Applications and Combinatorics in Algebraic Geometry

Frank Sottile

Summary

Algebraic Geometry is a deep and well-established field within pure mathematics that is increasingly finding applications outside of mathematics. These applications in turn are the source of new questions and challenges for the subject. Many applications flow from and contribute to the more combinatorial and computational parts of algebraic geometry, and this often involves real-number or positivity questions. The scientific development of this area devoted to applications of algebraic geometry is facilitated by the sociological development of administrative structures and meetings, and by the development of human resources through the training and education of younger researchers.

One goal of this project is to deepen the dialog between algebraic geometry and its applications. This will be accomplished by supporting the research of Sottile in applications of algebraic geometry and in its application-friendly areas of combinatorial and computational algebraic geometry. It will be accomplished in a completely different way by supporting Sottile's activities as an officer within SIAM and as an organizer of scientific meetings. Yet a third way to accomplish this goal will be through Sottile's training and mentoring of graduate students, postdocs, and junior collaborators.

The **intellectual merits** of this project include the development of applications of algebraic geometry and of combinatorial and combinatorial aspects of algebraic geometry. Specifically, Sottile will work to develop the theory and properties of orbitopes from the perspective of convex algebraic geometry, continue to investigate linear precision in geometric modeling, and apply the quantum Schubert calculus to linear systems theory. In combinatorial algebraic geometry, Sottile will work to clarify the foundations of tropical algebraic geometry, study equivariant cohomology of arithmetic toric varieties, and continue to investigate generalizations of the Shapiro conjecture for flag manifolds.

The **broader impacts** of this project include Sottile's training of graduate students, postdocs, and young researchers through his direct mentoring and web of collaboration (of 22 collaborators in active research related to this proposal, 9 are graduate students or postdocs). Other broader impacts include the wide dissemination of the research conducted by members of his research team at conferences and seminars where they will make presentations. A particularly important broader impact is the building of institutional infrastructure to support the applications of algebraic geometry—Sottile is the founding chair of the new SIAM Activity Group on Algebraic Geometry, and activities he carries out under this project will help to promote this activity group and make it relevant to the profession. This includes the organization of and Sottile's attendance at some key conferences, his spending (but not organizing) the Winter/Spring term of 2011 at the Institut Mittag-Leffler for the program on applicable algebraic geometry, and the Winter term of 2013 at the MSRI program in Commutative Algebra (if that is approved).

Project Description

1. Results from previous NSF support

Recently, my work has been supported by NSF grant DMS-0701050 "Applicable Algebraic Geometry: Real Solutions, Applications, and Combinatorics" (September 2007–August 2010). Another NSF grant, "Numerical Real Algebraic Geometry" DMS-0915211, has just started. This primarily supports a postdoc (Jon Hauenstein), a graduate student (Abraham Martín del Campo), and an undergraduate student (Christopher Brooks) to work on algorithms and software in Numerical Real Algebraic Geometry and a computational investigation of Galois groups of Schubert problems. The research to be carried out under DMS-0915211 is disjoint from the research that I am proposing here. For the period May 2008–May 2010 I hold a grant from Texas's Norman Hackerman Advanced Research Program, "Algebraic Geometry in Algebraic Statistics and Geometric Modeling" [AGS], which is joint with Luis García of Sam Houston State University. The Texas A&M portion supports graduate students Corey Irving and Weronika Buczynska, and some travel.

Since September 2007, I have published or completed 24 papers and one book, working with 27 different coauthors (12 who were postdocs or graduate students). These are listed at the end of this section, and are referenced in this proposal by their numbers enclosed in brackets, e.g. [3]. Other citations are alphanumeric and are found in the bibliography, e.g. [HL82]. The book, "Real Solutions to Equations From Geometry" [25], consists of expanded lecture notes which I am trying to get published.

1.1. Broader impacts.

This past support helped me to train junior scientists. In Summer 2008, I supported undergraduate Andy Howard who also worked with L. García. His poster won a prize at the MAA Undergraduate Student Poster Session at the January Joint Mathematical meetings in 2009. I partially supported four current graduate students, while also providing them with computers for their work and helping them to attend conferences. I supported the work of postdocs Chris Hillar, Zach Teitler and Aaron Lauve, by providing them with computers for their work and with travel funds. All three have received research grants that I helped them formulate and write, Hillar and Lauve receiving NSA Young Investigator grants and Teitler as a Co-PI on a SCREMS equipment grant.

Other broader impacts include support of my organizational and societal work. This last year, I led the successful effort to establish an activity group with SIAM devoted to the applications of algebraic geometry. I also helped to organize many scientific events, from regional workshops (yearly Texas Algebraic Geometry Seminars and CombinaTexas), to national events (MSRI workshop on Combinatorial, enumerative, and toric geometry in March 2009), to international meetings (Effective Methods in Algebraic Geometry, MEGA09 in Barcelona, and the semester and two workshops at the Bernoulli Centre at the EPFL in Lausanne in Winter 2008). I have also given or organized three short courses in Europe (January 2008, June 2009, and November 2009).

1.2. Intellectual merits.

I describe some of the research conducted under this previous support.

1.2.1. *From Fewnomial Bounds to Numerical Real Algebraic Geometry.*

Askold Khovanskii [Kho80] proved that a system of n polynomials in n variables

$$f_1(x_1,\dots,x_n) \;=\; f_2(x_1,\dots,x_n) \;=\; \cdots = \; f_n(x_1,\dots,x_n) \;=\; 0\,, \tag{1}$$

having a total of $n+l+1$ distinct monomial terms can have no more than

$$2^{\binom{l+n}{2}}(n+1)^{n+l} \tag{2}$$

non-degenerate positive solutions. Bihan and I [8] introduced a method to transform a polynomial system (1) into its *Gale Dual* system, which has the form

$$\prod_{j=1}^{n+l} p_j(y)^{\alpha_j^{(k)}} \;=\; 1 \qquad k=1,\dots,l\,, \tag{3}$$

where each $p_i(y)$ is an affine function on $\mathbb{R}^l$.

This transformation comes with a scheme-theoretic isomorphism between the solutions to (1) (when all $x_i \neq 0$) and to (3) (when all $p_j(y) \neq 0$), which restricts to an isomorphism where $x_i > 0$ and $p_j(y) > 0$. Using some ideas of Khovanskii's, we [2] gave a bound of

$$\tfrac{e^2+3}{4} 2^{\binom{l}{2}} n^l$$

for the number of positive ($p_j(y) > 0$) solutions to the Gale system (3), thereby obtaining the same bound for the number of positive solutions to the original system (1), which is smaller than (2). Rojas, Bihan, and I [3] showed that this bound is asymptotically sharp, for l fixed and n large. Bates, Bihan, and I [4] adapted these arguments to give the bound

$$\tfrac{e^4+3}{4} 2^{\binom{l}{2}} n^l$$

for all non-zero real solutions to (1), when the differences of the exponents span $\mathbb{Z}^n$. We also found bounds on the total Betti numbers of a fewnomial hypersurface [16], and gave a smaller bound when the polynomials f_i have pairwise distinct monomial terms [19].

