A Weyl character formula for Hessenberg varieties

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§0. Introduction

λ : regular dominant weight of $T^n \subset GL_n(\mathbb{C})$

- V_λ : irrep of $GL_n(\mathbb{C})$ with highest weight λ
- $P(\lambda) \subset \mathbb{R}^n$: permutohedron assoc. to λ

Weyl character formula :

$$\mathrm{char}(V_\lambda^*) = \boxed{\sum_{w \in \mathfrak{S}_n} \frac{e^{w\lambda}}{\prod_{\alpha:\mathrm{pos}}(1 - e^{-w\alpha})}}$$

The (formal) sum of weights appearing in V_λ

$$S(P(\lambda)) := \sum_{\mu \in P(\lambda) \cap (L+\lambda)} e^\mu = \boxed{\sum_{w \in \mathfrak{S}_n} \frac{e^{w\lambda}}{\prod_{\alpha:\mathrm{simp}}(1 - e^{-w\alpha})}}$$

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Goal of this talk:

Unify these two formulas.

The (full) flag variety (of type A_{n-1}) is the collection of complete flags of linear subspaces in $\mathbb{C}^n$:

$$Fl(\mathbb{C}^n) = \{ \ (\{0\} \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = \mathbb{C}^n) \ \}.$$

Hessenberg varieties are subvarieties of $Fl(\mathbb{C}^n)$.

S : regular semisimple $n \times n$ matrix / $\mathbb{C}$,
h : $[n] \to [n]$ a function satisfying the following for all i:

- $h(i+1) \geq h(i)$
- $h(i) \geq i+1$

The (regular semisimple) **Hessenberg variety** (associated to h) is

$$X(h) := \{V_\bullet \in Fl(\mathbb{C}^n) \mid SV_i \subset V_{h(i)} \text{ for all } i\}.$$

- non-singular, projective
- $T^n \curvearrowright Fl(\mathbb{C}^n)$ preserves $X(h) \subset Fl(\mathbb{C}^n)$

$$X(h) = \{V_\bullet \in Fl(\mathbb{C}^n) \mid SV_i \subset V_{h(i)} \text{ for all } i\}$$

<u>e.g.</u> $h(i) = n \quad (\forall i)$

$$\Longrightarrow \quad X(h) = Fl(\mathbb{C}^n),$$

$h(i) = i + 1 \quad (\forall i)$

$\Longrightarrow \quad X(h) =$ the toric variety assoc. to Permutohedron

$= \text{Perm}$

For $h:[n]\to[n]$, define

$$M_h := \{e_i - e_j \in \mathbb{R}^n \mid i < j \le h(i)\} \subset \Phi^+$$

(a collection of postive roots of type A_{n-1})

e.g.

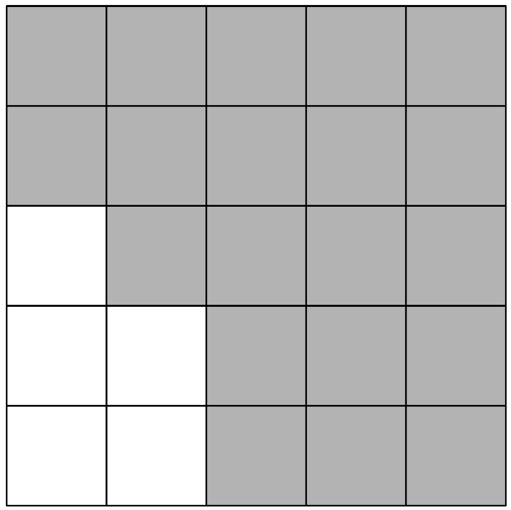
$h(i) = n$ for all i ($X(h) = Fl(\mathbb{C}^n)$) $\implies$ $M_h =$ positive roots

$h(i) = i + 1$ for all i ($X(h) =$ Perm) $\implies$ $M_h =$ simple roots

$\xi_h := \sum_{\alpha \in M_h} \alpha$: a weight of T^n

$$\xi_h = \sum_{\alpha \in M_h} \alpha = \sum_{i=1}^{n-1} (-a(i) - b(i) + 2)\varpi_i$$

e.g. $h = (2, 3, 5, 5, 5)$



$a(1) = 3 - 2 = 1, \;\; a(2) = 5 - 3 = 2, \; \ldots$

$b(1) = 5 - 5 = 0, \;\; b(2) = 5 - 4 = 1, \; \ldots$

$$\xi_h = \sum_{\alpha \in M_h} \alpha = \sum_{i=1}^{n-1} (-a(i) - b(i) + 2)\varpi_i \quad : \text{ a weight of } T^n$$

λ : a weight of T^n $\mapsto$ a line bundle L_λ on $Fl(\mathbb{C}^n) = G/B$

Proposition (A-Fujita-Lane)

Let λ be a weight of T^n. If $\lambda + \xi_h$ is regular dominant, then

$$\mathrm{char}_{T^n} H^0(X(h), L_\lambda) = \sum_{w \in \mathfrak{S}_n} \frac{e^{w\lambda}}{\prod_{\alpha \in M_h}(1 - e^{-w\alpha})}.$$

- $X(h) = Fl(\mathbb{C}^n) \implies M_h =$ positive roots
- $X(h) =$ Perm $\implies M_h =$ simple roots

Two extremal cases of this formula:

$$\text{char}_{T^n}\ H^0(Fl(\mathbb{C}^n), L_\lambda) = \sum_{w \in \mathfrak{S}_n} \frac{e^{w\lambda}}{\prod_{\alpha:\text{pos}}(1 - e^{-w\alpha})}$$

$$\text{char}_{T^n}\ H^0(\text{Perm}, L_\lambda) = \sum_{w \in \mathfrak{S}_n} \frac{e^{w\lambda}}{\prod_{\alpha:\text{simp}}(1 - e^{-w\alpha})}$$

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Thank you for your attention!