

# FRONTIERS OF REALITY IN SCHUBERT CALCULUS

FRANK SOTTILE

ABSTRACT. The theorem of Mukhin, Tarasov, and Varchenko (formerly the Shapiro conjecture for Grassmannians) asserts that all (*a priori* complex) solutions to certain geometric problems in the Schubert calculus are actually real. Their proof is quite remarkable, using ideas from integrable systems, Fuchsian differential equations, and representation theory. There is now a second proof of this result, and it has ramifications in other areas of mathematics, from curves to control theory to combinatorics. Despite this work, the original Shapiro conjecture is not yet settled. While it is false as stated, it has several interesting and not quite understood modifications and generalizations that are likely true, and the strongest and most subtle version of the Shapiro conjecture for Grassmannians remains open.

## INTRODUCTION

While it is not unusual for a univariate polynomial  $f$  with real coefficients to have *some* real roots—under reasonable assumptions we expect  $\sqrt{\deg f}$  real roots [37]—it is rare for a polynomial to have all of its roots real. In a sense, the only natural example of a polynomial with all of its roots real is the characteristic polynomial of a real symmetric matrix, as all eigenvalues of a symmetric matrix are real.

Similarly, when a system of real polynomial equations has finitely many (*a priori* complex) solutions, we expect some, but likely not all, solutions to be real. In fact, upper bounds on the number of real solutions [1, 33] sometimes ensure that not all solutions can be real. As before, the most natural example of a system with only real solutions is the system of equations for the eigenvectors and eigenvalues of a real symmetric matrix.

Here is another system of polynomial equations that has only real solutions. The Wronskian of univariate polynomials  $f_0, \dots, f_n \in \mathbb{C}[t]$  is the determinant

$$\det \begin{pmatrix} f_0(t) & f_1(t) & \cdots & f_n(t) \\ f_0'(t) & f_1'(t) & \cdots & f_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(n)}(t) & f_1^{(n)}(t) & \cdots & f_n^{(n)}(t) \end{pmatrix}.$$

Up to a scalar multiple, the Wronskian depends only upon the linear span  $P$  of the polynomials  $f_0, \dots, f_n$ . This scaling retains only the information of the roots and their multiplicities. Recently, Mukhin, Tarasov, and Varchenko [40] proved the remarkable (but seemingly innocuous) result.

---

2000 *Mathematics Subject Classification.* 14M15, 14N15.

*Key words and phrases.* Schubert calculus, Bethe ansatz, Wronskian, Calogero-Moser space.

The work of Sottile supported by NSF grant DMS-0701050.

Bulletin of the AMS, to appear.

**Theorem 1.** *If the Wronskian of a vector space  $P$  of polynomials has only real roots, then  $P$  has a basis of real polynomials.*

While not immediately apparent, those  $(n+1)$ -dimensional subspaces  $P$  of  $\mathbb{C}[t]$  with a given Wronskian  $W$  are the solutions to a system of polynomial equations that depend on the roots of  $W$ . In Section 1, we explain how the Shapiro conjecture for Grassmannians is equivalent to Theorem 1.

The proof of Theorem 1 uses the Bethe ansatz for the (periodic) Gaudin model on certain modules (representations) of the Lie algebra  $\mathfrak{sl}_{n+1}\mathbb{C}$ . The Bethe ansatz is a method to find pure states, called *Bethe vectors*, of quantum integrable systems [21]. Here, that means common eigenvectors for a family of commuting operators called the Gaudin Hamiltonians which generate a commutative Bethe algebra  $\mathcal{B}$ . As  $\mathcal{B}$  commutes with the action of  $\mathfrak{sl}_{n+1}\mathbb{C}$ , this also decomposes a module of  $\mathfrak{sl}_{n+1}\mathbb{C}$  into irreducible submodules. It includes a set-theoretic map from the Bethe eigenvectors to spaces of polynomials with a given Wronskian. A coincidence of numbers, from the Schubert calculus and from representation theory, implies that this map is a bijection. As the Gaudin Hamiltonians are symmetric with respect to the positive definite Shapovalov form, their eigenvectors and eigenvalues are real. Theorem 1 follows as eigenvectors with real eigenvalues must come from real spaces of polynomials. We describe this in Sections 2, 3, and 4.

There is now a second proof [45] of Theorem 1, also passing through integrable systems and representation theory. It provides a deep connection between the Schubert calculus and the representation theory of  $\mathfrak{sl}_{n+1}\mathbb{C}$ , strengthening Theorem 1 to include transversality.

The geometry behind the statement of Theorem 1 appears in many other guises, some of which we describe in Section 6. These include linear series on the projective line [9], rational curves with prescribed flexes [32], and the feedback control of a system of linear differential equations [5, 12]. A special case of the Shapiro conjecture concerns rational functions with prescribed critical points, and was proved in this form by Eremenko and Gabrielov [13]. They showed that a rational function whose critical points lie on a circle in the Riemann sphere maps that circle to another circle. Using the strengthening of Theorem 1 involving transversality, Purbhoo [49] discovered that the fundamental combinatorial algorithms on Young Tableaux come from the monodromy of the map that takes spaces of polynomials to their Wronskians.

