=\; F_a(\pi - \theta) \qquad\qquad \lambda_a(\theta) \;=\; (-1)^a \lambda_a(\pi-\theta)\,.$$

Thus if $a\mu$ is odd, the integral (4.15) vanishes, and it suffices to prove that

$$\int_0^{\frac{\pi}{2}} \lambda_a^\mu F_a \;>\; 0\,, \qquad \text{for all } a \geq 3 \text{ and } \mu \geq 4\,. \qquad (4.16)$$

As in Subsection D, we establish this inequality by breaking the integral into two pieces. This is based on the following lemma, whose proof is given below.

Lemma 4.13. *For $\theta \in [0, \frac{\pi}{a+1}]$, we have $\lambda_a(\theta) \geq 0$ and $F_a(\theta) \geq 0$.*

Thus we have,

$$\int_0^{\frac{\pi}{2}} \lambda_a^\mu F_a \;>\; \int_0^{\frac{\pi}{a+1}} \lambda_a^\mu F_a \;-\; \int_{\frac{\pi}{a+1}}^{\frac{\pi}{2}} |\lambda_a^\mu F_a| \,,$$

and Lemma 4.9 follows from the following estimate.

Lemma 4.14. *For every $a \geq 3$ and $\mu \geq 4$, we have*

$$\int_0^{\frac{\pi}{a+1}} \lambda_a^\mu F_a \;>\; \int_{\frac{\pi}{a+1}}^{\frac{\pi}{2}} |\lambda_a^\mu F_a| \,. \tag{4.17}$$

We prove this inequality (4.17) by induction, first establishing the inductive step in Subsection 1 and then computing the base case in Subsection 2.

Proof of Lemma 4.13. The statement for λ_a is immediate from its definition. For F_a, we use some calculus. Recall that $F_a(\theta) = 1 - 2\cos 2a\theta + \cos 2(a{+}1)\theta$, which equals

$$2(\sin^2 a\theta - \sin(2a{+}1)\theta \ \sin\theta)\,.$$

Since the first term is everywhere nonnegative and the second nonnegative on $[\frac{\pi}{2a+1}, \frac{2\pi}{2a+1}]$ (and $\frac{\pi}{a+1} < \frac{2\pi}{2a+1}$), we only need to show that F_a is nonnegative on $[0, \frac{\pi}{2a+1}]$. Since $F_a(0) = 0$, it will suffice to show that F_a' is nonnegative on $[0, \frac{\pi}{2a+1}]$.

As $F_a' = 4a\sin 2a\theta - 2(a{+}1)\sin 2(a{+}1)\theta$, we have $F_a'(0) = 0$, and so it will suffice to show that F_a'' is nonnegative on $[0, \frac{\pi}{2a+1}]$. Since $a > 2$, we have $8a^2 > 4(a+1)^2$, and so

$$\begin{aligned} F_a'' &= 8a^2\cos 2a\theta \;-\; 4(a{+}1)^2\cos 2(a{+}1)\theta \\ &> 4(a{+}1)^2(\cos 2a\theta \;-\; \cos 2(a{+}1)\theta \;=\; 8(a{+}1)^2\sin(2a{+}1)\theta \ \sin\theta\,. \end{aligned}$$

But this last expression is nonnegative on $[0, \frac{\pi}{2a+1}]$. ☐

Our proof of Lemma 4.14 will use the following well-known inequalities for the sine function.

Proposition 4.15. *If* $0 \le x \le \frac{\pi}{2}$*, then* $\frac{2}{\pi}x \le \sin x$*. If* $0 \le x \le \frac{\pi}{4}$*, then* $\frac{2\sqrt{2}}{\pi}x \le \sin x$*. For every* $x \ge 0$*, we have*

$$3\frac{x}{\pi} - 4\frac{x^3}{\pi^3} \;\le\; \sin x \;\le\; x\,.$$

The first two inequalities hold as the sine function is concave on the interval $[0, \frac{\pi}{2}]$, and the last is standard. The most interesting is the cubic lower bound for sine. It is the Mercer–Caccia inequality [13], which we illustrate in Figure 11.