Dan Bates and I realized that the method of proof in [2, 4] leads to a new numerical continuation algorithm that computes the real solutions of a Gale system (3) (and hence of (1)) without computing any complex solutions. We described this algorithm in [21], where we also gave a proof-of-concept implementation when $l = 2$. Further investigation and implementation of this algorithm is a major goal of DMS-0915211.

1.2.2. *Galois Groups.* Jordan [Jor70] first considered Galois groups of enumerative geometric problems, which are important invariants. The geometric problem has special structure when the Galois group is deficient (not equal to the full symmetric group on the solutions). Harris laid their modern foundations in 1979 [Har79], showing that the algebraic Galois group equals a geometric monodromy group. These Galois groups are very difficult to compute [BV08, Vak06]. However, Leykin and I realized that numerical homotopy continuation is ideally suited for computing elements of these Galois/monodromy groups. Using off-the-shelf implementations, we studied Schubert problems on Grassmannians that may be formulated as complete intersections [12]. In all problems we studied, the Galois group was the full symmetric group, indicating that there is no additional structure. This involved problems with as many as 17589 solutions, and our project demonstrated the potential of numerical computation as an aid to research in pure mathematics.

A second goal of DMS-0915211 is a multi-year project to compute Galois groups of perhaps millions of Schubert problems on Grassmannians, cataloging all deficient problems of reasonable size. There are both numeric and symbolic approaches, and different algorithms to develop and implement. As with the work in Numerical Real Algebraic Geometry, this work is disjoint from this proposal.

1.2.3. *Skew Littlewood-Richardson Rules From Hopf algebras.* Aaron Lauve, Thomas Lam, and I established a simple formula involving the harpoon ($\rightharpoonup$) action of a Hopf algebra on its dual (which is a standard in the theory of Hopf algebras [Mon93]), and showed how it implies new combinatorial formulas in the algebra of symmetric functions and other related combinatorial Hopf algebras [22]. In particular, we prove a version of the Littlewood-Richardson rule for skew Schur functions that was conjectured by Assaf and McNamara [AM], and which extends their skew Pieri formula. This is one of the first applications of ideas from Hopf algebras to prove a new formula in enumerative combinatorics (heretofore, Hopf algebras helped us to better understand combinatorics).

1.3. **Recent Grant-Supported Research.**

[1] *Real Hessian curves*, with A. Ortiz-Rodríguez. Boll. Soc. Math. Mexico, **13**, 2007.

[2] *New Fewnomial Upper Bounds from Gale dual polynomial systems*, with Frédéric Bihan, Moscow Mathematics Journal, **7** (2007), Number 3, 387–407.

[3] *Sharpness of fewnomial bound and the number of components of a fewnomial hypersurface*, with F. Bihan and M. Rojas, Algorithms in Algebraic Geometry, A. Dickenstein, F. Schreyer, and A. Sommese. eds. pp. 15–20, Springer, 2007.

[4] *Bounds on the number of real solutions to polynomial equations*, with Daniel J. Bates, Firédéric Bihan, IMRN, 2007, 2007:rnm114-7.

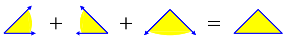
[5] *Line problems in non-linear computational geometry*, with T. Theobald. Surveys in Discr.and Comput. Geom., Contemp. Math., **453** AMS, 411–432, 2008.

[6] *The recursive nature of cominuscule Schubert calculus*, with K. Purbhoo. Advances in Mathematics, **217** (2008), pp. 1962–2004.

[7] *Convex hulls of orbits and orientations of a moving protein domain*, with M. Longinetti and L. Sgheri, Discrete and Comput. Geometry, DOI: 10.1007/s00454-008-9076-8.

[8] *Gale duality for complete intersections*, with Frédéric Bihan, Annales de l'Institut Fourier, Tome 58 (2008) fasicule 3, 877–891.

[9] *Equivariant Chow groups of the quot scheme*, with T. Braden and L. Chen. Pacific Journal of Mathematics, **238**, No. 2 (2008) 201–232.

[10]  (Theorems of Brion, Lawrence, and Varchenko on rational generating functions for cones), with M. Beck and Ch. Haase. Mathematical Intelligencer, **31**, No. 1, 9–17.

[11] *A Littlewood-Richardson rule for Grassmannian permutations*, with Kevin Purbhoo. Proc. Amer. Math. Soc., **137** (2009) 1875–1882.

[12] *Galois groups of Schubert problems via numerical homotopy continuation*, with A. Leykin. Mathematics of Computation, **78** (2009) 1749–1765.

[13] *Linear precision for parametric patches*, with Luis García-Puente. Advances in Computational Mathematics. DOI: 10.1007/s10444-009-9126-7.

[14] *Linear precision for toric surface patches*, with K. Ranestad and H-Ch. Graf von Bothmer. Found. Comput. Math., DOI: 10.1007/s10208-009-9052-6.

[15] *General isotropic flags are general (for Grassmannian Schubert calculus)*, Journal of Algebraic Geometry, to appear. `arXiv:0801.2611`.

[16] *Betti number bounds for fewnomial hypersurfaces via stratified Morse theory*, with Frédéric Bihan, `arXiv:0801.2554`. Proc. AMS, to appear.

[17] *Some geometrical aspects of control points for toric patches*, with G. Craciun and L. García-Puente, `arXiv:0812.1275`. Springer LNCS, to appear.

[18] *Frontiers of Reality in Schubert Calculus*, `arXiv:0907.1847`, 40 pages. Bulletin of the AMS, to appear.
[19] *Fewnomial bounds for completely mixed polynomial systems*, `arXiv:0905.4543`, with Frédéric Bihan, Advances in Geometry, to appear. 13 pages.
[20] *Experimentation at the Frontiers of Reality in Schubert Calculus*, with C. Hillar, L. García, A. Martín del Campo, J. Ruffo, Z. Teitler, and S. Johnson. 16 pages. `arXiv:0906.2497`.
[21] *Khovanskii-Rolle continuation for real solutions*, with Daniel J. Bates, 21 pages, `arXiv:0908.4579`.
[22] *A Skew Littlewood-Richardson rule from Hopf algebras*, with Thomas Lam and Aaron Lauve. 10 pages. `arXiv:0908.3714`.
[23] *New Hopf Structures on Binary Trees*, with Stefan Forcey and Aaron Lauve.
[24] *Arithmetic toric varieties*, with Javier Elizondo, Paulo Lima-Filho, and Zach Teitler.
[25] *Real Solutions to Equations From Geometry*, book, 120 pages. `arXiv:0609829`.

2. Proposed research

I propose research in two interrelated areas: applications of algebraic geometry and combinatorial algebraic geometry, with projects in each area. These are interrelated because the same objects and ideas appear in both areas. For example, real toric varieties are the central object in the work of § 2.1.2 on toric patches, but that work inspired the study of arithmetic toric varieties of § 2.2.3, and arithmetic real toric varieties appear also in § 2.1.1 on orbitopes. Similarly, the work on geometry and combinatorics of the Schubert calculus of § 2.2.5 reinforces my plan to apply quantum Schubert calculus to control theory (§ 2.1.3) and my further investigations of the Shapiro Conjecture in § 2.2.4. While I have a history of work in these areas, the main projects in each area, orbitopes and foundations of tropical geometry, represent new research directions.