A generalization of Theorem 1 by Mukhin, Tarasov, and Varchenko [44] implies the following attractive statement from matrix theory. Let  $b_0, b_1, \dots, b_n$  be distinct real numbers,  $\alpha_0, \dots, \alpha_n$  be complex numbers, and consider the matrix

$$Z := \begin{pmatrix} \alpha_0 & (b_0 - b_1)^{-1} & \cdots & (b_0 - b_n)^{-1} \\ (b_1 - b_0)^{-1} & \alpha_1 & \cdots & (b_1 - b_n)^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ (b_n - b_0)^{-1} & (b_n - b_1)^{-1} & \cdots & \alpha_n \end{pmatrix}.$$

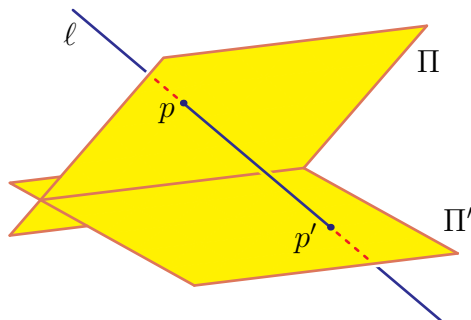
**Theorem 2.** *If  $Z$  has only real eigenvalues, then  $\alpha_1, \dots, \alpha_n$  are real.*

Unlike its proof, the statement of Theorem 2 has nothing to do with Schubert calculus or representations of  $\mathfrak{sl}_{n+1}\mathbb{C}$  or integrable systems, and it remains a challenge to prove it directly. We discuss this in Section 5.

The statement and proof of Theorem 1 is only part of this story. Theorem 1 settles (for Grassmannians) a conjecture in Schubert calculus made by Boris Shapiro and

Michael Shapiro in 1993/4. While this Shapiro conjecture is false for most other flag manifolds, there are appealing corrections and generalizations supported by theoretical evidence and by overwhelming computational evidence, and the strongest and most subtle form remains open. We sketch this in Section 7.

**First steps: the problem of four lines.** We close this Introduction by illustrating the Schubert calculus and the Shapiro conjecture with some beautiful geometry. Consider the set of all lines in three-dimensional space. This set (a Grassmannian) is four-dimensional, which we may see by counting the degrees of freedom for a line  $\ell$  as follows. Fix planes  $\Pi$  and  $\Pi'$  that meet  $\ell$  in points  $p$  and  $p'$  as shown.



Since each point  $p, p'$  has two degrees of freedom to move within its plane, we see that the line  $\ell$  enjoys four degrees of freedom.

Similarly, the set of lines that meet a fixed line is three-dimensional. More parameter counting tells us that if we fix four lines, then the set of lines that meet each of our fixed lines will be zero-dimensional. That is, it consists of finitely many lines. The Schubert calculus gives algorithms to determine this number of lines. We instead use elementary geometry to show that this number is 2.

The Shapiro conjecture asserts that if the four fixed lines are chosen in a particular way, then both solution lines will be real. This special choice begins by specifying a twisted cubic curve,  $\gamma$ . While any twisted cubic will do, we'll take the one with parametrization

$$(1) \quad \gamma : t \longmapsto (6t^2 - 1, \frac{7}{2}t^3 + \frac{3}{2}t, \frac{3}{2}t - \frac{1}{2}t^3).$$

Our fixed lines will be four lines tangent to  $\gamma$ .

We understand the lines that meet our four tangent lines by first considering lines that meet three tangent lines. We are free to fix the first three points of tangency to be any of our choosing, for instance,  $\gamma(-1)$ ,  $\gamma(0)$ , and  $\gamma(1)$ . Then the three lines  $\ell(-1)$ ,  $\ell(0)$ , and  $\ell(1)$  tangent at these points have parametrizations

$$(-5 + s, 5 - s, -1), \quad (-1, s, s), \quad \text{and} \quad (5 + s, 5 + s, 1) \quad \text{for } s \in \mathbb{R}.$$

These lines all lie on the hyperboloid  $H$  of one sheet defined by

$$(2) \quad x^2 - y^2 + z^2 = 1,$$

which has two rulings by families of lines. The lines  $\ell(-1)$ ,  $\ell(0)$ , and  $\ell(1)$  lie in one family, and the other family consists of the lines meeting  $\ell(-1)$ ,  $\ell(0)$ , and  $\ell(1)$ . This family is drawn on the hyperboloid  $H$  in Figure 1.

The lines that meet  $\ell(-1)$ ,  $\ell(0)$ ,  $\ell(1)$ , and a fourth line  $\ell(s)$  will be those in this second family that also meet  $\ell(s)$ . In general, there will be two such lines, one for each point of intersection of line  $\ell(s)$  with  $H$ , as  $H$  is defined by the quadratic polynomial (2). The

remarkable geometric fact is that every such tangent line,  $\ell(s)$  for  $s \notin \{-1, 0, 1\}$ , will meet the hyperboloid in two real points. We illustrate this when  $s = 0.31$  in Figure 1, highlighting the two solution lines.