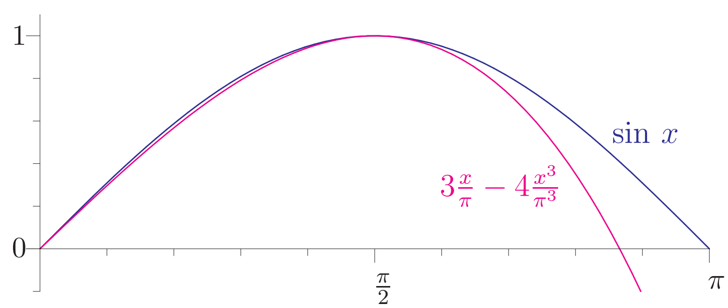

Fig. 11. The Mercer-Caccia inequality

1. Induction step of Lemma 4.14

Our main tool for the induction step is the following estimation.

Lemma 4.16. *For all* a*,* $\mu \ge 3$*, we have*

$$\int_0^{\frac{\pi}{a+1}} \lambda_a^{\mu+1} F_a \;\ge\; \frac{(a+1)^3}{3(a+1)^2 - 4} \int_0^{\frac{\pi}{a+1}} \lambda_a^{\mu} F_a\,. \tag{4.18}$$

Induction step of Lemma 4.14. Suppose that we have

$$\int_0^{\frac{\pi}{a+1}} \lambda_a^\mu F_a \;>\; \int_{\frac{\pi}{a+1}}^{\frac{\pi}{2}} |\,\lambda_a^\mu F_a\,|\,, \tag{4.19}$$

for some number μ. We use the Mercer-Caccia inequality at $x = \frac{\pi}{a+1}$ to obtain

$$\sin\tfrac{\pi}{a+1} \;\geq\; 3\frac{\frac{\pi}{a+1}}{\pi} \;-\; 4\frac{(\frac{\pi}{a+1})^3}{\pi^3} \;=\; \frac{3(a+1)^2-4}{(a+1)^3}\,.$$

For $\theta \in [\frac{\pi}{a+1}, \frac{\pi}{2}]$, we have $\sin\theta \geq \sin\frac{\pi}{a+1}$ and $|\sin(a+1)\theta| \leq 1$, and therefore

$$|\lambda_a(\theta)| \;=\; \left|\frac{\sin(a+1)\theta}{\sin\theta}\right| \leq \left|\frac{1}{\sin\frac{\pi}{a+1}}\right| \;\leq\; \frac{(a+1)^3}{3(a+1)^2-4}\,. \tag{4.20}$$

This last number, C_a, is the constant in Lemma 4.16. By Lemma 4.16, our induction hypothesis (4.19), and (4.20), we have

$$\int_0^{\frac{\pi}{a+1}} \lambda_a^{\mu+1} F_a \;\geq\; C_a \int_0^{\frac{\pi}{a+1}} \lambda_a^\mu F_a \;\geq\; C_a \int_{\frac{\pi}{a+1}}^{\frac{\pi}{2}} |\,\lambda_a^\mu F_a\,| \;\geq\; \int_{\frac{\pi}{a+1}}^{\frac{\pi}{2}} |\,\lambda_a^{\mu+1} F_a\,|\,,$$

which completes the induction step of Lemma 4.14. □

Our proof of Lemma 4.16 uses some linear bounds for λ_a. To gain an idea of the task at hand, in Figure 12 we show the integrand $\lambda_a^\mu F_a$ and λ_a on $[0, \frac{\pi}{a+1}]$, for $a = 4$ and $\mu = 2$.

We estimate λ_a. Define the linear function

$$\ell_a(\theta) \;:=\; \tfrac{(a+1)^2}{\pi}(\tfrac{\pi}{a+1} - \theta)\,,$$

which is the line through the points $(0, a{+}1)$ and $(\frac{\pi}{a+1}, 0)$ on the graph of λ_a.