One common theme in this work is its use of computers. In each of these projects (except for tropical geometry and arithmetic toric varieties), computation and visualization plays a key role, none more so than the work on the Shapiro conjecture which requires serious work on supercomputers, such as the calclabs of the TAMU math department (a 700 core supercomputer whose day job is calculus instruction) or the Brazos Cluster at TAMU, of which I jointly control 160 cores through a SCREMS grant. I, and the students and postdocs working with me, use our personal computers on a daily basis for testing and studying examples and writing software. It significantly increases the productivity of students and postdocs to provide them with laptops, which become indispensable for their research.

Another theme is the collaborative nature of my work. Other than survey articles, I have been the sole author of exactly two short papers in the past decade. One reason is that I use collaboration as a vehicle to help train students and postdocs; there is no better way to learn a topic than by doing it. Another reason is that much of my work has components beyond my expertise, and I actively seek out collaborators complement my expertise.

A significant portion of this project will support students and postdocs working with me, either directly or via travel reimbursement. I take my responsibility to help guide their professional development very seriously. My research team meets regularly to discuss our work and professional matters. This includes regular lunchtime meetings where we share advice on research, writing papers, refereeing, giving and watching talks, courses, mathematics, and other aspects of the profession.

I train those who work with me in all aspects of our profession. At scheduled weekly meetings, we discuss their research and other professional matters, from their classes, to their teaching, to talks they may be giving, to papers, job applications, and research grants. For all of these tasks, I critique their work and encourage their efforts. On top of these tutorials, everyone who starts with me is given books to aid their professional development, such as [BC99, Con96, Kra99, Kra96, Kra04, SW00, Zin05].

This is more than just talking. I get my students and postdocs invited to give presentatoins at seminars and conferences, help them plan, prepare, and practice their talks, and support their travel. I get them a referee job as a tool to discuss with them the ins and outs of refereeing. I have now helped four postdocs write research grants; three have individual research grants and fourth is a Co-PI on an equipment grant.

2.1. **Applications.** Algebraic geometry is a deep and theoretically important part of pure mathematics that provides tools and insights for the applications of mathematics. In turn, algebraic geometry is enriched by ideas and techniques developed in applications. The proposed work in applications of algebraic geometry will inform applications and exploit mathematical structures discovered in applications.

This proposal will not only support my research in applications, but also some of the activities I will carry out promoting applications of algebraic geometry. For the first half of the three-year duration of this proposal, I will serve as the chair of the newly formed SIAM Activity Group on Algebraic Geometry, whose creation I led. Besides working as the public face of this area, I am chairing the program committee for its first biennial conference scheduled for October 2011. I will also help organize a meeting in 2011/2 on Algebraic Geometry and Geometric Modeling, the fourth such since the original Vilnius meeting in 2002. I also plan to attend the full program during the semester at the Institut Mittag-Leffler in Winter/Spring 2011 on “Applicable Algebraic Geometry”.

Two graduate students work with me on applications. One, Weronika Buczynska, will graduate in August 2010, and the other, Corey Irving, is scheduled to graduate in May 2011. I expect at least one new student will join this part of my research team this year.

2.1.1. *Orbitopes.* An *orbitope* is the convex hull of an orbit of a compact group G acting linearly on a vector space. Many special cases of these highly symmetric convex bodies have been studied from different perspectives (primarily from convexity, combinatorics, or complexity [BB05, BN08, Car07, HL82, GLS05], [7]). With Bernd Sturmfels and Raman Sanyal (a Miller Fellow at Berkeley) we are initiating the systematic study of orbitopes from many perspectives, including a new one afforded by optimization.

Semidefinite programming is field within convex optimization that extends linear programming and provides efficient tools for optimizing linear objective functions over *spectrahedra*. These convex semialgebraic sets are affine sections of the cone of positive semi-definite matrices. As linear objective functions may be pulled back along linear projections, the natural domain for semidefinite programs are projections of spectrahedra. A motivation for the systematic study of orbitopes is that they form a rich class of convex semialgebraic sets, providing a testing ground for the fundamental question in convex algebraic geometry:

(4) Is every convex semialgebraic set the projection of a spectrahedron?

As a fundamental structural question about convex semialgebraic sets, this is both theoretically compelling and fundamentally important for optimization.

For me, this came from work with Longinetti and Sgheri [7] on a question that arose from work on algorithms for the structure of certain metallo-proteins [GLS05]. The orthogonal group $O(3)$ acts irreducibly by conjugation on the 5-dimensional vector space W_3 of symmetric 3×3 matrices with trace zero. We studied orbitopes in W_3 and in $W_3 \oplus W_3$, with the goal of determining their facets and Carathéodory numbers (Each point x in an orbitope $\mathcal{O}$ is a convex combination of points of the orbit. If d_x is the minimum number of points needed, then the *Carathéodory number* of $\mathcal{O}$ is $\max\{d_x \mid x \in \mathcal{O}\}$.) We completed this investigation for orbitopes in W_3, but obtained only partial answers for orbitopes in $W_3 \oplus W_3$.

It was clear to me that orbitopes are interesting objects which had not yet been systematically studied. The fundamental question (4) about the structure of convex semialgebraic sets spurred Sanyal, Sturmfels, and I to begin their systematic study. We are writing a foundational paper, which will treat several general families of orbitopes, including the $O(n)$-orbitopes in W_n, the $n \times n$ symmetric matrices of trace zero, and orbitopes for $SO(2)$, which were studied by Carathéodory [Car07] and Barvinok and Novik [BN08].

Basic questions from convexity include determining the facial structure of an orbitope, studying its polar body, and determining its Carathéodory number. The Carathéodory number is particularly subtle as it is a generalization of the problem of determining tensor rank [KB09], which is also generating much interest in algebraic geometry [Lan08, LT09]. Basic questions about orbitopes from algebraic geometry include describing the Zariski closure of the boundary of an orbitope $\mathcal{O}$ and in particular understanding its Fano subvarieties, as the affine span of a face of $\mathcal{O}$ is a linear space contained in this Zariski closure—this is the algebraic relaxation of determining the faces of $\mathcal{O}$. Not only are we interested in the fundamental question (4), but also in an arithmetic version—over which subfield of $\mathbb{R}$ does an orbitope have a representation as a spectrahedra? Quite often an orbitope is a spectrahedron over $\mathbb{R}$ but only a projection of a spectrahedron over its field of definition.

We already plan two additional papers. One, on *Schur-Weil orbitopes*, will study orbitopes like those in W_n, whose facial structures are determined by the lattice of subgroups of the group G. These include at least coadjoint orbits of G, and orbits of $O(n)$ and $SO(n)$ acting by translation on $n \times n$ matrices.

