FRONTIERS OF REALITY IN SCHUBERT CALCULUS

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Abstract. The theorem of Mukhin, Tarasov, and Varchenko (formerly the Shapiro conjecture for Grassmannians) asserts that all (*a priori* complex) solutions to certain geometric problems from the Schubert calculus are actually real. Their proof is quite remarkable, using ideas from integrable systems, Fuchsian differential equations, and representation theory. Despite this advance, the original Shapiro conjecture is not yet settled. While it is false as stated, it has several interesting and not quite understood modifications and generalizations that are likely true.

These notes will introduce the Shapiro conjecture for Grassmannians, give some idea of its proofs and consequences, its links to other subjects, and sketch its extensions.

Introduction

While it is not unusual for a univariate polynomial f with real coefficients to have *some* real roots—under reasonable assumptions we expect $\sqrt{\deg f}$ real roots [20]—it is rare for a polynomial to have all of its roots be real. In fact, the primary example from nature that comes to mind is when f is the characteristic polynomial of a symmetric matrix, as all eigenvalues of a symmetric matrix are real.

Similarly, when a system of real polynomial equations has finitely many (*a priori* complex) solutions, we expect some, but likely not all, solutions to be real. In fact, upper bounds on the number of real solutions [1, 18] sometimes ensure that not all solutions can be real. As before, the primary example that comes to mind of a system with only real solutions is the system of equations for the eigenvectors and eigenvalues of a real symmetric matrix.

Here is another system of polynomial equations which also turns out to have only real solutions. The Wronskian of univariate polynomials $f_0, \ldots, f_n \in \mathbb{C}[t]$ is the determinant

$$\det \begin{pmatrix} f_0(t) & f_1(t) & \cdots & f_n(t) \\ f_0'(t) & f_1'(t) & \cdots & f_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(n)}(t) & f_1^{(n)}(t) & \cdots & f_n^{(n)}(t) \end{pmatrix} .$$

Up to a scalar multiple, the Wronskian depends only upon the linear span P of the polynomials $f_0, \ldots, f_n$ in the vector space $\mathbb{C}[t]$ of all polynomials. This scaling retains only the information of the roots of the Wronskian and their multiplicities. Recently, Mukhin, Tarasov, and Varchenko [22] proved the remarkable (but seemingly innocuous) result.

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Theorem 1. *If the Wronskian of a space P of polynomials has only real roots, then P has a basis of real polynomials.*

While not immediately apparent, those $(n+1)$-dimensional subspaces P of $\mathbb{C}[t]$ with a given Wronskian W are the solutions to a system of polynomial equations which depend on the roots of W. In Section 1, we explain how the Shapiro conjecture for Grassmannians is equivalent to Theorem 1.

The proof of Theorem 1 uses the Bethe ansatz for the Gaudin model on certain modules (representations) of the Lie algebra $\mathfrak{sl}_{n+1}\mathbb{C}$. This is a method to decompose a module of $\mathfrak{sl}_{n+1}\mathbb{C}$ into irreducible submodules that is compatible with a family of commuting operators called the Gaudin Hamiltonians. It includes a set-theoretic map from the spaces of polynomials with a given Wronskian to certain Bethe eigenvectors for the Gaudin Hamiltonians. A coincidence of numbers, from the Schubert calculus on one hand and from representation theory on the other, implies that this map is a bijection. As the Gaudin Hamiltonians are symmetric with respect to the positive definite Shapovalov form, their eigenvectors and eigenvalues are real. Theorem 1 follows as eigenvectors with real eigenvalues must come from real spaces of polynomials. We describe this in Sections 2, 3, and 4.

The geometry behind the statement of Theorem 1 appears in many other guises, some of which we describe in Section 5. These include linear series on the projective line [5] and rational curves with prescribed flexes [17]. A special case of the Shapiro conjecture concerns a remarkable statement about rational functions with prescribed critical points, and was proved in this form by Eremenko and Gabrielov [8]. They showed that a rational function whose critical points lie on a circle in the Riemann sphere maps that circle to another circle.

A generalization of Theorem 1 by Mukhin, Tarasov, and Varchenko [24] implies the following particularly attractive statement from matrix theory. Let $\beta_1,\dots,\beta_n$ be distinct real numbers, $\alpha_1,\dots,\alpha_n$ be complex numbers, and consider the matrix

$$Z \ := \ \begin{pmatrix} \alpha_1 & (\beta_1-\beta_2)^{-1} & \cdots & (\beta_1-\beta_n)^{-1} \\ (\beta_2-\beta_1)^{-1} & \alpha_2 & \cdots & (\beta_2-\beta_n)^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ (\beta_n-\beta_1)^{-1} & (\beta_n-\beta_2)^{-1} & \cdots & \alpha_n \end{pmatrix} .$$

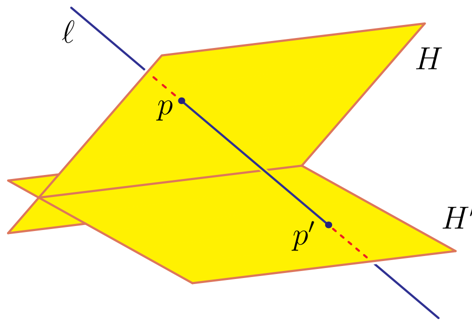
Theorem 2. *If Z has only real eigenvalues, then $\alpha_1,\dots,\alpha_n$ are real.*

Unlike its proof, the statement of Theorem 2 has nothing to do with Schubert calculus or representations of $\mathfrak{sl}_{n+1}\mathbb{C}$ or integrable systems, and it remains a challenge to prove it directly.

The statement and proof of Theorem 1 is only part of this story. Theorem 1 settles (for Grassmannians) a conjecture about Schubert calculus originally made by Boris Shapiro and Michael Shapiro in 1993/4. While this Shapiro conjecture is false for most other flag manifolds, there are appealing corrections and generalizations supported by theoretical evidence and also by overwhelming computational evidence. We sketch this in Section 6.

There are now three proofs [14, 22, 25] of the Shapiro conjecture for Grassmannians, all passing through integrable systems and representation theory. In the second proof [14], the solutions are identified with certain representations of a real rational Cherednik algebra [11], and reality follows as these representations are necessarily real. The third proof [25] provides a surprising and deep connection between the Schubert calculus and the representation theory of $\mathfrak{sl}_{n+1}\mathbb{C}$. We will only treat the first proof in these notes. During 2009, these notes will be expanded into a more complete treatment of this remarkable conjecture and its proofs.

First steps: the problem of four lines. We close this Introduction by illustrating the Schubert calculus and the Shapiro conjecture with some beautiful geometry. Consider the set of all lines in three-dimensional space. This set (a Grassmannian) is four-dimensional, which we may see by counting degrees of freedom for a line ℓ as follows. Fix planes H and H' that meet ℓ in points p and p'.



Since each point p and p' has two degrees of freedom to move within its plane, we see that the line ℓ enjoys four degrees of freedom.

Similarly, the set of lines that meet a fixed line is three-dimensional. More parameter counting tells us that if we fix four lines, then the set of lines that meet each of our fixed lines will be zero-dimensional. That is, it consists of finitely many lines. The Schubert calculus gives algorithms to determine this number of lines. We instead use elementary geometry to show that this number is 2.

The Shapiro conjecture asserts that if the four fixed lines are chosen in a particular way, then both solution lines will be real. This special choice begins by specifying a twisted cubic curve, γ. While any twisted cubic will do, we'll take the one with parameterization

$$\gamma\ :\ t\ \longmapsto (6t^2-1,\ \tfrac{7}{2}t^3+\tfrac{3}{2}t,\ \tfrac{3}{2}t-\tfrac{1}{2}t^3)\,. \tag{1}$$

Our fixed lines will be four lines tangent to γ.

We understand the lines that meet our four tangent lines by first considering lines that meet three tangent lines. We are free to fix the first three tangent points to be any of our choosing, for instance, $\gamma(-1)$, $\gamma(0)$, and $\gamma(1)$. Then the three lines $\ell(-1)$, $\ell(0)$, and $\ell(1)$ tangent at these points have parameterizations

$$(-5+s, 5-s, -1)\,,\quad (-1, s, s)\,,\quad \text{and}\quad (5+s, 5+s, 1)\quad \text{for } s\in\mathbb{R}.$$

These lines all lie on the hyperboloid H of one sheet defined by

$$x^2-y^2+z^2\ =\ 1\,, \tag{2}$$

which has two rulings by families of lines. The lines $\ell(-1)$, $\ell(0)$, and $\ell(1)$ lie in one family, and the other family consists of the lines meeting $\ell(-1)$, $\ell(0)$, and $\ell(1)$. This family is drawn on the hyperboloid H in Figure 1.

The lines that meet $\ell(-1)$, $\ell(0)$, $\ell(1)$, and a fourth line $\ell(s)$ will be those in this second family that also meet $\ell(s)$. In general, there will be two such lines, one for each point of intersection of line $\ell(s)$ with H, as H is defined by the quadratic polynomial (2). The remarkable geometric fact is that every such tangent line, $\ell(s)$ for $s \notin \{-1, 0, 1\}$, will meet the hyperboloid in two real points. We illustrate this when $s = 0.31$ in Figure 1, highlighting the two solution lines.

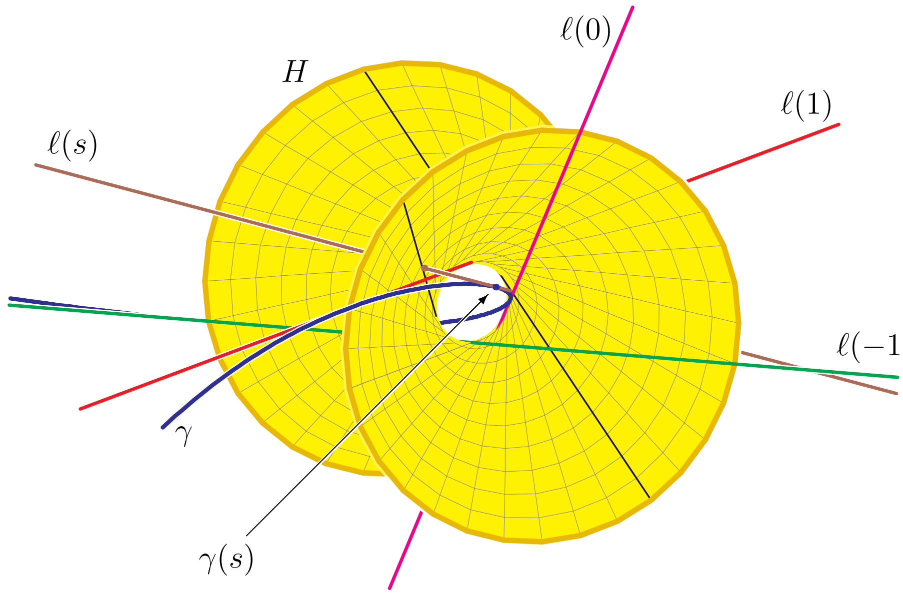


FIGURE 1. The problem of four lines.