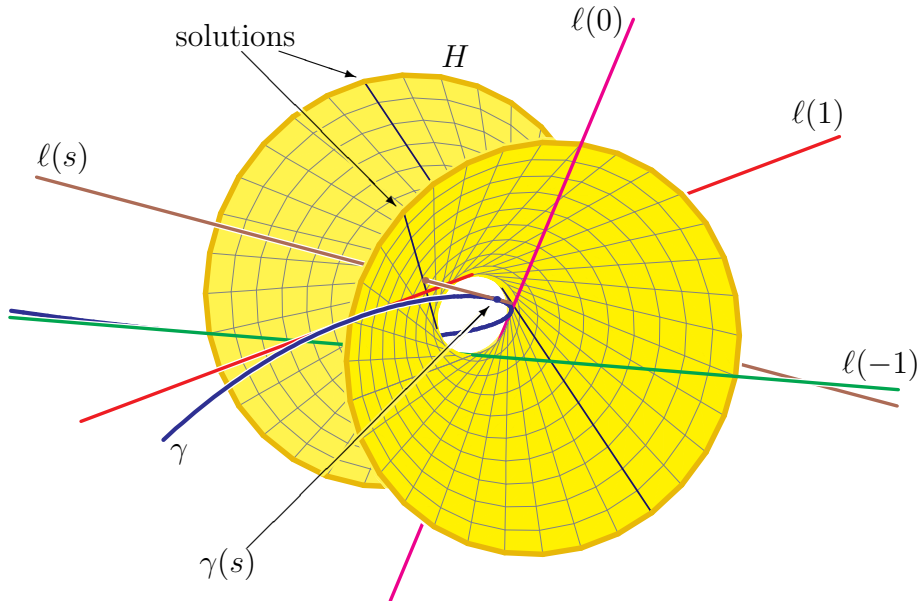


FIGURE 1. The problem of four lines.

The Shapiro conjecture and its extensions claim that this reality always happens: If the conditions for a Schubert problem are chosen in a particular way relative to a rational normal curve (here, tangent lines to the twisted cubic curve  $\gamma$  of (1)), then all solutions will be real. When the Schubert problem comes from a Grassmannian (like this problem of four lines), the Shapiro conjecture is true—this is the theorem of Mukhin, Tarasov, and Varchenko. For most other flag manifolds, it is known to fail, but in very interesting ways.

**Acknowledgments.** We thank those who have helped us to understand this story and to improve this exposition. In particular, we thank Eugene Mukhin, Alexander Varchenko, Milen Yakimov, Aaron Lauve, Zach Teitler, and Nickolas Hein.

## 1. THE SHAPIRO CONJECTURE FOR GRASSMANNIANS

Let  $\mathbb{C}_d[t]$  be the set of complex polynomials of degree at most  $d$  in the indeterminate  $t$ , a vector space of dimension  $d+1$ . Fix a positive integer  $n < d$  and let  $\mathbb{G}(n, d)$  be the set of all  $(n+1)$ -dimensional linear subspaces  $P$  of  $\mathbb{C}_d[t]$ . This *Grassmannian* is a complex manifold of dimension  $(n+1)(d-n)$  [23, Ch. 1.5].

The main character in our story is the Wronski map, which associates to a point  $P \in \mathbb{G}(n, d)$  the Wronskian of a basis for  $P$ . If  $\{f_0(t), \dots, f_n(t)\}$  is a basis for  $P$ , its Wronskian is the determinant of the derivatives of the basis,

$$(1.1) \quad \text{Wr}(f_0, \dots, f_n) := \det \begin{pmatrix} f_0 & f'_0 & \cdots & f_0^{(n)} \\ f_1 & f'_1 & \cdots & f_1^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ f_n & f'_n & \cdots & f_n^{(n)} \end{pmatrix},$$

which is a nonzero polynomial of degree at most  $(n+1)(d-n)$ . This does not quite define a map  $\mathbb{G}(n, d) \rightarrow \mathbb{C}_{(n+1)(d-n)}[t]$ , as choosing a different basis for  $P$  multiplies the Wronskian by a nonzero constant. If we consider the Wronskian up to a nonzero constant, we obtain the *Wronski map*

$$(1.2) \quad \text{Wr} : \mathbb{G}(n, d) \longrightarrow \mathbb{P}(\mathbb{C}_{(n+1)(d-n)}[t]) \simeq \mathbb{P}^{(n+1)(d-n)},$$

where  $\mathbb{P}(V)$  denotes the projective space consisting of all 1-dimensional linear subspaces of a vector space  $V$ .

We restate Theorem 1, the simplest version of the Theorem of Mukhin, Tarasov, and Varchenko [40].

**Theorem 1.** *If the Wronskian of a space  $P$  of polynomials has only real roots, then  $P$  has a basis of real polynomials.*

The problem of four lines in the Introduction is a special case of Theorem 1 when  $d = 3$  and  $n = 1$ . To see this, note that if we apply an affine function  $a + bx + cy + dz$  to the curve  $\gamma(t)$  of (1), we obtain a cubic polynomial in  $\mathbb{C}_3[t]$ , and every cubic polynomial comes from a unique affine function. A line  $\ell$  in  $\mathbb{C}^3$  (actually in  $\mathbb{P}^3$ ) is cut out by a two-dimensional space of affine functions, which gives a 2-dimensional space  $P_\ell$  of polynomials in  $\mathbb{C}_3[t]$ , and hence a point  $P_\ell \in \mathbb{G}(1, 3)$ .

It turns out that the Wronskian point  $P_\ell \in \mathbb{G}(1, 3)$  is a quartic polynomial with a root at  $s \in \mathbb{C}$  if and only if the corresponding line  $\ell$  meets the line  $\ell(s)$  tangent to the curve  $\gamma$  at  $\gamma(s)$ . Thus a line  $\ell$  meets four lines tangent to  $\gamma$  at real points if and only if the Wronskian of  $P_\ell \in \mathbb{G}(1, 3)$  vanishes at these four points. Since these points are real, Theorem 1 implies that  $P_\ell$  has a basis of real polynomials. Thus  $\ell$  is cut out by real affine functions, and hence it is real.