Another paper will be on *toric orbitopes*, which are convex hulls of orbits of $SO(2)^n$. Given a finite set $\mathcal{A} \subset \mathbb{N}^n$, $SO(2)^n$ acts on $\mathbb{R}^{2|\mathcal{A}|} = \mathbb{C}^{|\mathcal{A}|}$ with weights $\mathcal{A}$. Let $C_{\mathcal{A}}$ be the convex hull of the orbit of the vector $(1, \ldots, 1) \in \mathbb{C}^{|\mathcal{A}|}$. When $n = 1$, we are close to proving that $C_{\mathcal{A}}$ is a spectrahedron if and only if the corresponding projective toric variety $X_{\mathcal{A}}$ is normal. We will study the relation between normality of a toric variety and whether the corresponding orbitope is a spectrahedron.

We believe that orbitopes will be a rich class of examples within convex algebraic geometry, and foresee many other interesting results. We still hope to complete the unfinished business of [7]—describe the faces of orbitopes in $W_3 \oplus W_3$ and determine whether or not they are spectrahedra or projections of spectrahedra.

2.1.2. *Toric patches.* Geometric modeling builds computer models of curves and surfaces (and higher-dimensional objects) for industrial design and manufacture from basic units, called patches. It is both a source for and a consumer of interesting mathematical ideas.

For our discussion, a patch (scheme) is a collection $\beta := \{\beta_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\}$ of *blending functions*, where $\mathcal{A} \subset \mathbb{R}^d$ is a finite set and each $\beta_{\mathbf{a}}$ is a smooth, nonnegative function on the convex hull Δ of the indexing points $\mathcal{A}$. We assume that the blending functions are normalized in that they form a partition of unity, $\sum_{\mathbf{a} \in \mathcal{A}} \beta_{\mathbf{a}}(x) = 1$ for $x \in \Delta$.

Given *control points* $\mathbf{b} = \{\mathbf{b_a} \mid \mathbf{a} \in \mathcal{A}\} \subset \mathbb{R}^n$ the function (or rather its image $F(\Delta)$)

$$F : \Delta \longrightarrow \mathbb{R}^n \qquad \text{where} \qquad x \longmapsto \sum_{\mathbf{a}\in\mathcal{A}} \beta_{\mathbf{a}}(x)\mathbf{b_a}\,, \tag{5}$$

is used to represent objects in a computer. The control points provide an intuitive means to control the shape of $F(\Delta)$. The industry-standard Bézier patches (in which the bending functions are the Bernstein polynomials) are related to toric varieties $X_\mathcal{A}$ when $\mathcal{A}$ is the integer points in a product of simplices, such as the projective plane $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. For these, $\beta(\Delta) \subset \mathbb{RP}^\mathcal{A}$ is the positive part $X^+_\mathcal{A}$ of the toric variety $X_\mathcal{A}$ under the projective embedding given by $\mathcal{A}$, and $F(\Delta)$ is its image under a linear projection.

Noting this, Krasauskas [Kra02] proposed the mathematically appealing class of *toric patches*, which generalize the classical Bézier patches. The data for a toric patch are a collection of integer points $\mathcal{A} \subset \mathbb{Z}^n$ together with a positive weight for each point of $\mathcal{A}$. Their blending functions generalize the Bernstein polynomials, and they are connected in the same way to toric varieties. They also have many, but not all, of the useful properties of Bézier patches. While theoretically interesting, toric patches need to be better understood before they may be credibly added to the toolkit of geometric modeling.

One property of Bézier patches is *linear precision*, which is the ability to replicate affine functions. It is useful in approximation theory and for morphing and texture in modeling. For toric patches, linear precision is equivalent to the map (5) with control points $\mathbf{b_a} = \mathbf{a}$,

$$\Delta \ni x \longmapsto \sum_{\mathbf{a}\in\mathcal{A}} \beta_{\mathbf{a}}(x)\mathbf{a} \in \Delta\,,$$

being a homeomorphism of Δ. This is appealing because the map $\beta(\Delta) \to \Delta$ is the restriction of a linear projection $\mathbb{P}^\mathcal{A} \to \mathbb{P}^d$ which is the *algebraic moment map* μ for the toric variety $X_\mathcal{A}$ underlying the toric patch. The moment map in symplectic geometry is the composition of μ with the functorial projection $X_\mathcal{A}(\mathbb{C}) \to X^+_\mathcal{A}$ induced by the semigroup map $\mathbb{C} \ni z \mapsto z\bar{z} \in \mathbb{R}_{\geq 0}$. In algebraic statistics the algebraic moment map μ is the expectation map of the corresponding toric statistical model [13] and linear precision means maximum likelihood degree (MLD) 1 [CHKS06], and in chemical dynamical systems μ is the stoichiometric map of a toric dynamical system [CDSS07].

García and I [13] reduced the problem of which toric patches have linear precision to determining the sparse homogeneous polynomials f whose toric polar linear system, $[x_i \frac{\partial f}{\partial x_i} \mid i{=}1,\ldots,d]$, defines a birational map $\mathbb{P}^d \dashrightarrow \mathbb{P}^d$. Using the classification of plane birational maps, von Bothmer, Ranestad, and I classified the toric surface patches that have linear precision [14]. We found a completely new family of *trapezoidal patches* with underlying shape a trapezoid, which contains the classical Bézier patches as special cases.

These trapezoidal patches may find uses in modeling, if they are known within that community. I plan to explore the properties and modeling possibilities of these trapezoidal patches in papers aimed at the geometric modeling community. To ensure credibility, I will seek a coauthor from geometric modeling.

Trapezoidal patches suggest constructions of new patches with linear precision and algebraic statistics furnishes constructions of high-dimensional toric statistical models with MLD 1 (and therefore toric patches with linear precision). García, Sullivant, and I will study these and other constructions to produce 3-dimensional patches with linear precision, which may be used in space modeling as non-linear finite elements. We will study local restrictions (such as restrictions to faces or links of faces) to help classify higher-dimensional patches with linear precision. (Birational geometry is of little help above dimension 2.) Even partial results or new classes of patches with linear precision would be useful in geometric modeling.

With García and Craciun [17], we applied ideas from toric dynamical systems and toric geometry to obtain new global information on toric patches. In particular, toric degenerations give a qualitative and quantitative interpretation of regular triangulations of the control points. These are exactly the simplicial polytopes that can be limits of the patch, and thus are intuitive approximations to the shape of a patch. However, for tensor product patches, the geometric modeling community typically uses more general polyhedral subdivisions of the control points (such as rectangles) as an approximation to the shape of a patch. Toric degenerations should also give meaning to these structures; this is a project I plan with Chungang Zhu, who is coming from Dalian in China to work with me.