The Shapiro conjecture and its extensions claim that this reality always happens: If the conditions for a Schubert problem are chosen relative to a rational normal curve (here, the twisted cubic curve γ of (1)), then all solutions will be real. When the Schubert problem comes from a Grassmannian (like this problem of four lines), the Shapiro conjecture is true—this is the theorem of Mukhin, Tarasov, and Varchenko. For most other flag manifolds, it is known to fail, but in very interesting ways.

Our plan is to explain this conjecture more precisely for Grassmannians, outline the first proof of Mukhin, Tarasov, and Varchenko, and then give some of its consequences. Along the way we will discuss some special cases of the conjecture which at first glance do not seem to have any relation to Schubert calculus or representation theory. We conclude with a sketch of the emerging landscape of conjectures for other flag manifolds which generalize and correct the Shapiro conjecture.

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1. THE SHAPIRO CONJECTURE FOR GRASSMANNIANS

Let $\mathbb{C}_d[t]$ be the set of complex polynomials of degree at most d in the indeterminate t, a vector space of dimension $d+1$. Fix a positive integer $n < d$, and let $\text{Grass}_{n,d}$ be the set of all $(n+1)$-dimensional linear subspaces P of $\mathbb{C}_d[t]$. This *Grassmannian* is a complex manifold of dimension $(n+1)(d-n)$ [15, Ch. 1.5].

The main character in our story is the Wronski map, which associates a point $P \in \mathrm{Grass}_{n,d}$ to the Wronskian of a basis for P. If $\{f_0(t), \ldots, f_n(t)\}$ is a basis for P, its Wronskian is the determinant of the derivatives of the basis,

$$\mathrm{Wr}(f_0, \ldots, f_n) \;:=\; \det \begin{pmatrix} f_0 & f_1 & \cdots & f_n \\ f_0' & f_1' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(n)} & f_1^{(n)} & \cdots & f_n^{(n)} \end{pmatrix}, \tag{1.1}$$

which is a nonzero polynomial of degree of at most $(n{+}1)(d{-}n)$. This does not quite define a map $\mathrm{Grass}_{n,d} \to \mathbb{C}_{(n+1)(d-n)}[t]$, as choosing a different basis for P multiplies the Wronskian by a nonzero constant. If we consider the Wronskian up to a nonzero constant, we obtain the *Wronski map*

$$\mathrm{Wr} \;:\; \mathrm{Grass}_{n,d} \;\longrightarrow\; \mathbb{P}(\mathbb{C}_{(n+1)(d-n)}[t]) \;\simeq\; \mathbb{P}^{(n+1)(d-n)}, \tag{1.2}$$

where $\mathbb{P}(V)$ denotes the projective space consisting of all 1-dimensional linear subspaces of a vector space V.

We restate Theorem 1, the simplest version of the Theorem of Mukhin, Tarasov, and Varchenko [22].

Theorem 1. *If the Wronskian of a space P of polynomials has only real roots, then P has a basis of real polynomials.*

The problem of four lines is a special case of Theorem 1 when $d = 3$ and $n = 1$. To see this, note that if we apply an affine function $a + bx + cy + dz$ to the curve $\gamma(t)$ of (1), we obtain a cubic polynomial in $\mathbb{C}_3[t]$, and every cubic polynomial comes from a unique affine function. A line ℓ in $\mathbb{C}^3$ (actually in $\mathbb{P}^3$) is cut out by a two-dimensional space of affine functions, which gives a 2-dimensional space P_ℓ of polynomials in $\mathbb{C}_3[t]$, and hence a point $P_\ell \in \mathrm{Grass}_{n,d}$.

It turns out that the Wronskian of a point $P_\ell \in \mathrm{Grass}_{n,d}$ is a quartic polynomial with a root at $s \in \mathbb{C}$ if and only if the line ℓ corresponding to P_ℓ meets the line $\ell(s)$ tangent to the curve γ at $\gamma(s)$. Thus a line ℓ meets four lines tangent to γ at real points if and only if the corresponding point $P_\ell \in \mathrm{Grass}_{n,d}$ has a Wronskian whose roots are these four points. Since these points are real, Theorem 1 implies that P_ℓ has a basis of real polynomials. Thus ℓ is cut out by real affine functions, and hence is real.

1.1. **Geometric form of the Shapiro Conjecture.** Let $P \in \mathrm{Grass}_{n,d}$ be a subspace. We consider the order of vanishing at a point $s \in \mathbb{C}$ of polynomials in a basis for P. There will be a minimal order a_0 of vanishing for these polynomials. Suppose that f_0 vanishes to this order. Subtracting an appropriate multiple of f_0 from each of the other polynomials, we may assume that they vanish to order greater than a_0 at s. Let a_1 be the minimal order of vanishing at s of these remaining polynomials. Continuing in this fashion, we obtain a basis $f_0, \ldots, f_n$ of P and a sequence

$$0 \leq a_0 < a_1 < \cdots < a_n \leq d\,,$$

where f_i vanishes to order a_i at s. We call this sequence $\mathbf{a}_P(s)$ the *ramification* of P at s. For a sequence $\mathbf{a} : 0 \leq a_0 < a_1 < \cdots < a_n \leq d$, write $\Omega^\circ_{\mathbf{a}}(s)$ for the set of points $P \in \mathrm{Grass}_{n,d}$ with ramification $\mathbf{a}$ at s, which is a Schubert cell of $\mathrm{Grass}_{n,d}$. It has codimension

$$|\mathbf{a}| \ := \ a_0 \ + \ a_1 - 1 \ + \ \cdots \ + \ a_n - n\,,$$

as may be seen by expanding the basis $f_0, \ldots, f_n$ of P in the basis $\{(t-s)^i \mid i = 0, \ldots, d\}$ of $\mathbb{C}_d[t]$. Since $f_j^{(i)}$ vanishes to order at least $a_j - i$ at s and $f_i^{(i)}$ vanishes to order exactly $a_i - i$ at s, we see that the Wronskian of a subspace $P \in \Omega^\circ_{\mathbf{a}}(s)$ vanishes to order exactly $|\mathbf{a}|$ at s.

Let $\mathrm{Grass}^\circ_{n,d}$ consist of subspaces $P \in \mathrm{Grass}_{n,d}$ that have a basis $f_0, \ldots, f_n$ where f_i has degree $d-n+1$. This is an open subset of $\mathrm{Grass}_{n,d}$. When $P \in \mathrm{Grass}^\circ_{n,d}$, we obtain the Plücker formula for the total ramification of a general subspace P of $\mathbb{C}_d[t]$,

$$\dim \mathrm{Grass}_{n,d} \ = \ \sum_{s \in \mathbb{C}} |\mathbf{a}_P(s)|\,. \tag{1.3}$$

In general, the total ramification of P is bounded by the dimension of $\mathrm{Grass}_{n,d}$. (One may also define ramification at infinity for subspaces $P \notin \mathrm{Grass}^\circ_{n,d}$, to obtain the Plücker formula in full generality.) If $\mathbf{a}_P(s) = 0 < 1 < \cdots < n$, so that $|\mathbf{a}_P(s)| = 0$, then we say that P is *unramified* at s. In this language, Theorem 1 states that if a subspace $P \in \mathrm{Grass}_{n,d}$ is ramified only at real points, then P is real in that it has a basis of real polynomials.

Let us introduce some more geometry. Let W be the Wronskian of P. Then

$$P \ \in \ \bigcap_{s\colon W(s)=0} \Omega^\circ_{\mathbf{a}_P(s)}(s)\,,$$

and this intersection consists of all subspaces with Wronskian W. In particular, P lies in the intersection of the closures of these Schubert cells, which we now describe. For each $s \in \mathbb{C}$, $\mathbb{C}_d[t]$ has a complete flag of subspaces

$$F_\bullet(s) \ : \ \mathbb{C}\cdot(t-s)^d \ \subset \ \mathbb{C}_1[t]\cdot(t-s)^{d-1} \ \subset \ \cdots \ \subset \ \mathbb{C}_{d-1}[t]\cdot(t-s) \ \subset \ \mathbb{C}_d[t]\,.$$

More generally, a flag $F_\bullet$ is a sequence of subspaces

$$F_\bullet \ : \ F_1 \ \subset \ F_2 \ \subset \ \cdots \ \subset \ F_d \ \subset \mathbb{C}_d[t]\,,$$

where F_i has dimension i. For a sequence $\mathbf{a}$ and a flag $F_\bullet$, the *Schubert variety*

$$\{P \in \mathrm{Grass}_{n,d} \mid \dim\left(P \cap F_{d+1-a_j}\right) \geq n+1-j, \ \text{for } j = 0, 1, \ldots, n\}\,, \tag{1.4}$$

is a subvariety of $\mathrm{Grass}_{n,d}$, written $\Omega_{\mathbf{a}}F_\bullet$. It consists of linear subspaces P having special position (encoded by $\mathbf{a}$) with respect to the flag $F_\bullet$. Since $\dim(P \cap F_{d+1-i}(s))$ counts the number of linearly independent polynomials in P that vanish to order at least i at s, we see that $\Omega^\circ_{\mathbf{a}}(s) \subset \Omega_{\mathbf{a}}F_\bullet(s)$. More precisely, $\Omega_{\mathbf{a}}F_\bullet(s)$ is the closure of the Schubert cell $\Omega^\circ_{\mathbf{a}}(s)$ and it is the disjoint union of $\Omega^\circ_{\mathbf{b}}(s)$ for $\mathbf{b} \geq \mathbf{a}$, where $\geq$ is componentwise comparison.

Given ramification sequences $\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(m)}$ and flags $F_\bullet^{(1)}, \ldots, F_\bullet^{(m)}$, the intersection

$$\Omega_{\mathbf{a}^{(1)}}F_\bullet^{(1)} \ \bigcap \ \Omega_{\mathbf{a}^{(2)}}F_\bullet^{(2)} \ \bigcap \ \cdots \ \bigcap \ \Omega_{\mathbf{a}^{(m)}}F_\bullet^{(m)} \tag{1.5}$$

consists of those linear subspaces $P \in G$ having specified position $\mathbf{a}^{(i)}$ with respect to the flag $F_\bullet^{(i)}$, for each $i = 1, \ldots, m$. Kleiman [19] showed that if the flags $F_\bullet^{(i)}$ are general, then the intersection (1.5) is (generically) transverse.

