Exercises from Sottile's Lectures

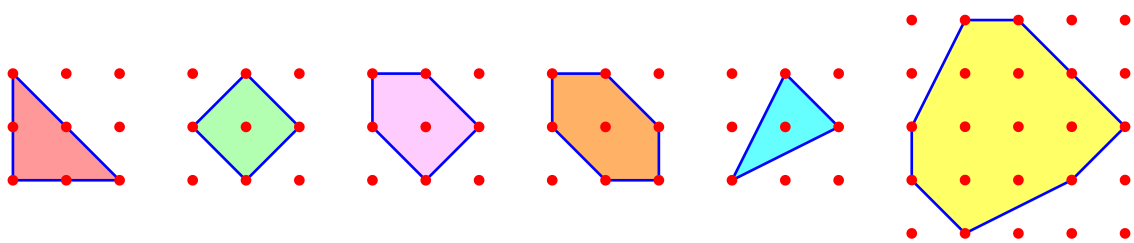
Exercises I.

1. Use the embedding $x \mapsto (x, x^{-1})$ of the torus $\mathbb{C}^*$ into $\mathbb{C}^2$, or any other method, to show that the coordinate ring of the torus $\mathbb{C}^*$ may be identified with $\mathbb{C}[x, x^{-1}]$. Deduce that the coordinate ring of $(\mathbb{C}^*)^n$ may be identified with $\mathbb{C}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$. Show that this is the complex group ring $\mathbb{C}[M]$ of the free abelian group of characters $M = \mathbb{Z}^n$ of the torus $(\mathbb{C}^*)^n$. Can you relate the structure of $\mathbb{C}[M]$ to the group structure on $(\mathbb{C}^*)^n$?
2. Show that the monomial map $\varphi_{\mathcal{A}} \colon (\mathbb{C}^*)^n \to (\mathbb{C}^*)^{\mathcal{A}}$ is injective if and only if $\mathcal{A}$ generates M. When $\mathcal{A}$ does not generate M, identify the kernel of $\varphi_{\mathcal{A}}$.
3. Let $\mathcal{A} \subset \mathbb{Z}^6 = (Z^2)^3$ be the columns of the 6×8 matrix

$$A \;=\; \begin{pmatrix} 1&1&1&1&0&0&0&0\\ 0&0&0&0&1&1&1&1\\ 1&1&0&0&1&1&0&0\\ 0&0&1&1&0&0&1&1\\ 1&0&1&0&1&0&1&0\\ 0&1&0&1&0&1&0&1 \end{pmatrix}$$

   What is $X_{\mathcal{A}}$ and $I_{\mathcal{A}}$? (Describe $X_{\mathcal{A}}$ and give a Gr obner basis for $I_{\mathcal{A}}$.) Hint: The even (respectively odd) numbered rows give the vertices of the 3-cube.
4. Show that the collection of 2×2 minors of the $k \times m$ matrix $(z_{ij})_{i=1,\dots,k}^{j=1,\dots,m}$ of indeterminates forms a reduced Gröbner basis for the toric ideal of the variety $X_{\mathcal{A}}$ of matrices of rank 1, where where the term order is degree reverse lexicographic with the variables ordered by $z_{a,b} \prec z_{c,d}$ if $a < c$ or $a = c$ and $b > d$.

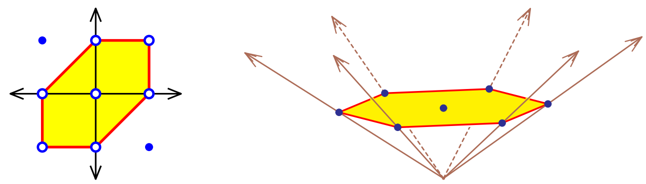
   What about a reduced Gröbner basis in other term orders?
5. Identify a generating set and a reduced Gröbner basis for the toric ideal $I_{\mathcal{A}}$ given by as many of the following subsets of $\mathbb{Z}^2$ as you can.



   Are toric ideals always generated by quadratic polynomials?
6. Work out the details of the suggested proof of Theorem 1.4 (From Sottile's Notes). Explain why 'reduced' is necessary for the conclusion.
7. Let S be a finitely generated subsemigroup of M. Show that every maximal ideal $\mathfrak{m}$ of $\mathbb{C}[S]$ restricts to a semigroup homomorphism from S to $\mathbb{C}$, and vice-versa. Hint: use that maximal ideals correspond to algebra maps $\mathbb{C}[S] \to \mathbb{C}$.
8. For $\mathcal{A} \subset M$ finite, show that $X_{\mathcal{A}}(\mathbb{R}) = \mathrm{Hom}_{sg}(\mathbb{N}\mathcal{A}, \mathbb{R})$ and $X_{\mathcal{A},\geq} = \mathrm{Hom}_{sg}(\mathbb{N}\mathcal{A}, \mathbb{R}_{\geq})$.

Exercises II.

1. Prove that if a finite set $\mathcal{A} \subset \mathbb{Z}^n$ is homogeneous with $\operatorname{rank}(\mathbb{Z}\mathcal{A}) = 1 + m$, then there is a basis for $\mathbb{Z}\mathcal{A}$ identifying it with $\mathbb{Z}^{1+m}$ and a subset $\mathcal{B} \subset \mathbb{Z}^m$ such that $\mathcal{A} = \mathcal{B}^+$.
2. Suppose that $\mathcal{A} \subset \mathbb{Z}^{1+n}$ is the set of column vectors of an integer matrix. Show that $\mathcal{A}$ is homogeneous if and only if its row space in $\mathbb{Q}^{\mathcal{A}}$ has a vector with every coordinate 1.
3. Give a spanning set of degree two generators for $I_{\mathcal{A}}$, where $\mathcal{A}$ is the lifted hexagon.



   Interpret each generator as a point common to the convex hulls of two disjoint subsets of $\mathcal{A}$.
4. Let $\mathcal{A} \subset \mathbb{Z}^n$. Show that the map $\varphi_{\mathcal{A}} \colon (\mathbb{C}^*)^n \to \mathbb{P}^{\mathcal{A}}$ is injective if and only if the integral affine span of $\mathcal{A}$ is $\mathbb{Z}^n$.
5. Prove the equivalence of the two definitions (2.3) and (2.3) of affine span.
6. Show that the Euclidean volume of the simplex $\operatorname{conv}\{0, e_1, \ldots, e_n\}$ is $\frac{1}{n!}$, where e_i is the standard coordinate unit vector in $\mathbb{R}^n$. Harder: Prove that this is the minimum volume of any lattice simplex, and that all others have volume an integer multiple of $\frac{1}{n!}$.