More basic than these patch schemes are barycentric coordinates of polytopes Δ, which are patch schemes in the previous sense where $\mathcal{A}$ consists of the vertices of Δ. *Wachspress coordinates* of Δ are the unique barycentric coordinates for Δ that are rational functions of minimal degree $n-d$, where $\Delta \subset \mathbb{R}^d$ has n facets [Wac75, War96]. Graduate student Corey Irving and I are studying Wachspress coordinates from the perspective of algebraic geometry. Wachspress coordinates for a polygon with general vertices $p_1, \ldots, p_n, p_{n+1} = p_1$ are a basis for the linear system of polynomials of degree $n-2$ that vanish at every point of intersection of non-adjacent edges of this polygon. The image of $\mathbb{P}^2$ under this linear system is the *Wachspress variety*, whose minimal resolution (of its ideal) we hope to describe explicitly. (The definition does not require the p_i to form a convex polygon or even to be real.) We also hope to determine its Cox ring. While this variety is close to other rational varieties that have been studied from these perspectives, it in fact appears to be new.

We will also look at Wachspress varieties for polytopes of dimensions 3 and higher. While Wachspress coordinates make sense for convex polytopes in $\mathbb{R}^d$ [War96], a first step is to define them for non-convex polytopes in $\mathbb{C}^d$.

2.1.3. *Feedback control and quantum cohomology.* In the theory of linear systems, the space of m-input, p-output $(m \times p)$ systems of linear differential equations with MacMillan degree n is identified with the space of rational curves of degree n in the Grassmannian of m-planes in $\mathbb{C}^{m+p}$. After Laplace transform, the system is represented by its *transfer function*, a $m \times p$ matrix $G(s)$ of rational functions, and $s \mapsto$ row space$[I : G(s)]$ is the corresponding rational curve. Many basic questions about linear feedback control of these systems are solved using the geometry and cohomology of the Grassmannian (for static control) [Byr89] and its quantum cohomology (for dynamic control) [Ros94].

Helmke, Rosenthal, and Wang [HRW06] considered $m \times m$ systems that were symmetric in that $G(s) = G(s)^T$ or Hamiltonian in that $G(s) = G(-s)^T$, showing that symmetric static control was governed by the cohomology of the Lagrangian Grassmannian.

Hillar and I observed two further symmetries, $G(s) = -G(s)^T$ (skew-symmetric) and $G(s) = -G(-s)^T$ (skew-Hamiltonian), which are naturally controlled with skew-symmetric feedback. We simplified the arguments of [HRW06] to show that static skew-symmetric feedback control is governed by the cohomology of the orthogonal Grassmannian. We also identify the $m \times m$ linear systems with symmetric (respectively skew-symmetric) transfer functions of MacMillan degree n as rational curves in the Lagrangian Grassmannian of degree n (respectively in the orthogonal Grassmannian of degree $2n$—the restriction of $\mathcal{O}(1)$ from the Grassmannian is twice the generator of the Picard group of the orthogonal Grassmannian).

Modulo a moving lemma, this implies that basic questions in dynamic symmetric and skew-symmetric output feedback control are answered through the quantum cohomology of the respective spaces. I will work to establish this non-trivial moving lemma.

Compactifications of spaces of systems whose transfer function is symmetric, Hamiltonian, or their skew analogs should be interesting. There is not a good analog of the quot scheme for the Lagrangian or orthogonal Grassmannians [KT03, KT04], but they do have Drinfel'd compactifications as spaces of quasi-maps [Bra06]. The Drinfel'd compactifications for the Grassmannian and Lagrangian Grassmannian have interesting structures [SS01, Ruf08] and their study has led to new results in enumerative geometry [Sot00a]. I plan to study Drinfel'd compactifications of the spaces of Hamiltonian and skew-Hamiltonian linear systems.

2.2. **Combinatorial algebraic geometry.** The second major research direction in this proposal involves combinatorial (and also computational) aspects of algebraic geometry. Two areas in this direction, toric varieties and Schubert calculus, form the background for two of the projects in Section 2.1, namely toric patches and symmetric feedback control. There is one student working with me on these questions, Nickolas Hein. While he is expected to graduate in the spring of 2011, I will soon take on additional students.

2.2.1. *Foundations of Tropical Geometry.* Tropical geometry is a new mathematical realm which may be viewed as the shadow of the geometry of the algebraic torus $\mathbb{T}_N(K)(= N \otimes_{\mathbb{Z}} K^* \simeq \mathbb{G}_m^n(K))$ under the valuation map. Here, $N \simeq \mathbb{Z}^n$ is a free abelian group and $\mathbb{T}_N$ is the associated algebraic torus whose lattice of 1-parameter subgroups is N and lattice of characters is $M := \mathrm{Hom}(N, \mathbb{Z})$, and K is a field with a valuation $\nu \colon K^* \to \Gamma$, where Γ is an ordered abelian group. The valuation induces a map $\mathrm{Trop}\colon \mathbb{T}_N(K) \to N_\Gamma := N \otimes_{\mathbb{Z}} \Gamma$. Remarkably, many properties of a subvariety $X \subset \mathbb{T}_N(K)$ are retained by its image under Trop. There are other ways to view tropical geometry and many, many surprising and spectacular results have come from this field.

I have attended and even organized several conferences on tropical geometry, but until now have abstained from working in it. This is about to change. While attending the opening workshop of the MSRI program on tropical geometry in August, and while preparing lectures for a course I am giving on the subject, I became frustrated with the state of its foundations, and have formulated a program to improve these foundations.

There are three related directions/foundational aspects of tropical geometry that I want to pursue initially. These are all towards gaining a better understanding of tropical varieties *in situ*, as objects within N_Γ. This is complementary to what I have seen of Mikhalkin's (much anticipated and delayed) project to define and understand abstract tropical varieties when the value group is $\mathbb{Q}$ or $\mathbb{R}$ and the ring $\mathcal{Z}$ (see below) is $\mathbb{Z}$. Some of what I plan to do should be clear to the experts, yet it is not currently written in any published or `arXiv`ed work (for example, this is found neither in the published book [IMS09] nor in the draft text [MS09]). The closest in spirit to what I propose (and an inspiration to me) is Payne's work, particularly "Fibers of tropicalization" [Pay09b] and also [Pay09a].

One shortcoming in the treatment of tropical geometry is the restriction that the field K be algebraically closed *and* its value group Γ be a subgroup of $\mathbb{R}$. These unnatural restrictions obscure the mathematics and miss some applications, for example when $K = \mathbb{Q}_p$ (the p-adic rationals with value group $\mathbb{Z}$) [AIR09]. Balanced rational polyhedral complexes make sense in N_Γ, when Γ is any ordered abelian group and N a finitely generated free abelian group. It is very natural to develop tropical geometry and establish its main results in the generality of arbitrary fields with valuation, and I believe that I can do this so that they are functorial under base change (field extension). While this should be mostly obvious to the experts, I have not seen tropical varieties discussed at this appropriate level of generality.

A second shortcoming is that tropical varieties are always cycles–sets (polyhedral complexes) with multiplicities. Algebraic geometry encodes multiplicities in the nilpotent elements of rings of functions on a variety. It seems possible to do this for tropical varieties.