A *Schubert problem* is a list $\mathbf{A} := (\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(m)})$ of sequences satisfying

$$(n{+}1)(d{-}n) \ (= \ \dim \mathrm{Grass}_{n,d}\,) \ = \ |\mathbf{a}^{(1)}| + \cdots + |\mathbf{a}^{(m)}|\,.$$

Given a Schubert problem, Kleiman's Theorem implies that a general intersection (1.5) will be zero-dimensional and thus consist of finitely many points. By transversality, the number $\delta(\mathbf{A})$ of these points is independent of choice of general flags. The Schubert calculus, through the Littlewood-Richardson rule [12], gives algorithms to determine $\delta(\mathbf{A})$.

We mention an important special case. Let $\boldsymbol{\iota} := 0 < 1 < \cdots < n{-}1 < n{+}1$ be the unique ramification sequence with $|\boldsymbol{\iota}| = 1$, and suppose that $(\boldsymbol{\iota}, \ldots, \boldsymbol{\iota})$ is a Schubert problem, so that $(n{+}1)(d{-}n)$ is the number of occurrences of $\boldsymbol{\iota}$. Write $\boldsymbol{\iota}_{n,d}$ for this sequence. Schubert [30] gave the following formula

$$\delta(\boldsymbol{\iota}_{n,d}) \ = \ [(n{+}1)(d{-}n)]!\,\frac{1!2!\cdots n!}{(d{-}n)!(d{-}n{+}1)!\cdots d!}\ . \tag{1.6}$$

By (1.3), the total ramification $(\mathbf{a}_P(s) \mid |\mathbf{a}_P(s)| > 0)$ of a subspace $P \in \mathrm{Grass}^\circ_{n,d}$ is a Schubert problem. Let W be the Wronskian of P. We would like the intersection containing P

$$\bigcap_{s\colon W(s)=0} \Omega_{\mathbf{a}_P(s)} F_\bullet(s) \tag{1.7}$$

to be transverse and zero-dimensional. However, Kleiman's Theorem does not apply, as the flags $F_\bullet(s)$ for s a root of W are not generic. For example, in the problem of four lines, if the Wronskian is $t^4 - t$, then the corresponding intersection (1.7) of Schubert varieties is not transverse. (This worked out in detail in [5, §9].)

We can see that this intersection (1.7) is however always zero-dimensional. Note that any positive-dimensional subvariety meets $\Omega_{\boldsymbol{\iota}} F_\bullet$, for any flag $F_\bullet$. (This is because, for example, $\Omega_{\boldsymbol{\iota}} F_\bullet$ is a hyperplane section of $\mathrm{Grass}_{n,d}$ in its Plücker embedding into projective space.) In particular, if the intersection (1.7) is not zero-dimensional, then given a point $s \in \mathbb{P}^1$ with $W(s) \neq 0$, there will be a point P' in (1.7) which also lies in $\Omega_\iota F_\bullet(s)$. But then the total ramification of P' does not satisfy the Plücker formula (1.3), as its ramification strictly contains the total ramification of P.

A consequence of this argument is that the Wronski map (1.2) is a finite map. In particular, all of its fibers are finite. The intersection number $\delta(\boldsymbol{\iota}_{n,d})$ in (1.6) is an upper bound for the cardinality of a fiber. By Sard's Theorem, this upper bound is obtained for generic Wronskians.

Theorem 1.8. *There are finitely many spaces of polynomials $P \in \mathrm{Grass}_{n,d}$ with a given Wronskian. For a general polynomial $W(t)$ of degree $(n{+}1)(d{-}n)$, there are exactly $\delta(\boldsymbol{\iota}_{n,d})$ spaces of polynomials with Wronskian $W(t)$.*

When W has distinct roots, these spaces of polynomials are exactly the points in the intersection (1.7), where $\mathbf{a}_P(s) = \boldsymbol{\iota}$ at each root s of W. A limiting argument, in which the roots of the Wronskian are allowed to collide one-by-one, proves a local form of Theorem 1.

Theorem 1.9 ([32])**.** *If the roots of a polynomial $W(t)$ of degree $(n+1)(d-n)$ are real, distinct, and sufficiently clustered together, then there are $\delta(\boldsymbol{\iota}_{n,d})$ spaces of polynomials with Wronskian $W(t)$, so that the intersection* (1.7) *is transverse, and each such space of polynomials is real.*

We noted that the intersection (1.7) is not transverse when $d = 3$, $n = 1$, and $W(t) = t^4 - t$. It turns out that it is always transverse when the roots of the Wronskian are distinct and real. This is the stronger form of the Theorem of Mukhin, Tarasov, and Varchenko.

Theorem 1.10 ([25])**.** *For any Schubert problem $(\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(m)})$ and any distinct real numbers $s_1, \ldots, s_m$, the intersection*

$$\Omega_{\mathbf{a}^{(1)}} F_\bullet(s_1) \bigcap \Omega_{\mathbf{a}^{(2)}} F_\bullet(s_2) \bigcap \cdots \bigcap \Omega_{\mathbf{a}^{(m)}} F_\bullet(s_m) \tag{1.11}$$

is transverse and consists solely of real points.

This theorem (without the transversality) is the original statement of the conjecture of Shapiro and Shapiro for Grassmannians, which was posed in exactly this form to the author in May 1995. The Shapiro conjecture was first discussed and studied in detail in [33], where significant computational evidence was presented (see also [35] and [28]). The results and computations in [33], as well as the result of Theorem 1.9, highlighted the key role that transversality seemed to play in the conjecture. This conjecture also appeared in [31].

We will not discuss the proof of Theorem 1.10, except to remark that its main ingredient is an isomorphism between algebraic objects associated to the intersection (1.11) and to certain representation-theoretic data. This isomorphism provides a very deep link between Schubert calculus for the Grassmannian and the representation theory of $\mathfrak{sl}_{n+1}\mathbb{C}$.

We will however sketch the proof of Theorem 1 in the next three sections.

2. Spaces of polynomials with given Wronskian

Theorem 1.8 enables the reduction of Theorem 1 to a special case. Since the Wronski map is finite, a standard limiting argument (given for example in Section 1.3 of [22] or Remark 3.4 of [33]) shows that it suffices to prove Theorem 1 when the Wronskian has distinct real roots that are sufficiently general. Since $\delta(\boldsymbol{\iota}_{n,d})$ is the upper bound for the number of spaces of polynomials with given Wronskian, it suffices to construct this number of distinct spaces of real polynomials with a given Wronskian, when the Wronskian has distinct real roots that are sufficiently general. In fact, this is exactly what Mukhin, Tarasov, and Varchenko do.

Theorem 1′. *If $s_1, \ldots, s_{(n+1)(d-n)}$ are generic real numbers, there exist exactly $\delta(\boldsymbol{\iota}_{n,d})$ distinct real vector spaces of polynomials P with Wronskian $\prod_i (t - s_i)$.*

The proof proceeds by first constructing $\delta(\boldsymbol{\iota}_{n,d})$ distinct spaces of polynomials with a given Wronskian having generic complex roots, which we describe in Section 2.1. This uses a Fuchsian differential equation given by the critical points of a remarkable symmetric function, called the master function. Critical points of the master function are also used in the Bethe ansatz for the Gaudin model, which is a method for decomposing a representation V of $\mathfrak{sl}_{n+1}\mathbb{C}$ into irreducibles that is compatible with the action of certain commuting operators, called the Gaudin Hamiltonians. In particular, a critical point of the master function gives a Bethe eigenvector of the Gaudin Hamiltonians which is also a highest weight vector for an irreducible submodule of V. This is described in Section 3, where the eigenvalues of the Gaudin Hamiltonians on a Bethe vector are shown to be the coefficients of the Fuchsian differential equation giving the corresponding spaces of polynomials. Finally, the reality of the space of polynomials follows as the Gaudin Hamiltonians are real symmetric operators when the Wronskian has only real roots. This implies that the eigenvalues are real, and thus the Fuchsian differential equation and the corresponding space of polynomials is also real. In all, this is an extraordinary proof.

2.1. **Critical points of master functions.** The construction of $\delta(\boldsymbol{\iota}_{n,d})$ spaces of polynomials with a given Wronskian begins with the critical points of a symmetric rational function that arose in the study of hypergeometric solutions to the Knizhnik-Zamolodchikov equations [29], as well as the Bethe ansatz method for the Gaudin model. (See §3.)

The master function $\Phi(\mathbf{x};\mathbf{s})$ depends upon a point $\mathbf{s} := (s_1,\dots,s_{(n+1)(d-n)}) \in \mathbb{C}^{(n+1)(d-n)}$, whose coordinates will be the roots of our Wronskian W, and an additional $\binom{n+1}{2}(d-n)$ complex variables

$$\mathbf{x} \;:=\; (x^{(0)}_1,\dots,x^{(0)}_{d-n},\, x^{(1)}_1,\dots,x^{(1)}_{2(d-n)},\,\dots\,,\, x^{(n-1)}_1,\dots,x^{(n-1)}_{n(d-n)})\,.$$

Each set of variables $\mathbf{x}^{(i)} := (x^{(i)}_1,\dots,x^{(i)}_{(i+1)(d-n)})$ will be the roots of certain intermediate Wronskians.

Define the *master function* $\Phi(\mathbf{x};\mathbf{s})$ by the (rather formidable) formula

$$(2.1)\qquad \frac{\displaystyle\prod_{i=0}^{n-1}\ \prod_{1\leq j<k\leq (i+1)(d-n)} (x^{(i)}_j - x^{(i)}_k)^2}{\displaystyle\prod_{j=1}^{n(d-n)}\ \prod_{k=1}^{(n+1)(d-n)} (x^{(n-1)}_j - s_k)\ \prod_{i=0}^{n-2}\ \prod_{j=1}^{(i+1)(d-n)}\ \prod_{k=1}^{(i+2)(d-n)} (x^{(i)}_j - x^{(i+1)}_k)}\,.$$

This is separately symmetric in each set of variables $\mathbf{x}^{(i)}$. This master function has a much simpler formulation which we give below (2.4).