Lemma 4.17. *For θ in the interval $[0, \frac{\pi}{a+1}]$, we have $\ell_a(\theta) \leq \lambda_a(\theta)$.*

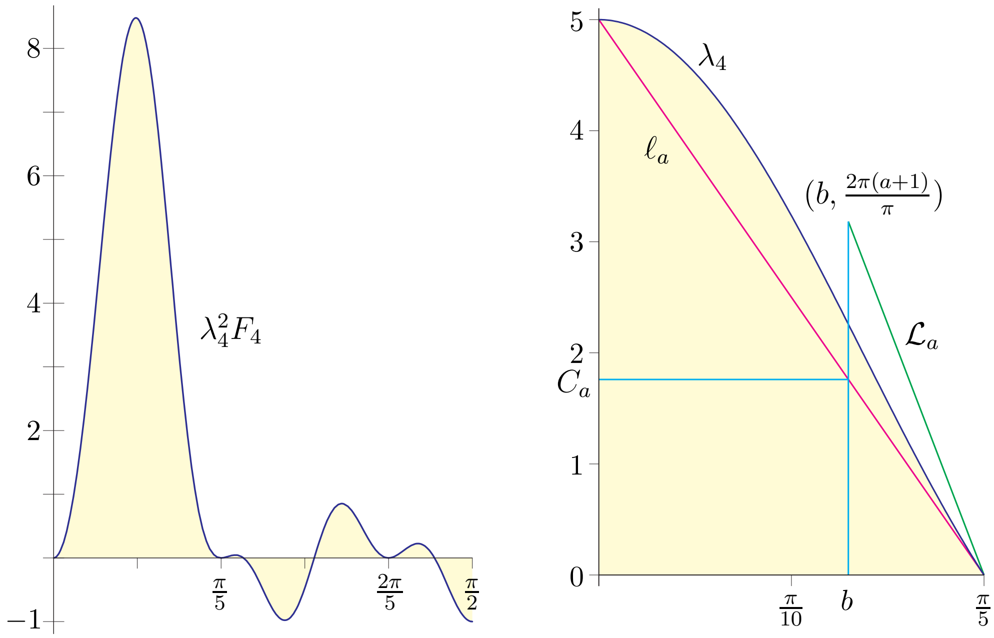

Fig. 12. The integrand $\lambda_4^2 F_4$ and λ_4

Proof. We need some information about the derivatives of $\lambda_a(\theta)$. First observe that

$$\begin{aligned}\lambda_a(\theta) &= \frac{\sin(a+1)\theta}{\sin\theta} = \frac{e^{i(a+1)\theta} - e^{-i(a+1)\theta}}{e^{i\theta} - e^{-i\theta}} = \sum_{j=0}^{a} e^{i(a-2j)\theta} \\ &= 2\cos a\theta + 2\cos(a-2)\theta + \cdots + \begin{cases} 2\cos\theta & \text{if } a \text{ is odd} \\ 1 & \text{if } a \text{ is even} \end{cases}.\end{aligned}$$

From this, we see that $\lambda_a'(0) = 0$ and λ_a' is negative on $(0, \frac{\pi}{a+1})$. Moreover, λ_a'' is a sum of terms of the form $-2(a-2j)^2\cos(a-2j)\theta$, for $0 \le j < \frac{a}{2}$. Thus λ_a'' is increasing on $[0, \frac{\pi}{a+1}]$, as each term is increasing on that interval.

Since ℓ_a has negative slope and $\lambda_a'(0) = 0$, we have $\ell_a(\theta) < \lambda_a(\theta)$ for $\theta \in [0, \frac{\pi}{a+1}]$ near 0. We compute $\lambda_a'(\frac{\pi}{a+1})$. Since

$$\lambda_a'(\theta) \;=\; \frac{(a{+}1)\cos(a{+}1)\theta}{\sin\theta} \;-\; \frac{\cos\theta\sin(a{+}1)\theta}{\sin^2\theta}\,,$$

we have

$$\lambda_a'(\tfrac{\pi}{a+1}) \;=\; \frac{-(a{+}1)}{\sin\frac{\pi}{a+1}} \;<\; \frac{-(a{+}1)^2}{\pi}\,,$$

as $0 < \sin\frac{\pi}{a+1} < \frac{\pi}{a+1}$. Thus at $\theta = \frac{\pi}{a+1}$, we have $\lambda_a(\theta) = \ell_a(\theta) = 0$ and $\lambda_a'(\theta) < \ell_a'(\theta)$ and so $\ell_a(\theta) < \lambda_a(\theta)$ for $\theta \in [0, \frac{\pi}{a+1}]$ near $\frac{\pi}{a+1}$.