The group M of characters gives linear functions on N_Γ. Given a *tropical polynomial*

$$f \;:=\; \bigoplus_{\mathbf{m}\in\mathcal{A}} \gamma_{\mathbf{m}} \odot \mathbf{m}\,, \tag{6}$$

where $\mathcal{A} \subset M$ is a finite set of characters and $\gamma_{\mathbf{m}} \in \Gamma$, its value at a point $u \in N_\Gamma$ is

$$f(u) \;=\; \min\{\gamma_{\mathbf{m}} + \mathbf{m}\cdot u \;\mid\; \mathbf{m}\in\mathcal{A}\} \;\in\; \Gamma\,. \tag{7}$$

This point u lies in the tropical hypersurface $\mathrm{Trop}(f)$ defined by f if this minimum is attained at least twice. As a set, $\mathrm{Trop}(f)$ is dual to a regular subdivision Δ of the point set $\mathcal{A}$ induced by f with the facets of $\mathrm{Trop}(f)$ dual to edges of Δ: Let $\pm\mathbf{m}_e$ be the difference of the endpoints of an edge e of Δ. Then the hyperplane parallel to the facet corresponding to e is annihilated by $\mathbf{m}_e$, and the multiplicity of that facet is the lattice length of the vector $\mathbf{m}_e$.

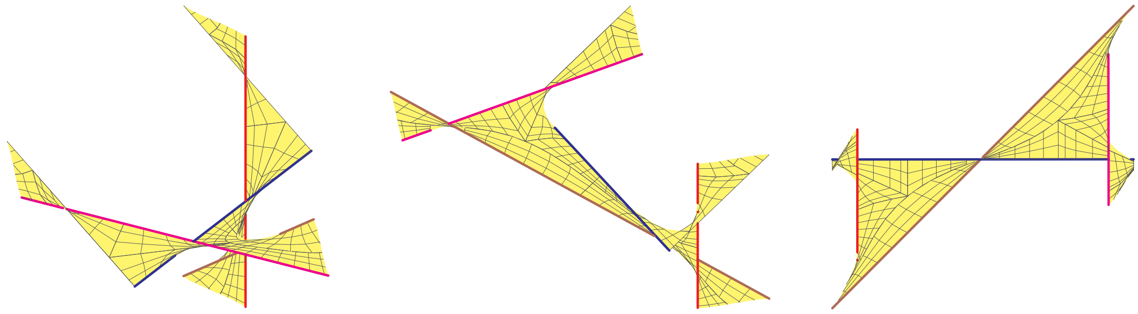
If we consider the quotient $M/\mathbf{m}_e\mathbb{Z}$ as 'functions' on the facet, and then this multiplicity is the order of the torsion subgroup of $M/\mathbf{m}_e\mathbb{Z}$, while the free part gives linear functions on hyperplanes parallel to the facet. Encoding multiplicities by torsion makes sense for tropical varieties that are not hypersurfaces, and it leads to a more refined version of the balanced condition for a tropical variety. While it is not yet clear the best way to package this structure, it is in the right direction towards defining *tropical schemes*.

With this encoding of multiplicities, a zero-dimensional tropical scheme is a point p with functions M/Λ_p, where Λ_p is a full-rank sublattice of M. Setting $\deg(p) := |M/\Lambda_p|$, we obtain Bernstein's Theorem [Ber75] for the degree of the intersection of $\mathrm{rank}(N)$ general hypersurface tropical schemes. This should also lead to an intersection theory for tropical schemes along the lines of [LJ07, Kat09b, Kat09a], which are for tropical cycles.

As non-archimedean amoebae, tropical varieties are rational polyhedral complexes because functions in the coordinate ring of $\mathbb{T}_N$ have exponents in the free abelian group M. If we do not require tropical varieties to reflect the geometry of $\mathbb{T}_N$, then it is possible to develop a version of tropical geometry in which the exponents M are rational or even real vectors. More generally, let $\mathcal{Z}$ be an ordered ring (e.g. $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$), Γ be an ordered $\mathcal{Z}$-module, and M, N be dual finitely generated free $\mathcal{Z}$-modules. Then (6) (with $\mathcal{A} \subset M$ finite) defines a tropical polynomial f whose zero set is where the minimum in (7) is attained twice.

This is a polyhedral complex dual to the triangulation Δ of the support $\mathcal{A}$ of f induced by the coefficients of f. Multiplicities should be encoded in the space of functions $M/\mathbf{m}_e\mathbb{Z}$ on a facet dual to an edge e of Δ. A zero-dimensional tropical scheme is a point p with functions M/Λ_p, where Λ_p is a free abelian subgroup of M with maximal rank. To define the multiplicity of such a scheme, (for intersection theory and Bernstein's Theorem), we need a measure on such quotients. When $\mathcal{Z} \subset \mathbb{R}$, Lesbegue measure (induced from $\mathbb{R}$) works, and we obtain exactly Bernstein's Theorem that the tropical intersection multiplicity of schemes with supports $\mathcal{A}_1, \ldots, \mathcal{A}_{\mathrm{rank(M)}}$ is the mixed volume of their convex hulls. If we measure each quotient to be 1, then this leads to a tropical fewnomial bound, where the tropical intersection number is bounded by a function of the number of monomials. I expect that these irrational tropical schemes will also have an intersection theory.

2.2.2. *Coamoebae.* The coamoeba of a subvariety $X \subset \mathbb{T}_n(\mathbb{C})$ is its image in $N_{S^1} := N \otimes_{\mathbb{Z}} S^1$ under the argument map. Coamoebae have only just begun to be studied, primarily for hypersurfaces [Nis08a, Nis08b, Nis09]. Mounir Nisse and I are studying coamoebae for varieties of larger codimension. They are already quite interesting for lines and linear spaces. Here are three views (in the fundamental domain $[-\pi, \pi]^3$) of the same highly symmetric coamoeba for the line $t \mapsto (t-1, t-\zeta, t-\zeta^2)$, where ζ is a primitive third root of unity.



When the field K has residue field $\mathbb{C}$, and when the valuation map admits a section (e.g. for the Puiseaux field or for $\mathbb{C}((t^{\mathbb{R}}))$), then we may define the *total tropical variety* of $X \subset \mathbb{T}_N(K)$, which lies in $N_{S^1} \times N_\Gamma$ and is fibered over the tropical variety of X, with fiber over a point u the coamoeba of the tropical degeneration X_u. This is related to Payne's exploded tropicalization [Pay09b] and to Mikhalkin's complex tropical varieties [Mik05]. Nisse and I are working on a structure theorem for these total tropical varieties.

2.2.3. *Arithmetic toric varieties.* Toric varieties are typically studied over an algebraically closed field, and in this setting enjoy an elegant classification: Normal varieties with a faithful action of split algebraic torus $\mathbb{T}_N$ having a dense orbit correspond to polyhedral fans Σ in N. (These are also normal equivariant compactifications of $\mathbb{T}_N$-torsors.) The toric variety X_Σ of a fan Σ in N makes sense as a scheme over spec$(\mathbb{Z})$, and extending scalars to any field k gives the *split toric variety X_Σ* over k. As noted in § 2.1.2, real split toric varieties arise in geometric modeling, algebraic statistics, and chemical dynamical systems. Split real toric varieties are also fundamental for my work on bounds for the number of real solutions to equations [SS06], [25], and § 1.1.