The critical points of the master function are solutions to the system of equations

$$(2.2)\qquad \frac{1}{\Phi}\frac{\partial}{\partial x^{(i)}_j}\Phi(\mathbf{x};\mathbf{s}) \;=\; 0 \qquad \text{for}\quad i=0,\dots,n{-}1,\quad j=1,\dots,(i{+}1)(d{-}n)\,.$$

When the parameters $\mathbf{s}$ are generic, these *Bethe ansatz equations* turn out to have finitely many solutions. The master function is invariant under the group

$$\mathcal{S} \;:=\; \mathcal{S}_{d-n}\times\mathcal{S}_{2(d-n)}\times\ \cdots\ \times\mathcal{S}_{n(d-n)}\,,$$

where $\mathcal{S}_m$ for the group of permutations of $1,\dots,n$, and the factor $\mathcal{S}_{(i+1)(d-n)}$ permutes the variables in $\mathbf{x}^{(i)}$. Thus $\mathcal{S}$ acts on the critical points. The invariants of this action are polynomials whose roots are the coordinates of the critical points. Given a critical point $\mathbf{x}$, define monic polynomials $\mathbf{p}_\mathbf{x} := (p_0,\dots,p_{n-1})$ where the components $\mathbf{x}^{(i)}$ of $\mathbf{x}$ are the roots of p_i,

$$p_i \ := \ \prod_{j=1}^{(i+1)(d-n)} (t - x_j^{(i)}) \qquad \text{for} \quad i = 0, \ldots, n{-}1\,. \tag{2.3}$$

Also write p_n for the Wronskian, the monic polynomial with roots $\mathbf{s}$. The master function is greatly simplified by this notation. Recall that the discriminant $\mathrm{Discr}(f)$ of a polynomial f is the square of the product of differences of its roots and the resultant $\mathrm{Res}(f,g)$ is the product of all differences of the roots of f and g [4]. Then the formula for the master function (2.1) is

$$\Phi(\mathbf{x};\mathbf{s}) \ = \ \prod_{i=0}^{n} \mathrm{Discr}(p_i) \Big/ \prod_{i=0}^{n-1} \mathrm{Res}(p_i, p_{i+1})\ . \tag{2.4}$$

The connection between the critical points of the master function and spaces of polynomials is through a Fuchsian differential equation of order $n{+}1$ that has only polynomial solutions. Given (an orbit of) a critical point $\mathbf{x}$ represented by the list of polynomials $\mathbf{p}_{\mathbf{x}}$ and write p_n for the Wronskian W, define the *fundamental differential operator $D_{\mathbf{x}}$ of the critical point $\mathbf{x}$* by

$$\Bigl(\frac{d}{dt} - \ln'\Bigl(\frac{p_n}{p_{n-1}}\Bigr)\Bigr) \ \cdots \ \Bigl(\frac{d}{dt} - \ln'\Bigl(\frac{p_1}{p_0}\Bigr)\Bigr)\Bigl(\frac{d}{dt} - \ln'(p_0)\Bigr)\,, \tag{2.5}$$

where $\ln'(f) := \frac{d}{dt}\ln f$. Write $V_{\mathbf{x}}$ for the kernel of $D_{\mathbf{x}}$, which we call the *fundamental space of the critical point $\mathbf{x}$*.

Example 2.6. Since

$$\Bigl(\frac{d}{dt} \ - \ \ln'(p)\Bigr)\ p \ = \Bigl(\frac{d}{dt} \ - \ \frac{p'}{p}\Bigr)\ p \ = \ p' - \frac{p'}{p}p \ = \ 0\,,$$

we see that p_0 is a solution of $D_{\mathbf{x}}$. It is instructive to look at $D_{\mathbf{x}}$ and $V_{\mathbf{x}}$ when $n=1$. Suppose that f a solution to $D_{\mathbf{x}}$ that is linearly independent from p_0. Then

$$0 \ = \ \Bigl(\frac{d}{dt} - \ln'\Bigl(\frac{W}{p_0}\Bigr)\Bigr)\Bigl(\frac{d}{dt} - \ln'(p_0)\Bigr)\ f \ = \ \Bigl(\frac{d}{dt} - \ln'\Bigl(\frac{W}{p_0}\Bigr)\Bigr)\bigl(f' - \frac{p_0'}{p_0}f\bigr)\,.$$

This implies that

$$\frac{W}{p_0} \ = \ f' - \frac{p_0'}{p_0}f\,,$$

or rather that $W = f'p_0 - p_0'f = \mathrm{Wr}(f, p_0)$, so that the kernel of $D_{\mathbf{x}}$ is a space of functions with Wronskian W.

Mukhin and Varchenko showed that what we just saw is always the case, and much more.

Theorem 2.7 ([26], Section 5)**.** *Suppose that $V_{\mathbf{x}}$ is the fundamental space of the critical point $\mathbf{x}$ of the master function Φ whose parameters $\mathbf{s}$ are roots of a polynomial W.*

(1) *The critical point $\mathbf{x}$ is recovered from $V_{\mathbf{x}}$ as follows. Suppose that $f_0, \ldots, f_n$ are monic polynomials in $V_{\mathbf{x}}$ with $\deg f_i = d{-}n+i$. Then, up to scalar multiples, the polynomials $p_0, \ldots, p_{n-1}$ in the sequence $\mathbf{p}_{\mathbf{x}}$ are*

$$f_0\,,\ \mathrm{Wr}(f_0, f_1)\,,\ \mathrm{Wr}(f_0, f_1, f_2)\,,\ \ldots\,,\ \mathrm{Wr}(f_0, \ldots, f_{n-1})\,.$$

(2) $V_{\mathbf{x}}$ *is a space of polynomials of degree* d *and dimension* $n+1$ *lying in* G° *with Wronskian* W*.*

Statement (1) is quite general; it generalizes Example 2.6 and gives a recipe for writing the differential operator with kernel generated by sufficiently differentiable functions $f_0(t), f_2(t), \ldots, f_n(t)$. It follows from some interesting identities among Wronskians shown in the Appendix of [26]. Statement 2 is the deeper of the two. Together these imply that the kernel V of an operator of the form (2.5) is a space of polynomials with Wronskian W if and only if the the polynomials $p_0, \ldots, p_{n-1}$ come from the critical points of the master function (2.1) corresponding to W.

Thus there is an injection from $\mathcal{S}$-orbits of critical points of the master function Φ with parameters $\mathbf{s}$ to spaces of polynomials in $\mathrm{Grass}^\circ_{n,d}$ whose Wronskian has roots $\mathbf{s}$. Mukhin and Varchenko also showed that when $\mathbf{s}$ is generic, this is in fact a bijection.

Theorem 2.8 (Theorem 6.1 in [27])**.** *For generic complex numbers* $\mathbf{s}$*, the master function* Φ *has* $\delta(\boldsymbol{\iota}_{n,d})$ *distinct orbits of critical points and all critical points are nondegenerate.*

The structure (but not of course the details) of their proof is remarkably similar to the structure of the proof of Theorem 1.9 (given in [32]); they allow the parameters to collide one-by-one, and show how the orbits of critical points behave. Ultimately, they obtain the same recursion as in [32], which mimics the Pieri formula for the branching rule for tensor products of representations of $\mathfrak{sl}_{n+1}$ with its fundamental representation V_{ω_n}. This same structure is also found in the main argument in [7]. In fact, this is the same recursion in $\mathbf{a}$ that Schubert established for intersection numbers $\delta(\mathbf{a}, \boldsymbol{\iota}, \ldots, \boldsymbol{\iota})$, and then solved to obtain the formula (1.6).

3. The Bethe ansatz for the Gaudin model

The Bethe ansatz is intended to give an explicit decomposition of a representation V of $\mathfrak{sl}_{n+1}\mathbb{C}$ into irreducible submodules that is also compatible with the action of a family of commuting operators on V, called the Gaudin Hamiltonians. These commuting operators constitute an integrable system. Its development, justification, and refinements are the subject of a large body of work, a small part of which we mention. One unintended consequence (besides the proof of the Shapiro conjecture) is a deeper link between Schubert calculus on the Grassmannian $\mathrm{Grass}_{n,d}$ and representation theory of $\mathfrak{sl}_{n+1}\mathbb{C}$ than had been known previously.

3.1. Representations of $\mathfrak{sl}_{n+1}\mathbb{C}$.

The Lie algebra $\mathfrak{sl}_{n+1}\mathbb{C}$ (or simply $\mathfrak{sl}_{n+1}$) is the space of $(n+1) \times (n+1)$-matrices with zero trace. It has a decomposition

$$\mathfrak{sl}_{n+1} \;=\; \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \,,$$

where $\mathfrak{n}_+$ ($\mathfrak{n}_-$) are the strictly upper (lower) triangular matrices, and $\mathfrak{h}$ consists of the diagonal matrices with zero trace.

As $\mathfrak{h}$ is commutative, any representation V of $\mathfrak{sl}_{n+1}$ decomposes into joint eigenspaces of $\mathfrak{h}$, called *weight spaces*,

$$V \;=\; \bigoplus_{\mu \in \mathfrak{h}^*} V[\mu] \,,$$

where, for $v \in V[\mu]$ and $h \in \mathfrak{h}$, we have $h.v = \mu(h)v$. The possible weights μ of representations lie in the integral *weight lattice*. Positive weights are those that are integral linear combinations of weights of the representation $\mathfrak{n}_+$. The weight lattice has a distinguished basis of *fundamental weights* $\omega_1, \ldots, \omega_n$ that generate the cone of *dominant weights* (a subcone of the positive weights).

The irreducible representations of $\mathfrak{sl}_{n+1}$ enjoy the following classification. An irreducible representation V has a unique highest weight μ. That is, if π is another weight of V, then $\mu - \pi$ is positive. Furthermore, μ is dominant. This highest weight space $V[\mu]$ is 1-dimensional and it generates V, and any two irreducible modules with the same highest weight are isomorphic. Write V_μ for the *highest weight module* with highest weight μ. Lastly, there is one highest weight module for each dominant weight.

The highest weight space $V_\mu[\mu]$ of V_μ is also distinguished as the set of vectors in V_μ that are annihilated by the nilpotent subalgebra $\mathfrak{n}_+$ of $\mathfrak{sl}_{n+1}$. More generally, if V is any representation of $\mathfrak{sl}_{n+1}$ and μ is a weight, then the *singular vectors* in V of weight μ, written $\mathrm{sing}(V[\mu])$, are the vectors in $V[\mu]$ annihilated by $\mathfrak{n}_+$. If $v \in \mathrm{sing}(V[\mu])$ is nonzero, then the submodule $\mathfrak{sl}_{n+1}.v$ it generates is isomorphic to the highest weight module V_μ. Thus V decomposes as a direct sum of submodules generated by the singular vectors,

$$V \;=\; \bigoplus_{\mu} \mathfrak{sl}_{n+1}.\mathrm{sing}(V[\mu])\,, \tag{3.1}$$

so that the multiplicity of the highest weight module V_μ in V is simply the dimension of this space of singular vectors of weight μ.

When V is a tensor product of highest weight modules, the Littlewood-Richardson rule [12] gives formulas for the dimensions of the spaces of singular vectors. Since this is formally the same rule as used to determine the number of points in an intersection (1.5) of Schubert varieties coming from a Schubert problem, these geometric intersection numbers are equal to the dimensions of spaces of singular vectors. In particular, if $V_{\omega_1} \simeq \mathbb{C}^{n+1}$ is the defining representation of $\mathfrak{sl}_{n+1}$ and $V_{\omega_n} = \wedge^n V_{\omega_1} = V^*_{\omega_1}$ (these are the first and last fundamental representations of $\mathfrak{sl}_{n+1}$), then

$$\dim \mathrm{sing}(V_{\omega_n}^{\otimes(n+1)(d-n)}[0]) \;=\; \delta(\boldsymbol{\iota}_{n,d})\,. \tag{3.2}$$

3.2. **The Gaudin model.** The Bethe ansatz is a conjectural method to obtain this decomposition (3.1) by giving an explicit basis for $\mathrm{sing}(V[\mu])$, which is also an eigenbasis for a family of commuting operators on V. For us, V is the tensor product $V_{\omega_n}^{\otimes m}$, and the family of commuting operators are the Gaudin Hamiltonians. These depend upon m distinct complex numbers $s_1, \ldots, s_m$ and a complex variable t.