**1.1. Geometric form of the Shapiro conjecture.** Let  $P \in \mathbb{G}(n, d)$  be a subspace. We consider the order of vanishing at a point  $s \in \mathbb{C}$  of polynomials in a basis for  $P$ . There will be a minimal order  $a_0$  of vanishing for these polynomials. Suppose that  $f_0$  vanishes to this order. Subtracting an appropriate multiple of  $f_0$  from each of the other polynomials, we may assume that they vanish to order greater than  $a_0$  at  $s$ . Let  $a_1$  be the minimal order of vanishing at  $s$  of these remaining polynomials. Continuing in this fashion, we obtain a basis  $f_0, \dots, f_n$  of  $P$  and a sequence

$$0 \leq a_0 < a_1 < \dots < a_n \leq d,$$

where  $f_i$  vanishes to order  $a_i$  at  $s$ . Call this sequence  $\mathbf{a}_P(s)$  the *ramification* of  $P$  at  $s$ . For a sequence  $\mathbf{a} : 0 \leq a_0 < \dots < a_n \leq d$ , write  $\Omega_{\mathbf{a}}^\circ(s)$  for the set of points  $P \in \mathbb{G}(n, d)$  with  $\mathbf{a}_P(s) = \mathbf{a}$ , which is a Schubert cell of  $\mathbb{G}(n, d)$ . It has codimension

$$|\mathbf{a}| := a_0 + a_1 - 1 + \dots + a_n - n,$$

as may be seen by expressing the basis  $f_0, \dots, f_n$  of  $P$  in terms of the basis  $\{(t-s)^i \mid i = 0, \dots, d\}$  of  $\mathbb{C}_d[t]$ . Since  $f_j^{(i)}$  vanishes to order at least  $a_j - i$  at  $s$  and  $f_i^{(i)}$  vanishes to order exactly  $a_i - i$  at  $s$ , the Wronskian of a subspace  $P \in \Omega_{\mathbf{a}}^\circ(s)$  vanishes to order exactly  $|\mathbf{a}|$  at  $s$ .

Let  $\mathbb{G}(n, d)^\circ$  be the dense open subset of  $\mathbb{G}(n, d)$  consisting of those  $P$  having a basis  $f_0, \dots, f_n$  where  $f_i$  has degree  $d-n+i$ . When  $P \in \mathbb{G}(n, d)^\circ$ , we obtain the Plücker

formula for the total ramification of a general subspace  $P$  of  $\mathbb{C}_d[t]$ ,

$$(1.3) \quad \dim \mathbb{G}(n, d) = \sum_{s \in \mathbb{C}} |\mathbf{a}_P(s)|.$$

In general, the total ramification of  $P$  is bounded by the dimension of  $\mathbb{G}(n, d)$ . (One may also define ramification at infinity for subspaces  $P \notin \mathbb{G}(n, d)^\circ$  to obtain the Plücker formula in its full generality.) If  $\mathbf{a}_P(s): 0 < 1 < \dots < n$ , so that  $|\mathbf{a}_P(s)| = 0$ , then  $P$  is *unramified* at  $s$ . Theorem 1 states that if a subspace  $P \in \mathbb{G}(n, d)$  is ramified only at real points, then  $P$  has a basis of real polynomials.

We introduce some more geometry. Let  $W = \prod_s (t - s)^{|\mathbf{a}_P(s)|}$  be the Wronskian of  $P$ . Then

$$P \in \bigcap_{s: W(s)=0} \Omega_{\mathbf{a}_P(s)}^\circ(s),$$

and this intersection consists of all subspaces with the same ramification as  $P$ . In particular,  $P$  lies in the intersection of the closures of these Schubert cells, which we now describe. For each  $s \in \mathbb{C}$ ,  $\mathbb{C}_d[t]$  has a complete flag of subspaces

$$F_\bullet(s) : \mathbb{C} \cdot (t-s)^d \subset \mathbb{C}_1[t] \cdot (t-s)^{d-1} \subset \dots \subset \mathbb{C}_{d-1}[t] \cdot (t-s) \subset \mathbb{C}_d[t].$$

More generally, a flag  $F_\bullet$  is a sequence of subspaces

$$F_\bullet : F_1 \subset F_2 \subset \dots \subset F_d \subset \mathbb{C}_d[t],$$

where  $F_i$  has dimension  $i$ . For a sequence  $\mathbf{a}$  and a flag  $F_\bullet$ , the *Schubert variety*

$$(1.4) \quad \{P \in \mathbb{G}(n, d) \mid \dim(P \cap F_{d+1-a_j}) \geq n+1-j, \text{ for } j = 0, 1, \dots, n\}$$

is a subvariety of  $\mathbb{G}(n, d)$ , written  $\Omega_{\mathbf{a}} F_\bullet$ . It consists of linear subspaces  $P$  having special position (encoded by  $\mathbf{a}$ ) with respect to the flag  $F_\bullet$ . Since  $\dim(P \cap F_{d+1-i}(s))$  counts the number of linearly independent polynomials in  $P$  that vanish to order at least  $i$  at  $s$ , we see that  $\Omega_{\mathbf{a}}^\circ(s) \subset \Omega_{\mathbf{a}} F_\bullet(s)$ . More precisely,  $\Omega_{\mathbf{a}} F_\bullet(s)$  is the closure of the Schubert cell  $\Omega_{\mathbf{a}}^\circ(s)$  and it is the disjoint union of cells  $\Omega_{\mathbf{b}}^\circ(s)$  for  $\mathbf{b} \geq \mathbf{a}$ , where  $\geq$  is componentwise comparison.