If $\ell_a(\theta) > \lambda_a(\theta)$ at some point $\theta \in (0, \frac{\pi}{a+1})$, then we would have $\ell_a(\theta) = \lambda_a(\theta)$ for at least two points θ in $(0, \frac{\pi}{a+1})$. Since $\ell_a(\theta) = \lambda_a(\theta)$ at the endpoints, Rolle's Theorem would imply that λ_a'' has at least two zeroes in $(0, \frac{\pi}{a+1})$, which is impossible as λ_a'' is increasing. $\square$

Proof of Lemma 4.16. By Lemma 4.17, we have

$$\int_0^{\frac{\pi}{a+1}} \lambda_a^{\mu+1} F_a \;\geq\; \int_0^{\frac{\pi}{a+1}} \ell_a \lambda_a^{\mu} F_a\,,$$

and so it suffices to prove

$$\int_0^{\frac{\pi}{a+1}} \ell_a \lambda_a^{\mu} F_a \;\geq\; C_a \int_0^{\frac{\pi}{a+1}} \lambda_a^{\mu} F_a\,.$$

This is equivalent to showing that

$$\int_0^{\frac{\pi}{a+1}} (\ell_a - C_a)\lambda_a^{\mu} F_a \;\geq\; 0. \tag{4.21}$$

As $L_a := \ell_a - C_a$ is linear, this is the difference of two integrals of positive functions. We establish the inequality (4.21) by estimating each of those integrals.

The function L_a is a line with slope $-\frac{(a+1)^2}{\pi}$ and zero at

$$b \ := \ \frac{2(a^2+2a-1)\pi}{(a+1)(3a^2+6a-1)} \ \in \ \left[\frac{\pi}{2(a{+}1)}\,,\,\frac{\pi}{a{+}1}\right].$$

The inequality (4.21) is equivalent to

$$\int_0^b L_a\,\lambda_a^\mu F_a \ \geq \ \int_b^{\frac{\pi}{a+1}} |L_a|\,\lambda_a^\mu F_a\,. \tag{4.22}$$

For $\theta \in [0, \frac{\pi}{2(a+1)}]$, the linear inequalities of Proposition 4.15 give

$$\sin{(a{+}1)\theta} \ \geq \ \frac{2}{\pi}(a{+}1)\theta \qquad \text{and} \qquad \sin\theta \ \leq \ \theta\,,$$

and thus

$$\lambda_a(\theta) \ = \ \frac{\sin{(a{+}1)\theta}}{\sin\theta} \ \geq \ \frac{2(a{+}1)}{\pi}\,.$$

Since $L_a\lambda_a^\mu F_a$ is nonnegative on $[0,b]$ and $\frac{\pi}{2(a+1)} < b$, we have

$$\int_0^b L_a\,\lambda_a^\mu F_a \ \geq \ \int_0^{\frac{\pi}{2(a+1)}} L_a\,\lambda_a^\mu F_a \ \geq \ \frac{2^\mu(a{+}1)^\mu}{\pi^\mu}\int_0^{\frac{\pi}{2(a+1)}} L_a\,F_a\,.$$

We may exactly compute this last integral to obtain

$$\begin{aligned}\int_0^{\frac{\pi}{2(a+1)}} L_a\,F_a \ = \ & \frac{1}{8\pi a^2(3a^2{+}6a{-}1)}\cdot\big((5\pi^2a^4{+}(10\pi^2{-}24)a^3{-}(7\pi^2{+}60)a^2{-}16a{+}4)\\ & +\cos\frac{a\pi}{a+1}\cdot(12a^4+48a^3+56a^2+16a-4)\\ & +\sin\frac{a\pi}{a+1}\cdot(-4\pi a^4-12\pi a^3+4\pi a^2+12\pi a)\big)\,.\end{aligned}$$

As $a>1$, we have $\cos\frac{a\pi}{a+1} > -1$ and $\sin\frac{a\pi}{a+1} > 0$. Substituting these values into this last formula and multiplying by $(2(a{+}1)/\pi)^\mu$ gives a lower bound for the integral on the left of (4.22),

$$A \ := \ \frac{2^\mu(a+1)^\mu((5\pi^2{-}12)a^4+(10\pi^2{-}72)a^3-(7\pi^2{+}116)a^2-32a+8)}{8\pi^{\mu+1}a^2(3a^2+6a-1)}\,. \tag{4.23}$$

For the integral on the right of (4.22), consider the line through the points $(\frac{\pi}{a+1}, 0)$

and $(b, \frac{2(a+1)}{\pi})$,

$$\mathcal{L}_a \ := \ \frac{2(3a^2+6a-1)}{\pi^2}\left(\frac{\pi}{a+1}-\theta\right).$$