For each $i, j = 1, \ldots, n{+}1$, let $E_{i,j} \in \mathfrak{sl}_{n+1}$ be the matrix that has all entries 0, except a 1 in row i and column j. For each such pair (i,j) consider the differential operator $X_{i,j}(t)$ acting on $V_{\omega_n}^{\otimes m}$-valued functions of t,

$$X_{i,j}(t) \;:=\; \delta_{i,j}\frac{d}{dt} \;-\; \sum_{k=1}^{m} \frac{E_{j,i}^{(k)}}{t - s_k}\ ,$$

where $E^{(k)}_{j,i}$ acts on tensors in $V^{\otimes m}_{\omega_n}$ by $E_{j,i}$ in the kth factor and by the identity in other factors. Define a differential operator acting on $V^{\otimes m}_{\omega_n}$-valued functions of t,

$$\mathbf{M} \ := \sum_{\sigma\in\mathcal{S}} |\sigma|\ X_{1,\sigma(1)}(t)\ X_{2,\sigma(2)}(t)\ \cdots\ X_{n+1,\sigma(n+1)}(t)\ ,$$

where $\mathcal{S}$ is the group of permutations of $\{1,\dots,n{+}1\}$ and $|\sigma| = \pm$ is the sign of a permutation $\sigma\in\mathcal{S}$. Write $\mathbf{M}$ in standard form

$$\mathbf{M} \ = \ \frac{d^{n+1}}{dt^{n+1}} \ + \ M_1(t)\frac{d^n}{dt^n} \ + \ \cdots \ + \ M_{n+1}(t)\,.$$

These coefficients $M_1(t),\dots,M_{n+1}(t)$ are called the *Gaudin Hamiltonians*. They are linear operators that depend rationally on t and act on $V^{\otimes m}_{\omega_n}$. We collect together some of their properties.

Theorem 3.3. *Suppose that $s_1,\dots,s_m$ are distinct complex numbers. Then*

(1) *The Gaudin Hamiltonians commute, that is, $[M_i(u), M_j(v)] = 0$ for all $i,j = 1,\dots,n{+}1$ and $u,v\in\mathbb{C}$.*
(2) *The Gaudin Hamiltonians centralize the action of $\mathfrak{sl}_{n+1}$ on $V^{\otimes m}_{\omega_n}$.*

Proofs of these statements may be found in [21], as well as Propositions 7.2 and 8.3 in [23]. A consequence of the second assertion is that the Gaudin Hamiltonians preserve the weight space decomposition of the singular vectors of $V^{\otimes m}_{\omega_n}$. Since they commute with each other, the singular vectors of $V^{\otimes m}_{\omega_n}$ have a basis of common eigenvalues. The Bethe ansatz is a method to write down these joint eigenvectors and their eigenvalues.

3.3. **The Bethe ansatz for the Gaudin model.** The Bethe ansatz for the Gaudin model begins with a rational function, called a universal weight function, that takes values in a weight space $V^{\otimes m}_{\omega_n}[\mu]$,

$$v\ :\ \mathbb{C}^l\times\mathbb{C}^m \ \longmapsto\ V^{\otimes m}_{\omega_n}[\mu]\ .$$

This *universal weight function* was introduced in [29] to solve the Knizhnik-Zamolodchikov equations with values in $V^{\otimes m}_{\omega_n}[\mu]$. When the arguments $(\mathbf{x},\mathbf{s})$ are the critical points of a master function, the vector $v(\mathbf{x},\mathbf{s})$ is both singular and an eigenvector of the Gaudin Hamiltonians. (This master function is a generalization of the one defined by (2.1).) The Bethe ansatz conjecture asserts that the vectors $v(\mathbf{x},\mathbf{s})$ form a basis for the space of singular vectors.

For us, $m = (n{+}1)(d{-}n)$, $l = \binom{n+1}{2}(d{-}n)$, and $\mu = 0$. Then the universal weight function is a map

$$v\ :\ \mathbb{C}^{\binom{n+1}{2}(d-n)} \ \longmapsto\ V^{\otimes(n+1)(d-n)}_{\omega_n}[0]\,.$$

For these notes, we omit the definition of $v(\mathbf{x},\mathbf{s})$.

While $v(\mathbf{x},\mathbf{s})$ is a rational function of $\mathbf{x}$ and hence not globally defined, it turns out (Lemma 2.1 of [27]) that if the coordinates of $\mathbf{s}$ are distinct and $\mathbf{x}$ is a critical point of the master function (2.1), then the vector $v(\mathbf{x},\mathbf{s})\in V^{\otimes(n+1)(d-n)}_{\omega_n}[0]$ is well-defined and it is in fact a singular vector. Such a vector $v(\mathbf{x},\mathbf{s})$ when $\mathbf{x}$ is a critical point of the master function a *Bethe vector*. Mukhin and Varchenko also prove the following, which is the second part of Theorem 6.1 in [27].

Theorem 3.4. *When* $\mathbf{s} \in \mathbb{C}^{(n+1)(d-n)}$ *is general, the Bethe vectors form a basis of the space* $\operatorname{sing}(V_{\omega_n}^{\otimes(n+1)(d-n)}[0])$.

The reason to introduce these Bethe vectors is that they are the joint eigenvectors of the Gaudin Hamiltonians.

Theorem 3.5 (Theorem 9.2 in [23])**.** *For any critical point* $\mathbf{x}$ *of the master function* (2.1)*, the Bethe vector* $v(\mathbf{x}, \mathbf{s})$ *is a joint eigenvector of the Gaudin Hamiltonians* $M_1(t), \ldots, M_{n+1}(t)$. *The corresponding eigenvalues* $\mu_1(t), \ldots, \mu_{n+1}(t)$ *are given by the formula*

$$\begin{aligned} (3.6)\qquad & \frac{d^{n+1}}{dt^{n+1}} + \mu_1(t)\frac{d^n}{dt^n} + \cdots + \mu_n(t)\frac{d}{dt} + \mu_{n+1}(t) = \\ & \Big(\frac{d}{dt} + \ln'(p_0)\Big)\Big(\frac{d}{dt} + \ln'\Big(\frac{p_1}{p_0}\Big)\Big) \cdots \Big(\frac{d}{dt} + \ln'\Big(\frac{p_n}{p_{n-1}}\Big)\Big)\Big(\frac{d}{dt} + \ln'\Big(\frac{W}{p_n}\Big)\Big), \end{aligned}$$

where $(p_0(t), \ldots, p_n(t))$ *are the polynomials* (2.3) *associated to the critical point* $\mathbf{x}$ *and* $W(t)$ *is the polynomial with roots* $\mathbf{s}$.

Observe that (3.6) is similar to the formula (2.5) for the differential operator $D_{\mathbf{x}}$ of the critical point $\mathbf{x}$. This similarity is made more precise if we replace the Gaudin Hamiltonians by a different set of operators. Consider the differential operator formally conjugate to $(-1)^{n+1}M$,

$$\begin{aligned} K &= \frac{d^{n+1}}{dt^{n+1}} - \frac{d^n}{dt^n}M_1(t) + \cdots + (-1)^n\frac{d}{dt}M_n(t) + (-1)^{n+1}M_{n+1}(t) \\ &= \frac{d^{n+1}}{dt^{n+1}} + K_1(t)\frac{d^n}{dt^n} + \cdots + K_n(t)\frac{d}{dt} + K_{n+1}(t)\ . \end{aligned}$$

These coefficients $K_i(t)$ are operators on $V_{\omega_n}^{\otimes(n+1)(d-n)}$ that depend rationally on t, and are also called the Gaudin Hamiltonians. Here are the first three,

$$\begin{aligned} K_1(t) &= -M_1(t)\,, \qquad\qquad K_2(t) = M_2(t) - nM_1'(t)\,, \\ K_3(t) &= -M_3(t) + (n{-}1)M_2''(t) - \binom{n}{2}M_1'''(t)\,, \end{aligned}$$

and in general $K_i(t)$ is a differential polynomial in $M_1(t), \ldots, M_{n+1}(t)$.

These operators also commute, $[K_i(u), K_j(v)] = 0$ for all i, j, u, v, and they also commute with the $\mathfrak{sl}_{n+1}$-action on $V_{\omega_n}^{\otimes(n+1)(d-n)}$, and the Bethe vector $v(\mathbf{x}, \mathbf{s})$ is also a joint eigenvector of these new Gaudin Hamiltonians $K_i(t)$. The corresponding eigenvalues $\lambda_1(t), \ldots, \lambda_{n+1}(t)$ are given by the formula

$$\begin{aligned} (3.7)\qquad & \frac{d^{n+1}}{dt^{n+1}} + \lambda_1(t)\frac{d^n}{dt^n} + \cdots + \lambda_n(t)\frac{d}{dt} + \lambda_{n+1}(t) = \\ & \Big(\frac{d}{dt} - \ln'\Big(\frac{W}{p_{n-1}}\Big)\Big)\Big(\frac{d}{dt} - \ln'\Big(\frac{p_{n-1}}{p_n}\Big)\Big) \cdots \Big(\frac{d}{dt} - \ln'\Big(\frac{p_1}{p_0}\Big)\Big)\Big(\frac{d}{dt} - \ln'(p_0)\Big), \end{aligned}$$

which is just the fundamental differential operator $D_{\mathbf{x}}$ of the critical point $\mathbf{x}$.

Corollary 3.8. *Suppose that* $\mathbf{s} \in \mathbb{C}^{(n+1)(d-n)}$ *is generic.*

1. *The set of Bethe vectors form an eigenbasis of* $\operatorname{sing}(V_{\omega_n}^{\otimes(n+1)(d-n)}[0])$ *for the Gaudin Hamiltonians* $K_1(t), \ldots, K_{n+1}(t)$.

(2) *The Gaudin Hamiltonians* $K_1(t), \ldots, K_{n+1}(t)$ *have simple spectrum in that eigenvalues of the Gaudin Hamiltonians separate the basis of eigenvectors.*

Statement (1) follows from Theorems 3.4 and 3.5. For Statement (2), suppose that two Bethe vectors $v(\mathbf{x}, \mathbf{s})$ and $v(\mathbf{x}', \mathbf{s})$ have the same eigenvalues. By (3.7), the corresponding fundamental differential operators would be equal, $D_{\mathbf{x}} = D_{\mathbf{x}'}$. But this implies that the fundamental spaces coincide, $V_{\mathbf{x}} = V_{\mathbf{x}'}$. By Theorem 2.7 the fundamental space determines the orbit of critical points, so the critical points $\mathbf{x}$ and $\mathbf{x}'$ lie in the same orbit, which implies that $v(\mathbf{x}, \mathbf{s}) = v(\mathbf{x}', \mathbf{s})$.