An *arithmetic toric variety* Y is a normal variety over a field k with a faithful action of a (not necessarily split) algebraic torus $\mathcal{T}$ having a dense orbit. Extending scalars to the splitting field K of $\mathcal{T}$ (so that $\mathcal{T}_K \simeq \mathbb{T}_N(K)$), Y_K becomes isomorphic to the split toric variety X_Σ, and the Galois group Gal(K/k) acts on N and on Σ. With Elizondo, Lima-Filho, and Teitler [24], we study these and classify them (when Y is quasi-projective or K/k is quadratic) using Galois cohomology. The Cox quotient construction of X_Σ helps compute Galois cohomology, we classify arithmetic affine (not necessarily normal) toric varieties, and show that normal equivariant compactifications of arithmetic torsors correspond to $\mathcal{G}$-invariant fans in N, which extends the classification of split torsors.

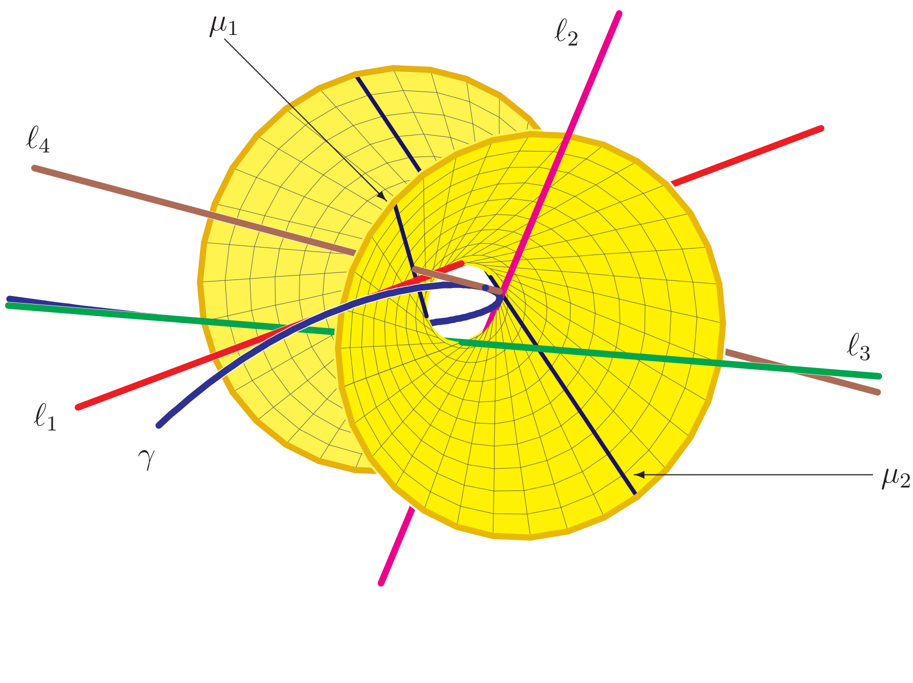
Arithmetic toric varieties were studied by Voskresenskiĭ [Vos82] and by Batyrev and Tschinkel [BT95], who were interested in rational points of bounded height on compactifications of anisotropic tori. Real arithmetic affine toric varieties appeared in the discussion on orbitopes as orbits of $SO(2)^n$. Delaunay studied compact real arithmetic toric varieties [Del03, Del04], classifying smooth real toric surfaces and Krasauskas [Kra01, KK05] proposed them as the basis for a family of patch schemes.

This work of Krasauskas led us and Clarence Wilkerson to collaborate on computing the Galois and torus equivariant cohomology of real arithmetic toric varieties. Our need to properly classify real arithmetic toric varieties led to [24]. We plan now to complete this computation of the equivariant cohomology of real arithmetic toric varieties. This is only the first step towards our goal of computing a richer and more subtle cohomology theory—Bredon cohomology [CW03]—of these real arithmetic toric varieties. While more general than ordinary equivariant cohomology, there are few spaces for which we know their Bredon cohomology, and this invariant is particularly well-suited for real varieties [dSLF09].

We will explore other aspects of arithmetic toric varieties, such as Galois-torus equivariant cohomology/Chow groups of smooth arithmetic toric varieties. This is more challenging than for real arithmetic toric varieties as the cohomology of a point involves the integer representation theory of Galois groups. There are also arithmetic analogs of many constructions in toric geometry, some of which may be compelling to study.

2.2.4. *Investigation of the Shapiro Conjecture.* This project best illustrates the potential for mathematics of questions motivated by applications. While real solutions are important for applications, they depend subtly on the coefficients of the equations, and there is a dearth of theory to understand them. A motivating problem for the investigation of reality is a curious conjecture of Boris and Michael Shapiro in the Schubert calculus. Given a *Schubert problem* (enumerative problem in the Schubert calculus on a flag manifold), they conjectured that if the flags imposing the Schubert conditions osculate a real rational normal curve, then all of the solutions would be real.

Here is an idea of the vivid geometry behind this conjecture. The first non-trivial Schubert problem asks for the number of lines in 3-space that meet four given lines. (The answer is two.) The Shapiro Conjecture asserts that if the four lines. $\ell_1, \ldots, \ell_4$, are all tangent to a twisted cubic curve γ, then both lines μ_1, μ_2 meeting them will be real. To see this, note that the set of lines meeting the three tangent lines ℓ_1, ℓ_2, and ℓ_3 will form one ruling of the hyperboloid of one sheet (shown below). Given a fourth tangent line, ℓ_4, each point where is meets the hyperboloid gives a solution line μ_i. As we see, ℓ_4 necessarily meets the hyperboloid, giving two real solutions.



The Shapiro conjecture for Grassmannians was initially thought too strong to be true, but early investigations while studying a question in control theory [RS98], and then a comprehensive computational and theoretical study [Sot00b, Ver00], found overwhelming evidence for it, when the flag manifold is a Grassmannian. Eremenko and Gabrielov [EG02] proved the Shapiro Conjecture for Grassmannians of codimension 2 planes using complex analysis, and Mukhin, Tarasov, and Varchenko gave two proofs for the Grassmannian [MTV05, MTV07]. Both proofs used ideas from geometry, representation theory, differential equations, and mathematical physics (the Bethe Ansatz). This story was the subject of an AMS Current Events Bulletin Lecture that I gave in January 2009, and a forthcoming article in the Bulletin of the AMS [18].

The early investigations [Sot00b, Sot00c] also found counterexamples to the Shapiro conjecture for the classical (type-A) flag manifold and the Lagrangian Grassmannian, but none for the orthogonal Grassmannian. These investigations also suggested that it was possible to repair the Shapiro conjecture when it fails, thereby obtaining refinements that were likely true. I began a long-term project to study the Shapiro conjecture computationally, seeking refinements that may hold and amassing evidence for these new conjectures. This project involves large-scale computer use, and I have assembled vertically-integrated research teams of students and postdocs to write the software, administer the computation, and analyze our data, and I acquired access to computers to carry out the computation. Members of the team train and mentor each other, and I use this research program as a vehicle to get my students involved with me in research and to train them in the use of computers in mathematical investigations.