We claim that $\lambda_a < \mathcal{L}_a$ in the interval $[b, \frac{\pi}{a+1}]$. To see this, first note that the slope of a secant line through $(\frac{\pi}{a+1}, 0)$ and a point $(\theta, \lambda_a(\theta))$ on the graph of λ_a is

$$\frac{\sin{(a+1)\theta}}{(\theta-\frac{\pi}{a+1})\sin\theta}. \tag{4.24}$$

Observe that $\sin{(a+1)\theta}$ is bounded above by the parabola

$$\sin{(a+1)\theta} \ \le \ \frac{4(a+1)^2}{\pi^2}\theta\left(\frac{\pi}{a+1}-\theta\right),$$

We use this bound and the Mercer–Caccia inequality for $\sin\theta$ to bound the slope (4.24),

$$\frac{\sin{(a+1)\theta}}{(\theta-\frac{\pi}{a+1})\sin\theta} \ \le \ \frac{4\pi(a+1)^2}{(3\pi^2-4\theta^2)} \ \le \ \frac{4(a+1)^4}{\pi(3a^2+6a-1)},$$

with the second equality holding as the minimum of the denominator $(3\pi^2-4\theta^2)$ on the interval $[b, \frac{\pi}{a+1}]$ occurs at $\theta = \frac{\pi}{a+1}$. When $a \ge 3$ we have,

$$\frac{4(a+1)^4}{\pi(3a^2+6a-1)} \ < \ \frac{2(3a^2+6a-1)}{\pi^2},$$

which so it follows that $\lambda_a < \mathcal{L}_a$ on $[b, \frac{\pi}{a+1}]$.

Using this and the easy inequality $F_a < 4$, we bound the integral on the right of (4.22),

$$\int_b^{\frac{\pi}{a+1}} |L_a|\,\lambda_a^\mu F_a \ < \ \int_b^{\frac{\pi}{a+1}} |L_a|\,\mathcal{L}^\mu F_a \ < \ \int_b^{\frac{\pi}{a+1}} 4|L_a|\,\mathcal{L}^\mu.$$

The last integral is not hard to compute,

$$B \ := \ \int_b^{\frac{\pi}{a+1}} 4|L_a|\,\mathcal{L}_a^\mu \ = \ \frac{2^{\mu+2}(a+1)^{\mu+3}[\mu+1-(a+1)(\mu+2)]}{\pi^{\mu-1}(\mu+1)(\mu+2)(3a^2+6a-1)^2}.$$

We claim that $A-B>0$, which will complete the proof of Lemma 4.16 and therefore the induction step for Lemma 4.14. For this, we observe that if multiply $A-B$ by their common (positive) denominator, we obtain an expression of the form $2^\mu(a+1)^\mu P(a,\mu)$, where P is a polynomial of degree six in a and two in μ. After making the substitution $P(3+x,3+y)$, we obtain a polynomial in x and y in which every coefficient in positive, which implies that $A-B>0$ when $a,m\geq 3$, and completes the proof. $\square$

2. Base of the induction for Lemma 4.14

We establish the inequality (4.17) of Lemma 4.14 when $\mu=4$, which is the base case of our inductive proof. This inequality is

$$\int_0^{\frac{\pi}{a+1}} \lambda_a^4\, F_a \;>\; \int_{\frac{\pi}{a+1}}^{\frac{\pi}{2}} |\lambda_a^4\, F_a| \qquad \text{for every } a\geq 3\,. \tag{4.25}$$

We establish this inequality by replacing each integral in (4.25) by one which we may evaluate in elementary terms, and then compare the values.