All that remains is to show that the space $V_{\mathbf{x}}$ is real.

4. Shapovalov form and the proof of the Shapiro conjecture

The last step in the proof of Theorem 1 is to show that if $\mathbf{s} \in \mathbb{R}^{(n+1)(d-n)}$ is generic and $\mathbf{x}$ a critical point of the master function (2.1), then the fundamental space $V_{\mathbf{x}}$ of the critical point $\mathbf{x}$ has a basis of real polynomials. As promised in the introduction, the reason for this reality is that the eigenvectors and eigenvalues of a symmetric matrix are real.

We begin with the Shapovalov form. The map $\tau \colon E_{ij} \mapsto E_{ji}$ induces an antiautomorphism on $\mathfrak{sl}_{n+1}$. Given a highest weight module V_μ and a nonzero vector v in $V_\mu[\mu]$, the *Shapovalov form* $\langle\cdot,\cdot\rangle$ on V_μ is defined recursively by

$$\langle v, v\rangle \;=\; 1 \qquad \text{and} \qquad \langle g.u, v\rangle \;=\; \langle u, \tau(g).v\rangle\,,$$

for $g \in \mathfrak{sl}_{n+1}$ and $u, v \in V$.

For example, if $V_{\omega_1} = \mathbb{C}^{n+1}$ is the defining representation of $\mathfrak{sl}_{n+1}$ with basis $e_0, \ldots, e_n$, and we set $v := e_n$, then $\langle e_i, e_j\rangle = \delta_{ij}$. Thus the Shapovalov form is the standard Euclidean inner product on V_{ω_1}. As V_{ω_n} is the linear dual of V_{ω_1}, the Shapovalov form on V_{ω_n} is also the standard Euclidean inner product. In general, this Shapovalov form is nondegenerate on V_μ and positive definite on the real part of V_μ.

The Shapovalov form on V_{ω_n} induces a symmetric bilinear form, also called the Shapovalov form, on the tensor product $V_{\omega_n}^{\otimes(n+1)(d-n)}$. This tensor Shapovalov form is also positive definite on the real part of $V_{\omega_n}^{\otimes(n+1)(d-n)}$.

Theorem 4.1 (Proposition 9.1 in [23])**.** *The Gaudin Hamiltonians are symmetric with respect to the tensor Shapovalov form,*

$$\langle K_i(t).u,\, v\rangle \;=\; \langle u,\, K_i(t).v\rangle\,,$$

for all $i = 1, \ldots, n{+}1$, $t \in \mathbb{C}$, *and* $u, v \in V_{\omega_n}^{\otimes(n+1)(d-n)}$.

We give the most important consequence of this result for our story.

Corollary 4.2. *When the parameters* $\mathbf{s}$ *and variable* t *are real, the Gaudin Hamiltonians* $K_1(t), \ldots, K_{n+1}(t)$ *are real linear operators which are simultaneously diagonalizable with real spectrum.*

Proof. From the definition of the Gaudin Hamiltonians $M_1(t), \ldots, M_{n+1}(t)$, we see that they are real linear operators which act on the real part of $V_{\omega_n}^{\otimes(n+1)(d-n)}$. The same is then also true of the Gaudin Hamiltonians $K_1(t), \ldots, K_{n+1}(t)$. But these are symmetric with respect to the positive definite Shapovalov form. Consequently, the are simultaneously diagonalizable with real spectrum.

Proof of Theorem 1. Suppose that $\mathbf{s} \in \mathbb{R}^{(n+1)(d-n)}$ is general. By Corollary 4.2, the Gaudin Hamiltonians for $t \in \mathbb{R}$ acting on $\mathrm{sing}(V_{\omega_n}^{(n+1)(d-n)}[0])$ are symmetric operators on a Euclidean space, and so have real eigenvectors and eigenvalues. The Bethe vectors $v(\mathbf{x}, \mathbf{s})$ for critical points $\mathbf{x}$ of the master function with parameters $\mathbf{s}$ form an eigenbasis for the Gaudin Hamiltonians. As $\mathbf{s}$ is general, the eigenvalues are distinct by Corollary 3.8 (2), and so the Bethe vectors must be real.

Given a critical point $\mathbf{x}$, the eigenvalues $\lambda_1(t), \ldots, \lambda_{n+1}(t)$ of the Bethe vectors are then real rational functions, and so the fundamental differential operator $D_{\mathbf{x}}$ has real coefficients. But then the fundamental space $V_{\mathbf{x}}$ of polynomials is real.

In this way, we see that each of the $\delta(\boldsymbol{\iota}_{n,d})$ spaces of polynomials $V_{\mathbf{x}}$ whose Wronskian has roots $\mathbf{s}$ that were constructed in Section 2 is in fact real. This proves Theorem 1.

5. Applications of the Shapiro conjecture

Theorem 1 and its stronger version, Theorem 1.10, have a number of other applications in mathematics. Some are straightforward, such as linear series on $\mathbb{P}^1$ with real ramification. Others are much less so, such as Schützenberger evacuation in algebraic combinatorics. Here, we discuss two applications which are in the first class, namely maximally inflected curves and rational functions with real critical points.

5.1. **Maximally inflected curves.** One of the earliest occurrences of the central mathematical object of these notes, spaces of polynomials with prescribed ramification, was in algebraic geometry, as these are linear series $P \subset H^0(\mathbb{P}^1, \mathcal{O}(d))$ on $\mathbb{P}^1$ with prescribed ramification. Their connection to Schubert calculus originated in work of Castelnuovo in 1889 [3] on g-nodal rational curves, and this was important in Brill-Noether theory (see Ch. 5 of [16]) and the Eisenbud-Harris theory of limit linear series [5, 6].

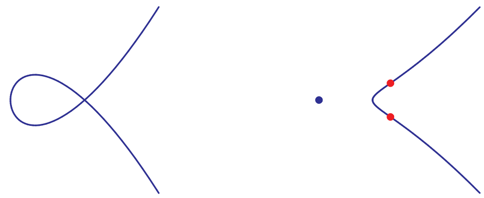
A linear series P on $\mathbb{P}^1$ of degree d and dimension $n{+}1$ (subspace in $\mathrm{Grass}_{n,d}$) gives rise to a degree d map

$$\varphi \,:\, \mathbb{P}^1 \longrightarrow \mathbb{P}^n = \mathbb{P}(P^*) \tag{5.1}$$

of $\mathbb{P}^1$ to projective space. We will call this map a curve. The linear series if ramified at points $s \in \mathbb{P}^1$ where the curve φ is not convex (the jets $\varphi(s), \varphi'(s), \ldots, \varphi^{(n)}(s)$ do not span $\mathbb{P}^n$). Call such a point s a *flex* of the curve (5.1).

A curve is real when P is real. It is *maximally inflected* if all of its flexes are real. The study of these curves was initiated in [17], where restrictions on the topology of plane maximally inflected curves were established. Specifically, there is a lower bound on the number of isolated singularities (and hence an upper bound on the number of nodes) of a maximally inflected plane curve.

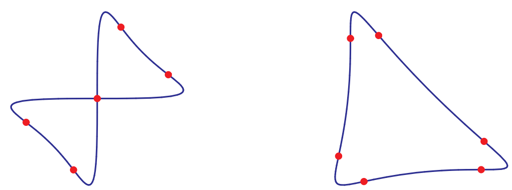
For example, there are two types of cubic curves, which are distinguished by their singular points. The singular point of the curve on the left is a node and connected to the rest of the curve, while the singular point on the other curve is isolated from the rest of the curve.



$y^2 = x^3 + x^2$ $\qquad$ $y^2 = x^3 - x^2$

While both curves have one of their three flexes at infinity, only the curve on the right has its other two flexes real (the dots) and is therefore maximally inflected. A nodal cubic cannot be maximally inflected.

Similarly, a maximally inflected quartic has either 1 or 0 of its (necessarily 3) singular points a node, and necessarily 2 or 3 solitary points. We draw the two types of maximally inflected quartics having six flexes, without their solitary points.



While many constructions of maximally inflected curves were known, Theorem 1, and in particular Theorem 1.10, show that there are many maximally inflected curves: Any curve $\varphi\colon \mathbb{P}^1 \to \mathbb{P}^n$ whose ramification lies in $\mathbb{RP}^1$ must be real and is therefore maximally inflected.

5.2. **Rational functions with real critical points.** A special case of Theorem 1, proved earlier by Eremenko and Gabrielov, serves to illustrate the breadth of mathematical areas touched by this Shapiro conjecture. When $n = 1$, we may associate a rational function $\varphi_P := f_1(t)/f_2(t)$ to a basis $\{f_1(t), f_2(t)\}$ of a vector space $P \in \text{Grass}_{n,d}$ of polynomials. Different bases give different rational functions, but they all differ from φ_P by a fractional linear transformation of the image $\mathbb{P}^1$. We say that such rational functions are *equivalent*.

The critical points of any such rational function are the points of the domain $\mathbb{P}^1$ where the derivative of φ_P,

$$d\varphi_P \ := \ \frac{f_1' f_2 - f_1 f_2'}{f_2^2} \ = \ \frac{1}{f_2^2} \cdot \det \begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix} \ ,$$

vanishes. That is, at the roots of the Wronskian. Eremenko and Gabrielov [8] prove the following result about the critical points of rational functions.

Theorem 5.2. *A rational function φ whose critical points lie on a circle in $\mathbb{P}^1$ maps that circle to a circle.*

To see that this is equivalent to Theorem 1 when $n = 1$, note that we may apply a change of variables to φ so that its critical points lie on the circle $\mathbb{RP}^1 \subset \mathbb{P}^1$. Similarly, the image circle may be assumed to be $\mathbb{RP}^1$. Reversing these coordinate changes establishes the equivalence.

The proof used methods specific to rational functions. Goldberg showed [13] that there are at most $c_d := \frac{1}{d}\binom{2d-2}{d-1}$ rational functions of degree d with a given collection of $2d-2$ simple critical points. If the critical points of a rational function φ of degree d lie on a circle $C \subset \mathbb{CP}^1$ and if φ maps C to C, then $\varphi^{-1}(C)$ forms a graph on the Riemann sphere with nodes the $2d-2$ critical points, each of degree 4, and each having two edges along C and one edge on each side of C. It turns out that there are also c_d such abstract graphs. (In fact, c_d is Catalan number, which counts many objects in combinatorics.) Eremenko and Gabrielov essentially constructed such a rational function φ for each such graph and choice of critical points on C. Since c_d is the upper bound for the number of such rational functions, this construction gives all rational functions with given set of critical points and thus proves Theorem 5.2. More recently, Eremenko and Gabrielov have found an elementary proof of this result, which uses an induction similar to that described after Theorem 2.8, but that has unfortunately never been published [9].