Given sequences  $\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(m)}$  and flags  $F_\bullet^{(1)}, \dots, F_\bullet^{(m)}$ , the intersection

$$(1.5) \quad \Omega_{\mathbf{a}^{(1)}} F_\bullet^{(1)} \bigcap \Omega_{\mathbf{a}^{(2)}} F_\bullet^{(2)} \bigcap \dots \bigcap \Omega_{\mathbf{a}^{(m)}} F_\bullet^{(m)}$$

consists of those linear subspaces  $P \in G$  having specified position  $\mathbf{a}^{(i)}$  with respect to the flag  $F_\bullet^{(i)}$ , for each  $i = 1, \dots, m$ . Kleiman [34] showed that if the flags  $F_\bullet^{(i)}$  are general, then the intersection (1.5) is (generically) transverse.

A *Schubert problem* is a list  $\mathbf{A} := (\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(m)})$  of sequences satisfying

$$|\mathbf{a}^{(1)}| + \dots + |\mathbf{a}^{(m)}| = (n+1)(d-n) (= \dim \mathbb{G}(n, d)).$$

Given a Schubert problem, Kleiman's Theorem implies that a general intersection (1.5) will be zero-dimensional and thus consist of finitely many points. By transversality, the number  $\delta(\mathbf{A})$  of these points is independent of choice of general flags. The Schubert calculus [35], through the Littlewood-Richardson rule [18], gives algorithms to determine  $\delta(\mathbf{A})$ .

We mention an important special case. Let  $\boldsymbol{\iota}: 0 < 1 < \dots < n-1 < n+1$  be the unique ramification sequence with  $|\boldsymbol{\iota}| = 1$ , and write  $\boldsymbol{\iota}_{n,d}$  for the Schubert problem in



which  $\iota$  occurs  $(n+1)(d-n)$  times. Schubert [54] gave the formula

$$(1.6) \quad \delta(\iota_{n,d}) = [(n+1)(d-n)]! \frac{1!2! \cdots n!}{(d-n)!(d-n+1)! \cdots d!}.$$

By the Plücker formula (1.3), the total ramification ( $\mathbf{a}_P(s): |\mathbf{a}_P(s)| > 0$ ) of a subspace  $P \in \mathbb{G}(n, d)^\circ$  is a Schubert problem. Let  $W$  be the Wronskian of  $P$ . We would like the intersection containing  $P$ ,

$$(1.7) \quad \bigcap_{s: W(s)=0} \Omega_{\mathbf{a}_P(s)} F_\bullet(s),$$

to be transverse and zero-dimensional. However, Kleiman's Theorem does not apply, as the flags  $F_\bullet(s)$  for  $s$  a root of  $W$  are not generic. For example, in the problem of four lines, if the Wronskian is  $t^4 - t$ , then the corresponding intersection (1.7) of Schubert varieties is not transverse. (This has been worked out in detail in [9, §9].)

We can see that this intersection (1.7) is always zero-dimensional. Note that any positive-dimensional subvariety meets  $\Omega_\iota F_\bullet$ , for any flag  $F_\bullet$ . (This is because, for example,  $\Omega_\iota F_\bullet$  is a hyperplane section of  $\mathbb{G}(n, d)$  in its Plücker embedding into projective space.) In particular, if the intersection (1.7) is not zero-dimensional, then given a point  $s \in \mathbb{P}^1$  with  $W(s) \neq 0$ , there will be a point  $P'$  in (1.7) which also lies in  $\Omega_\iota F_\bullet(s)$ . But then the total ramification of  $P'$  does not satisfy the Plücker formula (1.3), as its ramification strictly contains the total ramification of  $P$ .

A consequence of this argument is that the Wronski map (1.2) is a flat, finite map. In particular, it has finite fibers. The intersection number  $\delta(\iota_{n,d})$  in (1.6) is an upper bound for the cardinality of a fiber. By Sard's Theorem, this upper bound is obtained for generic Wronskians. An argument that proves this in somewhat greater generality was given by Eisenbud and Harris [9].

**Theorem 1.8.** *There are finitely many spaces of polynomials  $P \in \mathbb{G}(n, d)$  with a given Wronskian. For a general polynomial  $W(t)$  of degree  $(n+1)(d-n)$ , there are exactly  $\delta(\iota_{n,d})$  spaces of polynomials with Wronskian  $W(t)$ .*

When  $W$  has distinct roots, these spaces of polynomials are exactly the points in the intersection (1.7), where  $\mathbf{a}_P(s) = \iota$  at each root  $s$  of  $W$ . A limiting argument, in which the roots of the Wronskian are allowed to collide one-by-one, proves a local form of Theorem 1. We say that the roots  $\mathbf{s} = s_1, \dots, s_{(n+1)(d-n)}$  of the Wronskian are *clustered* if, up to an automorphism of  $\mathbb{R}\mathbb{P}^1$ , they satisfy

$$(1.9) \quad 0 < s_1 \ll s_2 \ll \cdots \ll s_{(n+1)(d-n)}.$$

**Theorem 1.10** ([61]). *If the roots of a polynomial  $W(t)$  of degree  $(n+1)(d-n)$  are real, distinct, and clustered, then there are  $\delta(\iota_{n,d})$  real spaces of polynomials with Wronskian  $W(t)$  and the intersection (1.7) is transverse.*

We noted that the intersection (1.7) is not transverse when  $d = 3$ ,  $n = 1$ , and  $W(t) = t^4 - t$ . It turns out that it is always transverse when the roots of the Wronskian are distinct and real. This is the stronger form of the Theorem of Mukhin, Tarasov, and Varchenko, proven in [45].