We first find an upper bound for the integral on the right. Recall that

$$\lambda_a(\theta) \;=\; \frac{\sin(a{+}1)\theta}{\sin\theta} \qquad \text{and} \qquad F_a(\theta) \;=\; 1-2\cos 2a\theta+\cos 2(a{+}1)\theta\,.$$

Since $|\lambda_a(\theta)|\leq \frac{1}{\sin\theta}$ and $|F_a(\theta)|\leq 4$ for $\theta\in[\frac{\pi}{a+1},\frac{\pi}{2}]$, we have

$$\int_{\frac{\pi}{a+1}}^{\frac{\pi}{2}} |\lambda_a^4\, F_a| \;\leq\; 4\int_{\frac{\pi}{a+1}}^{\frac{\pi}{2}} \frac{1}{\sin^4\theta} \;=\; \frac{4}{3}\cot\tfrac{\pi}{a+1}\big(2+\csc^2\tfrac{\pi}{a+1}\big)\,.$$

For $a\geq 3$, we have $0<\frac{\pi}{a+1}\leq\frac{\pi}{4}$. As we observed in Proposition 4.15, this implies that $\sin\frac{\pi}{a+1}\geq\frac{\pi}{a+1}\frac{2\sqrt{2}}{\pi}=\frac{2\sqrt{2}}{a+1}$, and so $\frac{1}{\sin\frac{\pi}{a+1}}\geq\frac{a+1}{2\sqrt{2}}$. Since $0\leq\cos\frac{\pi}{a+1}\leq 1$, we have

$$\frac{4}{3}\cot\tfrac{\pi}{a+1}\big(2+\csc^2\tfrac{\pi}{a+1}\big) \;\leq\; \frac{4(a{+}1)}{3\sqrt{2}}+\frac{(a{+}1)^3}{12\sqrt{2}} \;=:\; B\,. \tag{4.26}$$

We now find a lower bound for the integral on the left. We use the estimate from

Lemma 4.17, that for $\theta \in [0, \frac{\pi}{a+1}]$, we have

$$\lambda_a(\theta) \ \geq \ \ell_a(\theta) \ = \ \frac{(a+1)^2}{\pi}\left(\frac{\pi}{a+1} - \theta\right) .$$

Using this gives the estimate,

$$\int_0^{\frac{\pi}{a+1}} \lambda_a^4 F_a \ > \ \frac{(a+1)^8}{\pi^4}\int_0^{\frac{\pi}{a+1}} \left(\tfrac{\pi}{a+1} - \theta\right)^4 \left(1 - 2\cos 2a\theta + \cos 2(a{+}1)\theta\right) .$$

This may be evaluated in elementary terms to give

$$\frac{3(a{+}1)^8}{2a^5\pi^4}\sin\tfrac{2\pi}{a+1} + \frac{\pi(a+1)^3}{5} - \frac{2(a+1)^5}{\pi a^2} + \frac{3(a+1)^7}{\pi^3 a^4} + \frac{(a+1)^3}{\pi} - \frac{3(a+1)^3}{2\pi^3} \ . \quad (4.27)$$

Since, for $a \geq 3$, $0 \leq \frac{2\pi}{a+1} \leq \frac{\pi}{2}$, we have the bound from Proposition 4.15 of $\sin\frac{2\pi}{a+1} \geq \frac{4}{a+1}$. Thus the expression (4.27) is bounded below by

$$A \ := \ \frac{6(a{+}1)^7}{\pi a^5} + \frac{\pi(a+1)^3}{5} - \frac{2(a+1)^5}{\pi a^2} + \frac{3(a+1)^7}{\pi^3 a^4} + \frac{(a+1)^3}{\pi} - \frac{3(a+1)^3}{2\pi^3} \ . \quad (4.28)$$

Then the difference $A - B$ of the expressions from (4.28) and (4.26) is a rational function of the form

$$\frac{(a+1)\cdot P(a)}{120\pi^4 a^5} \ ,$$

where $P(a)$ is a polynomial of degree seven. If we expand $P(3{+}x)$ in powers of x, then we obtain a polynomial of degree seven in x with poisitve coefficients. This establishes the inequality (4.25) for all $a \geq 3$, which is the base case of the induction proving Lemma 4.14. This completes the proof of Lemma 4.14 and therefore of Lemma 4.9, and ultimately of Theorem 3.5. $\square$

CHAPTER V

CONCLUSION

Galois groups of enumerative problems are algebraic invariants that encode some structure of the geometry of the problems. In general, we expect these groups to be the full symmetric group, and when they are not, the geometric problem posses some intrinsic structure. We show that the Galois group of Schubert problems involving lines are either the alternating group or the full symmetric group. In addition, we conjecture these groups should be the full symmetric group. We provided two infinite families of Schubert problems with full symmetric Galois group, which provides evidence for our conjecture.