6. Extensions of the Shapiro conjecture

The proofs of different Bethe ansätze for other models (other integrable systems) and other Lie algebras, which is ongoing work of Mukhin, Tarasov, and Varchenko, and others, leads to generalizations of Theorem 1. One such is given in an appendix of [22], where it is conjectured that orbits of critical points of generalized master functions are real. This is the analog of the consequence of Theorem 1 and Theorem 2.7 (1) that the polynomials p_i are real, which is that new conjecture for the Lie algebra $\mathfrak{sl}_{n+1}$. In that appendix, it is noted that this generalization of the Shapiro conjecture is true for $\mathfrak{sp}_{2n}$ and $\mathfrak{so}_{2n+1}$, by the results in Section 7 of [26].

In [24], the Bethe ansatz for the XXX model is used to prove an analog of Theorem 1 for spaces of quasipolynomials (functions of the form $e^{\lambda_i x} f_i(x)$ with $\lambda_i \in \mathbb{R}$) whose discrete Wronskian has only simple real roots separated by at least the step size used in the discrete Wronskian. There surely is more to come.

Likewise the Shapiro conjecture, that an intersection of Schubert varieties in the Grassmannian given by the special flags $F_\bullet(s)$ consists only of real points, makes sense for other flag manifolds. In this more general setting, it is known to fail, but in a very interesting way. When it fails, we can modify it to give a conjecture that holds under scrutiny, and the Shapiro conjecture also admits some appealing generalizations. We briefly describe some of this story.

6.1. Lagrangian and Orthogonal Grassmannians.

Lagrangian and orthogonal Grassmannians are two varieties closely related to the classical Grassmannian. For each of these, the Shapiro conjecture is particularly easy to state.

The (odd) orthogonal Grassmannian, begins with a non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathbb{C}^{2n+1}$. This vector space has a basis $e_1, \ldots, e_{2n+1}$ such that

$$\langle e_i,\, e_{2n+2-j}\rangle \;=\; \delta_{i,j}\,.$$

The *(odd) orthogonal Grassmannian $OG(n)$* is the set of all n-dimensional subspaces V of $\mathbb{C}^{2n+1}$ that are *isotropic* in that $\langle V, V\rangle = 0$. These subspaces have maximal dimension among all isotropic vector spaces. This variety has dimension $\binom{n+1}{2}$.

The Shapiro conjecture for $OG(n)$ begins with a particular rational normal curve γ having parametrization

$$t \longmapsto e_1 + te_2 + \frac{t^2}{2}e_3 + \cdots + \frac{t^n}{n!}e_{n+1} - \frac{t^{n+1}}{(n+1)!}e_{n+2} + \frac{t^{n+2}}{(n+2)!}e_{n+3} - \cdots + (-1)^n\frac{t^{2n}}{(2n)!}e_{2n+1}\,.$$

This has special properties with respect to the form $\langle\cdot,\cdot\rangle$. For $t \in \mathbb{C}$, define the flag $F_\bullet(t)$ in $\mathbb{C}^{2n+1}$ by

$$F_i(t) \;:=\; \mathrm{Span}\{\gamma(t),\, \gamma'(t)\,, \ldots,\, \gamma^{(i-1)}(t)\}\,.$$

Then $F_\bullet(t)$ is *isotropic* in that

$$\langle F_i(t),\, F_{2n+1-i}(t)\rangle \;=\; 0\,.$$

More generally, an isotropic flag $F_\bullet$ of $\mathbb{C}^{2n+1}$ is a flag such that $\langle F_i, F_{2n+1-i}\rangle = 0$. The *Schubert variety* $X_{\boldsymbol\lambda}F_\bullet$ of $OG(n)$ is defined by a Schubert index $\boldsymbol\lambda$ and an isotropic flag $F_\bullet$. Write $|\boldsymbol\lambda|$ for its codimension. A Schubert problem is a list $(\boldsymbol\lambda_1, \ldots, \boldsymbol\lambda_m)$ of Schubert indices such that

$$|\boldsymbol\lambda_1| + |\boldsymbol\lambda_2| + \cdots + |\boldsymbol\lambda_m| \;=\; \dim OG(n) \;=\; \binom{n+1}{2}.$$

We state the Shapiro conjecture for $OG(n)$.

Conjecture 6.1. *If $(\boldsymbol\lambda_1, \ldots, \boldsymbol\lambda_m)$ is a Schubert problem for $OG(n)$ and $s_1, \ldots, s_m$ are distinct real numbers, then the intersection*

$$X_{\boldsymbol\lambda_1}F_\bullet(s_1) \bigcap X_{\boldsymbol\lambda_2}F_\bullet(s_2) \bigcap \cdots \bigcap X_{\boldsymbol\lambda_m}F_\bullet(s_m)$$

is transverse with all points real.

Besides optimism based upon the validity of the Shapiro conjecture for Grassmannians, the evidence for Conjecture 6.1 comes in two forms. A local version, analogous to Theorem 1.9, is true [34], and several tens of thousands of instances have been checked with a computer.

There is a similar story but with a different outcome for the Lagrangian Grassmannian. Let $\langle\cdot,\cdot\rangle$ be a nondegenerate skew symmetric bilinear form on $\mathbb{C}^{2n}$. This vector space has a basis $e_1, \ldots, e_{2n}$ such that

$$\langle e_i,\, e_{2n+1-j}\rangle \;=\; \begin{cases} \delta_{i,j} & \text{if } i \leq 2n \\ -\delta_{i,j} & \text{if } i > 2n \end{cases}\,.$$

The *Lagrangian Grassmannian* $LG(n)$ is the set of all *isotropic* n-dimensional subspaces V of $\mathbb{C}^{2n}$. These subspaces have maximal dimension among all isotropic vector spaces, and are typically called Lagrangian subspaces. This variety has dimension $\binom{n+1}{2}$.

For the Shapiro conjecture for $LG(n)$, we have the rational normal curve γ with parametrization

$$t \longmapsto e_1 + te_2 + \frac{t^2}{2}e_3 + \cdots + \frac{t^n}{n!}e_{n+1} - \frac{t^{n+1}}{(n+1)!}e_{n+2} + \frac{t^{n+2}}{(n+2)!}e_{n+3} - \cdots + (-1)^{n-1}\frac{t^{2n-1}}{(2n-1)!}e_{2n}\,.$$

For $t \in \mathbb{C}$, define the flag $F_\bullet(t)$ in $\mathbb{C}^{2n+1}$ by

$$F_i(t) \ := \ \mathrm{Span}\{\gamma(t),\, \gamma'(t)\,, \ldots,\, \gamma^{(i-1)}(t)\}\,.$$

Then $F_\bullet(t)$ is *isotropic* in that

$$\langle F_i(t),\, F_{2n-i}(t)\rangle \ = \ 0\,.$$

More generally, an isotropic flag $F_\bullet$ of $\mathbb{C}^{2n}$ is a flag such that $\langle F_i, F_{2n-i}\rangle = 0$. The Schubert variety $X_{\boldsymbol{\lambda}}F_\bullet$ of $LG(n)$ is defined by a Schubert index $\boldsymbol{\lambda}$ and an isotropic flag $F_\bullet$. It has codimension $|\boldsymbol{\lambda}|$. A Schubert problem is a list $(\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_m)$ such that

$$|\boldsymbol{\lambda}_1| + |\boldsymbol{\lambda}_2| + \cdots + |\boldsymbol{\lambda}_m| \ = \ \dim LG(n) \ = \ \binom{n+1}{2}\,.$$

Belkale and Kumar [2] define a notion of Levi movability for Schubert conditions, which has the following geometric interpretation. Each Schubert variety $X_{\boldsymbol{\lambda}}F_\bullet$ of $LG(n)$ is the intersection of $LG(n)$ with a Schubert variety $\Omega_{\mathbf{a}(\boldsymbol{\lambda})}F_\bullet$ of the Grassmannian of n planes in $\mathbb{C}^{2n}$. The index $\boldsymbol{\lambda}$ is *Levi movable* when these two Schubert varieties have the same codimension in their respective Grassmannians. A *Levi movable* Schubert problem is one made up of Levi movable Schubert indices.

The obvious generalization of Theorem 1 and Conjecture 6.1 to $LG(n)$ turns out to be false. We offer a modification that we believe is true.

Conjecture 6.2. *If $(\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_m)$ is a Schubert problem for $LG(n)$ and $s_1, \ldots, s_m$ are distinct real numbers, then the intersection*

$$X_{\boldsymbol{\lambda}_1}F_\bullet(s_1) \bigcap X_{\boldsymbol{\lambda}_2}F_\bullet(s_2) \bigcap \cdots \bigcap X_{\boldsymbol{\lambda}_m}F_\bullet(s_m)$$

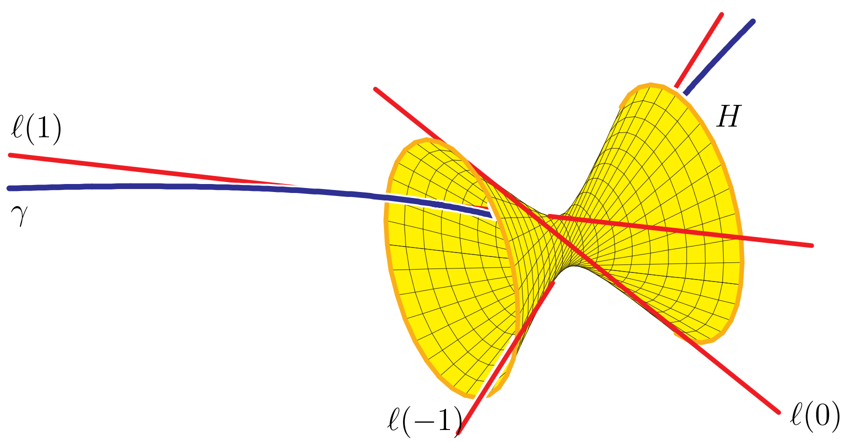
is transverse. If $(\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_m)$ is Levi movable, then all points of intersection are real, but if it is not Levi movable, then no point in the intersection is real.

The strongest evidence in favor of Conjecture 6.2 is that it is true when the Schubert problem $(\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_m)$ is Levi movable. This follows from the definition of Levi movable and the Shapiro conjecture for Grassmannians. Further evidence is that if each $\boldsymbol{\lambda}_i$ is simple in that $|\boldsymbol{\lambda}| = 1$, then a local version, similar to Theorem 1.9 but without transversality, is true. That is, if the s_i are sufficiently clustered, then no point in the intersection is real [34]. Lastly, several tens of thousands of instances have been checked with a computer.