**Theorem 1.11.** *For any Schubert problem  $\mathbf{A} = (\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(m)})$  and any distinct real numbers  $s_1, \dots, s_m$ , the intersection*

$$(1.12) \quad \Omega_{\mathbf{a}^{(1)}} F_\bullet(s_1) \bigcap \Omega_{\mathbf{a}^{(2)}} F_\bullet(s_2) \bigcap \cdots \bigcap \Omega_{\mathbf{a}^{(m)}} F_\bullet(s_m)$$

*is transverse and consists solely of real points.*

This theorem (without the transversality) is the original statement of the conjecture of Boris Shapiro and Michael Shapiro for Grassmannians, which was posed in exactly this form to the author in May 1995. The Shapiro conjecture was first discussed and studied in detail in [62], where significant computational evidence was presented (see also [67] and [50]). These results and computations, as well as Theorem 1.10, highlighted the key role that transversality plays in the conjecture. Apparently, this Shapiro conjecture was in part an attempt to propose a reason for the results in the thesis [59] which showed that for  $\mathbb{G}(1, d)$ , there are choices of real flags  $F_{\bullet}^i$  in (1.12) so that the intersection is transverse with all points real. This was extended to all problems in the special Schubert calculus on all Grassmannians [61]. Later, Vakil [66] showed that this was true for all Schubert problems on all Grassmannians.

The main ingredient in the proof of Theorem 1.11 is an isomorphism between algebraic objects associated to the intersection (1.12) and to certain representation-theoretic data. This isomorphism provides a very deep link between Schubert calculus for the Grassmannian and the representation theory of  $\mathfrak{sl}_{n+1}\mathbb{C}$ .

We sketch the proof of Theorem 1 in the next three sections.

## 2. SPACES OF POLYNOMIALS WITH GIVEN WRONSKIAN

Theorem 1.8 enables the reduction of Theorem 1 to a special case. Since the Wronski map is finite, a standard limiting argument (given, for example, in Section 1.3 of [40] or Remark 3.4 of [62]) shows that it suffices to prove Theorem 1 when the Wronskian has distinct real roots that are sufficiently general. Since  $\delta(\boldsymbol{\iota}_{n,d})$  is the upper bound for the number of spaces of polynomials with a given Wronskian, it suffices to construct this number of distinct spaces of real polynomials with a given Wronskian, when the Wronskian has distinct real roots that are sufficiently general. In fact, this is exactly what Mukhin, Tarasov, and Varchenko do [40].

**Theorem 1'.** *If  $s_1, \dots, s_{(n+1)(d-n)}$  are generic real numbers, there are  $\delta(\boldsymbol{\iota}_{n,d})$  distinct real vector spaces of polynomials  $P$  with Wronskian  $\prod_i (t - s_i)$ .*

The proof first constructs  $\delta(\boldsymbol{\iota}_{n,d})$  distinct spaces of polynomials with a given Wronskian having generic complex roots, which we describe in Section 2.1. This uses a Fuchsian differential equation given by the critical points of a remarkable symmetric function, called the master function. The next step uses the Bethe ansatz in a certain representation  $V$  of  $\mathfrak{sl}_{n+1}\mathbb{C}$ : each critical point of the master function gives a Bethe eigenvector of the Gaudin Hamiltonians which turns out to be a highest weight vector for an irreducible submodule of  $V$ . This is described in Section 3, where the eigenvalues of the Gaudin Hamiltonians on a Bethe vector are shown to be the coefficients of the Fuchsian differential equation giving the corresponding spaces of polynomials. This is the germ of the new, deep connection between representation theory and Schubert calculus that led to Theorem 1.11. Finally, the Gaudin Hamiltonians are real symmetric operators when the Wronskian has only real roots, so their eigenvalues are real, and thus the Fuchsian differential equation has real coefficients and the corresponding space of polynomials is also real. Figure 2 presents a schematic of this extraordinary proof.



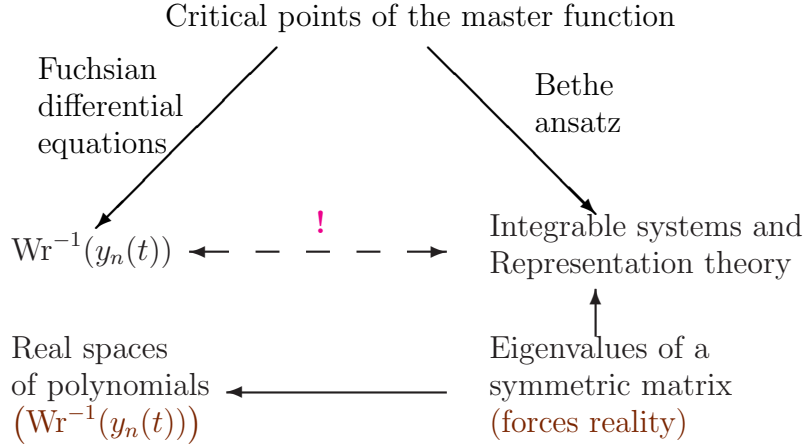


FIGURE 2. Schematic of proof of Shapiro conjecture.

**2.1. Critical points of master functions.** The construction of  $\delta(\mathfrak{t}_{n,d})$  spaces of polynomials with a given Wronskian begins with the critical points of a symmetric rational function that arose in the study of hypergeometric solutions to the Knizhnik-Zamolodchikov equations [52] and the Bethe ansatz method for the Gaudin model.