An immediate consequence of our theorem is that Schubert problems of lines do not have intrinsic structure. Moreover, the technique developed in the proof of Lemma 3.4 is by itself interesting. By interpreting the number of solutions (Kostka numbers) as integrals of certain trigonometric functions, we were able to bring continuous and analytical tools to a purely discrete and combinatorial problem. We believe our method generalizes to other Schubert problems in higher dimensional Grassmannians. We are working on the details of this generalization.

In his paper, Vakil conjectured that Schubert problems involving planes in $\mathbb{P}^3$ have Galois groups at least alternating. Vakil tested his criterion for all Schubert problems in $\mathbb{G}(2,n)$ for $n \leq 8$, verifying this conjecture. We are developing software to test Vakil's criterion for Schubert problems in $\mathbb{G}(k,n)$ in general, trying to extend Vakil's investigation in a more general setting. This is part of a long-term project devoted to understand the Galois groups of Schubert problems in general.

REFERENCES

[1] S. Billey and R. Vakil, *Intersections of Schubert varieties and other permutation array schemes*, in Algorithms in algebraic geometry, vol. 146 of IMA Vol. Math. Appl., Springer, New York, 2008, pp. 21–54.

[2] C. I. Byrnes, *Pole assignment by output feedback*, in Three decades of mathematical system theory, vol. 135 of Lecture Notes in Control and Inform. Sci., Springer, Berlin, 1989, pp. 31–78.

[3] C. I. Byrnes and P. K. Stevens, *Global properties of the root-locus map*, in Feedback control of linear and nonlinear systems (Bielefeld/Rome, 1981), vol. 39 of Lecture Notes in Control and Inform. Sci., Springer, Berlin, 1982, pp. 9–29.

[4] A. Cayley, *On the triple tangent planes of surfaces of the third order*, Cambridge and Dublin Math. J., 4 (1849), pp. 118–138.

[5] W. Fulton, *Young tableaux*, vol. 35 of London Mathematical Society Student Texts, Cambridge University Press, Cambridge, 1997.

[6] J. Harris, *Galois groups of enumerative problems*, Duke Math. J., 46 (1979), pp. 685–724.

[7] J. Harris, *Algebraic geometry*, vol. 133 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1992. A first course.

[8] C. Jordan, *Traité des Substitutions*, Gauthier-Villars, Paris, 1870.

[9] S. Kleiman, *The transversality of a general translate*, Compositio Math., 28 (1974), pp. 287–297.

[10] S. Kleiman and D. Laksov, *Schubert calculus*, Amer. Math. Monthly, 79 (1972), pp. 1061–1082.

[11] A. Leykin and F. Sottile, *Galois groups of Schubert problems via homotopy computation*, Math. Comp., 78 (2009), pp. 1749–1765.

[12] L. Manivel, *Symmetric functions, Schubert polynomials and degeneracy loci*, vol. 6 of SMF/AMS Texts and Monographs, American Mathematical Society, Providence, RI, 2001. Translated from the 1998 French original by John R. Swallow, Cours Spécialisés [Specialized Courses], 3.

[13] A. M. Mercer, U. Abel, and D. Caccia, *A sharpening of Jordan's inequality*, The American Mathematical Monthly, 93 (1986), pp. 568–569.

[14] J. Ruffo, Y. Sivan, E. Soprunova, and F. Sottile, *Experimentation and conjectures in the real Schubert calculus for flag manifolds*, Experiment. Math., 15 (2006), pp. 199–221.

[15] G. Salmon, *On the triple tangent planes to a surface of the third order*, Cambridge and Dublin Math. J., 4 (1849), pp. 252–260.

[16] H. Schubert, *Die n-dimensionalen Verallgemeinerungen der fundamentalen Anzahlen unseres Raume*, Math. Ann., 26 (1886), pp. 26–51. (dated 1884).

[17] A. Sommese and C. Wampler, *The numerical solution of systems of polynomials*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005.

[18] F. Sottile, *Enumerative geometry for the real Grassmannian of lines in projective space*, Duke Math. J., 87 (1997), pp. 59–85.

[19] R. Vakil, *A geometric Littlewood-Richardson rule*, Ann. of Math. (2), 164 (2006), pp. 371–421. Appendix A written with A. Knutson.

[20] R. Vakil, *Schubert induction*, Ann. of Math. (2), 164 (2006), pp. 489–512.