6.2. **Monotone conjecture for flag manifolds.** The original Shapiro conjecture was for Schubert varieties in the classical (type-A) flag manifold. This conjecture fails for the first non-trivial Schubert problem on a flag variety that is not a Grassmannian. Consider the geometric problem of partial flags $\ell \subset \Lambda$ in 3-dimensional space where ℓ is required to meet three fixed lines and Λ is required to contain two fixed points.

This is just the problem of four lines in disguise. Suppose that p and q are the two fixed points that Λ is required to contain. Then Λ contains the secant line $\overline{p,q}$ spanned by these two points. Since $\ell \subset \Lambda$, we see that ℓ must meet $\overline{p,q}$. As ℓ must also meet three lines, this problem reduces to the problem of four lines. In this way, there are two solutions to this Schubert problem.

Now let us investigate the original Shapiro conjecture for this Schubert problem, which posits that both flags $\ell \subset \Lambda$ will be real, if we require that ℓ meets three fixed tangent lines to a rational curve and Λ contains two fixed points of the rational curve. Let γ be the rational normal curve (1) from the Introduction and suppose that the three fixed lines of our problem are its tangent lines $\ell(-1)$, $\ell(0)$, and $\ell(1)$. These line lie on the hyperboloid H with equation (2). Here is another view of these lines, the curve γ, and the hyperboloid.



If we require ℓ to meet the three tangent lines $\ell(-1)$, $\ell(0)$, and $\ell(1)$ and Λ to contain the two points $\gamma(v)$ and $\gamma(w)$ of γ, then ℓ also meets the line $\lambda(v,w)$ spanned by these two points. As in the Introduction, the lines ℓ that we seek will come from points where the secant line $\lambda(v,w)$ meets H.

Figure 2 shows an expanded view down the throat of the hyperboloid, with a secant line $\lambda(v,w)$ that meets the hyperboloid in two points. For these points $\gamma(v)$

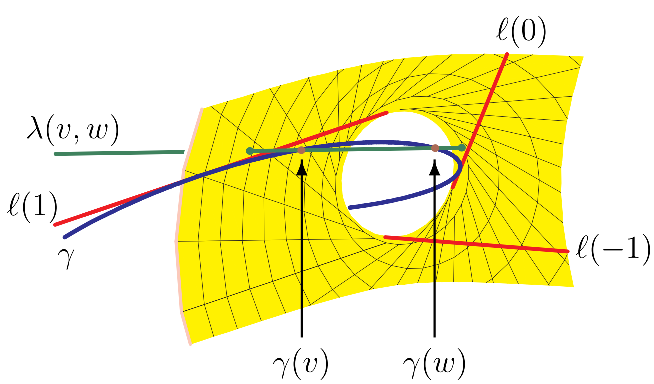


FIGURE 2. A secant line meeting H

and $\gamma(w)$ there will be two real flags $\ell \subset \Lambda$ satisfying our conditions. This is consistent with the Shapiro conjecture.

In contrast, Figure 3 shows a secant lines $\lambda(v,w)$ that does not meet the hyperboloid in any real points. For these points $\gamma(v)$ and $\gamma(w)$, neither flag $\ell \subset \Lambda$

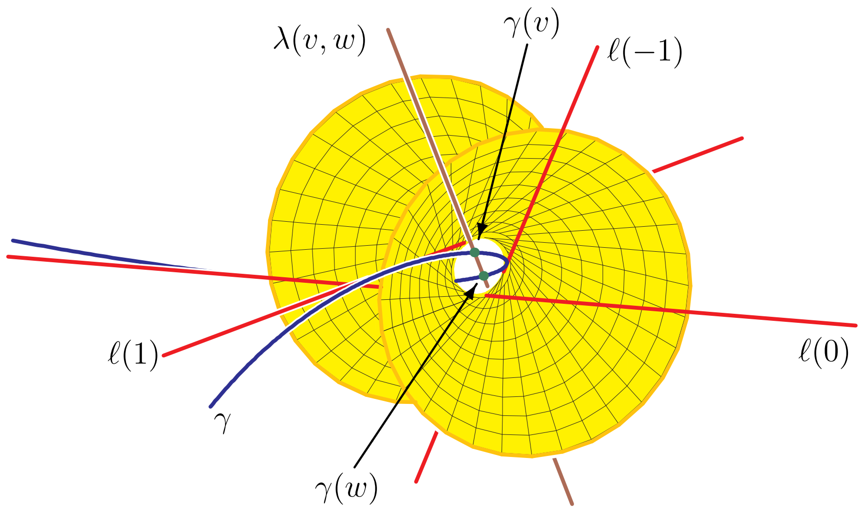


FIGURE 3. A secant line not meeting H

satisfying our conditions is real. This is not consistent with the Shapiro conjecture, so we see that Shapiro conjecture does not hold for this Schubert problem, and so it is false.

This failure is however quite interesting. If we label the points $-1, 0, 1$ with 1 (conditions on the line) and v, w by 2 (conditions on the plane), then along γ they occur in order

$$11122 \text{ in Figure 2} \quad \text{and} \quad 11212 \text{ in Figure 3}.$$

The sequence for Figure 2 is *monotone* and in this case both solutions are always real. This example suggests a way to correct the Shapiro conjecture, which we call the *monotone conjecture*.

Specifically, let $\mathbf{n}\colon 0 < n_1 < \cdots < n_m < d$ be a sequence of integers. The manifold $\mathbb{F}\ell_{\mathbf{n},d}$ of flags of type $\mathbf{n}$ is the set of all sequences of subspaces

$$E_\bullet \ : \ E_{n_1} \ \subset \ E_{n_2} \ \subset \ \cdots \ \subset \ E_{n_m} \ \subset \ \mathbb{C}_d[t]$$

with $\dim E_{n_i} = n_i$. The forgetful map $E_\bullet \mapsto E_{n_i}$ induces a projection

$$\pi_i \ : \ \mathbb{F}\ell_{\mathbf{n},d} \ \longmapsto \ \mathrm{Grass}_{n_i,d}$$

to a Grassmannian. A *Grassmannian Schubert variety* is a subvariety of $\mathbb{F}\ell_{\mathbf{n},d}$ of the form $\pi_i^{-1}\Omega_{\mathbf{a}}F_\bullet$. Write $X_{(\mathbf{a},n_i)}F_\bullet$ for this Grassmannian Schubert variety and call $(\mathbf{a}, n_i)$ a Grassmannian Schubert condition.

A *Grassmannian Schubert problem* is a list

$$(\mathbf{a}^{(1)}, n^{(1)}),\ (\mathbf{a}^{(2)}, n^{(2)}),\ \ldots,\ (\mathbf{a}^{(m)}, n^{(m)}), \tag{6.3}$$

of Grassmannian Schubert conditions satisfying $|\mathbf{a}^{(1)}| + \cdots + |\mathbf{a}^{(m)}| = \dim \mathbb{F}\ell_{\mathbf{n},d}$. We assume that the conditions (6.3) of a Grassmannian Schubert problem are sorted in that

$$n^{(1)} \leq n^{(2)} \leq \cdots \leq n^{(m)}.$$

We state the monotone conjecture.

Conjecture 6.4. *Let* $((\mathbf{a}^{(1)}, n^{(1)}), \ldots, (\mathbf{a}^{(m)}, n^{(m)}))$ *be a Grassmannian Schubert problem for the flag variety* $\mathbb{F}\ell_{\mathbf{n},d}$. *Whenever* $s_1 < s_2 < \cdots < s_m$ *are real numbers, the intersection*

$$X_{(\mathbf{a}^{(1)},n^{(1)})}F_\bullet(s_1) \bigcap X_{(\mathbf{a}^{(2)},n^{(2)})}F_\bullet(s_2) \bigcap \cdots \bigcap X_{(\mathbf{a}^{(m)},n^{(m)})}F_\bullet(s_m),$$

is transverse with all points of intersection real (when it is nonempty).

There is a lot of evidence in support of this monotone conjecture. First, the Shapiro conjecture for Grassmannians is the special case of the monotone conjecture when $m = 1$, for then $\mathbb{F}\ell_{\mathbf{n},d} = \mathrm{Grass}_{n_1,d}$, and the monotonicity condition $s_1 < \cdots < s_m$ is empty as any reordering of the list of Schubert conditions remains sorted.

But there is more. This conjecture was formulated in [28], where the failure of reality in our example was noted. That project utilized some serious computer investigation of the monotone conjecture. This computer experimentation used over 15 gigaHertz-years of computing, solving over 500 million polynomial systems representing intersections of Schubert varieties in over 1100 different enumerative problems on 27 different flag manifolds. Some of this computation studied intersections of Schubert varieties that were not necessarily monotone and that did not always involve Grassmannian Schubert conditions. This experimentation discovered that such an intersection is not necessarily transverse if the monotone condition is violated. More interesting, the intersection may not be zero-dimensional (for any $s_1, \ldots, s_m \in \mathbb{C}$) if the Schubert problem does not involve Grassmannian Schubert conditions.

A third piece of evidence for the monotone conjecture was provided by Eremenko, et. al [10], who showed that it is true for two-step flag manifolds, when $\mathbf{n} = d-1 < d$. This result is a special case of their main theorem, which asserts the reality of a rational function φ with prescribed critical points on $\mathbb{RP}^1$ and prescribed coincidences $\varphi(v) = \varphi(w)$, when v, w are real. Their proof was based on the results of [8].

Phrasing their result in terms of $\mathrm{Grass}_{d-1,d}$ shows that it is a generalization of the Shapiro conjecture, where we replace the flags $F_\bullet(s)$ by more general secant flags. Geometrically, the flag $F_\bullet(s)$ is the flag of subspaces osculating the rational normal curve γ. A *secant flag* $F_\bullet$ is one where every subspace F_i of $F_\bullet$ is spanned by its points of intersection with γ. Secant flags $F^1_\bullet, \ldots, F^m_\bullet$ are *disjoint* if there exist disjoint intervals $I_1, \ldots, I_m$ of γ such that the subspaces in flag $F^i_\bullet$ meet γ at points of I_i. The main result of [8] is that an intersection of Schubert varieties in $\mathrm{Grass}_{d-1,d}$ given by disjoint secant flags is transverse with all points real.

This result motivates the following *secant conjecture*.

Conjecture 6.5. *If* $(\mathbf{a}_1, \ldots, \mathbf{a}_m)$ *is a Schubert problem for* $\mathrm{Grass}_{n,d}$ *and* $F^1_\bullet, \ldots, F^m_\bullet$ *are disjoint secant flags, then the intersection*

$$\Omega_{\mathbf{a}_1}F^1_\bullet \bigcap \Omega_{\mathbf{a}_2}F^2_\bullet \bigcap \cdots \bigcap \Omega_{\mathbf{a}_m}F^m_\bullet$$

is transverse with all points real.

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