Problems on Sums of Squares and Positive Polynomials I

(1) Suppose $f, g_1, \ldots, g_k \in \mathbb{R}[X]$ and $f = g_1^2 + \ldots g_k^2$. Show that $\deg f = 2\max\{\deg g_i\}$. Show that if f is homogeneous of degree $2d$, then each g_i is homogeneous of degree d.

(2) Write $t^6 + 2t^5 - t^4 - 2t^2 - 8t + 8$ as a sum of two squares in $\mathbb{R}[t]$.

(3) Prove that $X^2Y^2 + X^2Z^2 + Y^2Z^2 - 4XYZ + 1$ is psd and not sos.

(4) Let $\mathbb{R}[t]$ denote the polynomial ring over $\mathbb{R}$ in one variable and consider $S(t^3) = [0, \infty)$ and $P = P(t^3)$. Prove that for $f \in \mathbb{R}[t]$, $f \geq 0$ does not always imply $f \in P$. Hint: Consider $1 + t \in \mathbb{R}[t]$ and assume that a representation exists. Now check the degrees of each side of the equation you wrote down.

(5) Let $f = X_1X_3^3 + X_2X_3^3 + X_1^2X_2^2 - X_1X_2X_3^2$ and $g = X_1^2X_2 + X_2^2X_3 + X_3^2X_1 - X_1X_2X_3$.

(a) Show that $f, g \geq 0$ on Δ_3 with zeros only at the vertices.

(b) Use computer algebra software to show that g satisfies the conclusion of Pólya's Theorem (with "strictly positive coefficents" replaced by "nonnegative coefficients").

(c) Prove that for any $N \in \mathbb{N}$, $(X_1 + X_2 + X_3)^N f$ has negative coefficients. Hint: Consider the coefficient of $X_1^n X_2 X_3^2$.