The first project found and studied the *Monotone Conjecture*, which is a subtle reformulation of the Shapiro Conjecture for the type-A flag manifold [RSSS06]. Our computational study looked at over 500 million instances of 1124 Schubert problems on 29 flag manifolds, and found the Monotone Conjecture to hold in each of the over 132 million instances tested. This consumed 15.76 Gigahertz-years of computing. Besides testing the Monotone Conjecture, this study found the smallest known Schubert problem with deficient Galois group and a counterexample to dimensional properness of flag linear systems on curves.

Almost immediately, Eremenko, *et. al* [EGSV06] proved a special case of the Monotone Conjecture, which, when interpreted for the Grassmannian of codimension 2 planes, suggested a generalization of the Shapiro Conjecture for Grassmannians—if the reference flags are secant to the rational normal curve along disjoint intervals, then all solutions to the Schubert problem are real. Instances of this *Secant Conjecture* are quite difficult to compute, and it was clear that a proper study of the Secant Conjecture would require the use of a supercomputer, as well as modern software tools to automate the computation.

With postdoc Chris Hillar in charge, our team designed and built the infrastructure for large-scale computations using instructional computer labs which are configured to moonlight as a supercomputer outside of class hours. This is organized around a MySQL database and we use shell scripts and a special-built perl software package to run and organize the computation. It calls Singular [GPS07] and Maple for the core mathematical routines of elimination and real-root counting. Other software tools help to administer the computation, and we may view the results on php webpages from anywhere on the planet [FRS]. This experimental design is explained in the methods paper [20]. The computation has been running since November 2008, and it is now nearly completed, having consumed 750

Gigahertz-years of computing to study 1.5 billion instances of Schubert problems, and finding the Secant conjecture to hold in each of the over 300 million instances tested. We intend to rerun about 10% of this experiment on a different supercomputer, using different core mathematical software (Macaulay2 [GS] for elimination and the SARAG library [Car06] for real root counting) later this Autumn, once we have access to the Brazos Cluster (through a SCREMS grant that Teitler and I hold).

With this infrastructure (research team, experimental design, software, and dedicated supercomputers) for large-scale computational experiments in place, we plan future experiments. After we write up our conclusions from our study of the Secant Conjecture, we plan to run a similar experiment studying the *Monotone Secant Conjecture*, which is the common extension of the Monotone and the Secant Conjectures. This will run for about one calendar year.

The next experiment will be started in the middle of this grant period, and I expect it to be devoted to studying extensions of the Shapiro conjecture for flag manifolds for the symplectic group. There is a symplectic moment curve whose osculating flags are isotropic (it comes from exponentiating a principal nilpotent element in the Lie algebra $\mathfrak{sp}$). New interpretations of calculations from 2000 suggest that the version of the Shapiro conjecture for the Lagrangian Grassmannian is that Schubert problems given by flags osculating the symplectic moment curve will have no real solutions, unless the Schubert problem is Levi-movable (a notion introduced by Belkale and Kumar [BK06]), in which case all the solutions are real. The proof of the Shapiro conjecture by Mukhin, Tarasov, and Varchenko implies that all solutions are real for Levi-movable Schubert problems. I do not know what to expect for other symplectic flag manifolds, except other interesting and subtle refinements of the Shapiro Conjecture. Since all students and postdocs currently working with me will have moved on by that time, this project will require me to train a whole new research team.

2.2.5. *Schubert calculus.* Understanding and extending the Littlewood-Richardson rule is the main theme of the Schubert calculus. The Littlewood-Richardson coefficients are the structure constants for multiplication in the cohomology ring of a flag variety with respect to its basis of Schubert classes. Suitably interpreted, these are positive for many flavors of cohomology, but combinatorial formulas are elusive. I have two programs to better understand Littlewood-Richardson coefficients. One begins with a new approach to Littlewood-Richardson coefficients for Grassmannians and the other is to prove formulas in the branching Schubert calculus in the greatest possible generality.

A *Schubert variety* in a Grassmannian is given by a partition λ and a flag. Write $X_\lambda(s)$ for the Schubert variety of type λ given by the flag of subspaces osculating the rational normal curve γ at the point $\gamma(s)$. Eisenbud and Harris [EH87] established the cycle-theoretic formula for the scheme-theoretic limit

$$\left[\lim_{s\to 0}(X_\lambda(0)\cap X_\mu(s))\right] \;=\; \sum_\nu c^\nu_{\lambda,\mu}\,[X_\nu(0)]\ , \tag{8}$$

where $c^\nu_{\lambda,\mu}$ is the Littlewood-Richardson coefficient. They showed that the limit scheme is supported on Schubert varieties $X_\nu(0)$, which implies that the scheme-theoretic multiplicity of the Schubert variety $X_\nu(0)$ in the limit scheme $\lim_{s\to 0}(X_\lambda(0)\cap X_\mu(s))$ is the Littlewood-Richardson coefficient $c^\nu_{\lambda,\mu}$. With graduate student Nickolas Hein, we are working to compute the scheme-theoretic limit directly using combinatorics and commutative algebra. This will give a new proof of the Littlewood-Richardson formula based on commutative algebra and geometry in which the coefficients $c^{\nu}_{\lambda,\mu}$ are interpreted as scheme-theoretic multiplicities.

This approach to the Littlewood-Richardson formula should work in greater generality. The limit (8) makes sense in any flag manifold G/P, as the family of Schubert varieties $X_\lambda(s)$ are translates of $X_\lambda(0)$ by a 1-parameter subgroup of G obtained by exponentiating a principal nilpotent element of its Lie algebra. There is also a similar limit which makes sense in the Drinfel'd compactification of spaces of rational curves in flag manifolds (this was used for the Grassmannian in [Sot00a] to obtain results in quantum cohomology), and may yield formulas in quantum cohomology.

A different direction is what some call the *branching Schubert calculus* [RR09]. Here, we begin with an embedding $\iota\colon G'/P' \hookrightarrow G/P$ of one flag manifold into another, and seek a formula for the pullback $\iota^*[X_\lambda]$, where X_λ is a Schubert variety in G/P. Bergeron and I gave such formulas involving Littlewood-Richardson coefficients in ordinary cohomology of type A [BS98] and C [BS02] flag varieties, and Lenart, Robinson, and I gave similar formulas in K-theory of type A flag varieties [LRS06]. This last paper used a general framework for embeddings $\iota\colon G'/P' \hookrightarrow G/P$ due to Billey and Braden [BB03] related to permutation patterns. Elizondo, Lima-Filho, and I wrote a manuscript explaining these restriction formulas for all such embeddings ι and cohomology theories from ordinary cohomology through torus-equivariant K-theory. We have not released this, for we wanted to do it in the greatest generality, which it seems is afforded by algebraic cobordism [LP09]. With the recent development of the rudiments of Schubert calculus for cobordism [HK09], we plan now to complete this once we complete work in progress on arithmetic toric varieties.

This is a way for me to learn algebraic cobordism, which may very well be the next (and final?) frontier for Schubert calculus.

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