The master function depends upon parameters  $\mathbf{s} := (s_1, \dots, s_{(n+1)(d-n)})$ , which are the roots of our Wronskian  $W$ , and an additional  $\binom{n+1}{2}(d-n)$  variables

$$\mathbf{x} := (x_1^{(1)}, \dots, x_{d-n}^{(1)}, x_1^{(2)}, \dots, x_{2(d-n)}^{(2)}, \dots, x_1^{(n)}, \dots, x_{n(d-n)}^{(n)}).$$

Each set of variables  $\mathbf{x}^{(i)} := (x_1^{(i)}, \dots, x_{i(d-n)}^{(i)})$  will turn out to be the roots of certain intermediate Wronskians.

Define the *master function*  $\Phi(\mathbf{x}; \mathbf{s})$  by the (rather formidable) formula

$$(2.1) \quad \frac{\prod_{i=1}^n \prod_{1 \leq j < k \leq i(d-n)} (x_j^{(i)} - x_k^{(i)})^2 \cdot \prod_{1 \leq j < k < (n+1)(d-n)} (s_j - s_k)^2}{\prod_{i=1}^{n-1} \prod_{j=1}^{i(d-n)} \prod_{k=1}^{(i+1)(d-n)} (x_j^{(i)} - x_k^{(i+1)}) \cdot \prod_{j=1}^{n(d-n)} \prod_{k=1}^{(n+1)(d-n)} (x_j^{(n)} - s_k)}.$$

This is separately symmetric in each set of variables  $\mathbf{x}^{(i)}$ . The Cartan matrix for  $\mathfrak{sl}_{n+1}$  appears in the exponents of the factors  $(x_*^{(i)} - x_*^{(j)})$  in (2.1). This hints at the relation of these master functions to Lie theory, which we do not discuss.

The critical points of the master function are solutions to the system of equations

$$(2.2) \quad \frac{1}{\Phi} \frac{\partial}{\partial x_j^{(i)}} \Phi(\mathbf{x}; \mathbf{s}) = 0 \quad \text{for } i = 1, \dots, n, \quad j = 1, \dots, i(d-n).$$

When the parameters  $\mathbf{s}$  are generic, these *Bethe ansatz equations* turn out to have finitely many solutions. The master function is invariant under the group

$$\mathcal{S} := \mathcal{S}_{d-n} \times \mathcal{S}_{2(d-n)} \times \dots \times \mathcal{S}_{n(d-n)},$$

where  $\mathcal{S}_m$  is the group of permutations of  $\{1, \dots, m\}$ , and the factor  $\mathcal{S}_{i(d-n)}$  permutes the variables in  $\mathbf{x}^{(i)}$ . Thus  $\mathcal{S}$  acts on the critical points. The invariants of this action are polynomials whose roots are the coordinates of the critical points.

Given a critical point  $\mathbf{x}$ , define monic polynomials  $\mathbf{p}_\mathbf{x} := (p_1, \dots, p_n)$  where the components  $\mathbf{x}^{(i)}$  of  $\mathbf{x}$  are the roots of  $p_i$ ,

$$(2.3) \quad p_i := \prod_{j=1}^{i(d-n)} (t - x_j^{(i)}) \quad \text{for } i = 1, \dots, n.$$

Also write  $p_{n+1}$  for the Wronskian, the monic polynomial with roots  $\mathbf{s}$ . The discriminant  $\text{Discr}(f)$  of a polynomial  $f$  is the square of the product of differences of its roots and the resultant  $\text{Res}(f, g)$  is the product of all differences of the roots of  $f$  and  $g$  [8]. Then the formula for the master function (2.1) becomes

$$(2.4) \quad \Phi(\mathbf{x}; \mathbf{s}) = \prod_{i=1}^{n+1} \text{Discr}(p_i) \Big/ \prod_{i=1}^n \text{Res}(p_i, p_{i+1}).$$

The connection between the critical points of  $\Phi(\mathbf{x}; \mathbf{s})$  and spaces of polynomials with Wronskian  $W$  is through a Fuchsian differential equation. Given (an orbit of) a critical point  $\mathbf{x}$  represented by the list of polynomials  $\mathbf{p}_\mathbf{x}$ , define the *fundamental differential operator*  $D_\mathbf{x}$  of the critical point  $\mathbf{x}$  by

$$(2.5) \quad \left( \frac{d}{dt} - \ln' \left( \frac{W}{p_n} \right) \right) \cdots \left( \frac{d}{dt} - \ln' \left( \frac{p_2}{p_1} \right) \right) \left( \frac{d}{dt} - \ln'(p_1) \right),$$

where  $\ln'(f) := \frac{d}{dt} \ln f$ . The kernel  $V_\mathbf{x}$  of  $D_\mathbf{x}$  is the *fundamental space of the critical point*  $\mathbf{x}$ .

**Example 2.6.** Since

$$\left( \frac{d}{dt} - \ln'(p) \right) p = \left( \frac{d}{dt} - \frac{p'}{p} \right) p = p' - \frac{p'}{p} p = 0,$$

we see that  $p_1$  is a solution of  $D_\mathbf{x}$ . It is instructive to look at  $D_\mathbf{x}$  and  $V_\mathbf{x}$  when  $n = 1$ . Suppose that  $f$  a solution to  $D_\mathbf{x}$  that is linearly independent from  $p_1$ . Then

$$0 = \left( \frac{d}{dt} - \ln' \left( \frac{W}{p_1} \right) \right) \left( \frac{d}{dt} - \ln'(p_1) \right) f = \left( \frac{d}{dt} - \ln' \left( \frac{W}{p_1} \right) \right) \left( f' - \frac{p_1'}{p_1} f \right).$$

This implies that

$$\frac{W}{p_1} = f' - \frac{p_1'}{p_1} f,$$

so  $W = \text{Wr}(f, p_1)$ , and the kernel of  $D_\mathbf{x}$  is a 2-dimensional space of functions with Wronskian  $W$ .

What we just saw is always the case. The following result is due to Scherbak and Varchenko [53] for  $n = 1$  and to Mukhin and Varchenko [47, §5] for all  $n$ .

**Theorem 2.7.** *Suppose that  $V_\mathbf{x}$  is the fundamental space of a critical point  $\mathbf{x}$  of the master function  $\Phi$  with generic parameters  $\mathbf{s}$  which are the roots of  $W$ .*

- (1) *Then  $V_\mathbf{x}$  is an  $(n+1)$ -dimensional space of polynomials of degree  $d$  lying in  $\mathbb{G}(n, d)^\circ$  with Wronskian  $W$ .*
- (2) *The critical point  $\mathbf{x}$  is recovered from  $V_\mathbf{x}$  in some cases as follows. Suppose that  $f_0, \dots, f_n$  are monic polynomials in  $V_\mathbf{x}$  with  $\deg f_i = d - n + i$ , each  $f_i$  is square-free, and that the pairs  $f_i$  and  $f_{i+1}$  are relatively prime. Then, up to scalar multiples, the polynomials  $p_1, \dots, p_n$  in the sequence  $\mathbf{p}_\mathbf{x}$  are*

$$f_0, \text{Wr}(f_0, f_1), \text{Wr}(f_0, f_1, f_2), \dots, \text{Wr}(f_0, \dots, f_{n-1}).$$

Statement (2) includes a general result about factoring a linear differential operator into differential operators of degree 1. Linearly independent  $C^\infty$  functions  $f_0, \dots, f_n$  span the kernel of the differential operator of degree  $n+1$ ,

$$\det \begin{pmatrix} f_0 & f_1 & \cdots & f_n & 1 \\ f'_0 & f'_1 & \cdots & f'_n & \frac{d}{dt} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_0^{(n+1)} & f_1^{(n+1)} & \cdots & f_n^{(n+1)} & \frac{d^{n+1}}{dt^{n+1}} \end{pmatrix}.$$

If we set  $p_{i+1} := \text{Wr}(f_0, \dots, f_i)$ , then (2.5) is a factorization over  $\mathbb{C}(t)$  of this determinant into differential operators of degree 1. This follows from some interesting identities among Wronskians shown in the Appendix of [47].

Theorem 2.7 is deeper than this curious fact. When the polynomials  $p_1, \dots, p_n, W$  are square-free, consecutive pairs are relatively prime, and  $\mathbf{s}$  is generic, it implies that the kernel  $V$  of an operator of the form (2.5) is a space of polynomials with Wronskian  $W$  having roots  $\mathbf{s}$  if and only if the polynomials  $p_1, \dots, p_n$  come from the critical points of the master function (2.1) corresponding to  $W$ .

This gives an injection from  $\mathcal{S}$ -orbits of critical points of the master function  $\Phi$  with parameters  $\mathbf{s}$  to spaces of polynomials in  $\mathbb{G}(n, d)^\circ$  whose Wronskian has roots  $\mathbf{s}$ . Mukhin and Varchenko showed that this is a bijection when  $\mathbf{s}$  is generic.

**Theorem 2.8** (Theorem 6.1 in [48]). *For generic complex numbers  $\mathbf{s}$ , the master function  $\Phi$  has nondegenerate critical points that form  $\delta(\mathbf{t}_{n,d})$  distinct orbits.*

The structure (but not of course the details) of their proof is remarkably similar to the structure of the proof of Theorem 1.10; they allow the parameters to collide one-by-one, and study how the orbits of critical points behave. Ultimately, they obtain the same recursion as in [61], which mimics the Pieri formula for the branching rule for tensor products of representations of  $\mathfrak{sl}_{n+1}$  with its fundamental representation  $V_{\omega_n}$ . This same structure is also found in the main argument in [11]. In fact, this is the same recursion in  $\mathbf{a}$  that Schubert established for intersection numbers  $\delta(\mathbf{a}, \mathbf{t}, \dots, \mathbf{t})$ , and then solved to obtain the formula (1.6) in [54].

### 3. THE BETHE ANSATZ FOR THE GAUDIN MODEL

The Bethe ansatz is a general (conjectural) method to find pure states, called *Bethe vectors*, of quantum integrable systems. The (periodic) Gaudin model is an integrable system consisting of a family of commuting operators called the Gaudin Hamiltonians that act on a representation  $V$  of  $\mathfrak{sl}_{n+1}\mathbb{C}$ . In this Bethe ansatz, a vector-valued rational function is constructed so that for certain values of the parameters it yields a complete set of Bethe vectors. As the Gaudin Hamiltonians commute with the action of  $\mathfrak{sl}_{n+1}\mathbb{C}$ , the Bethe vectors turn out to be highest weight vectors generating irreducible submodules of  $V$ , and so this also gives a method for decomposing some representations  $V$  of  $\mathfrak{sl}_{n+1}\mathbb{C}$  into irreducible submodules. The development, justification, and refinements of this Bethe ansatz are the subject of a large body of work, a small part of which we mention.

**3.1. Representations of  $\mathfrak{sl}_{n+1}\mathbb{C}$ .** The Lie algebra  $\mathfrak{sl}_{n+1}\mathbb{C}$  (or simply  $\mathfrak{sl}_{n+1}$ ) is the space of  $(n+1) \times (n+1)$ -matrices with trace zero. It has a decomposition

$$\mathfrak{sl}_{n+1} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+,$$

where  $\mathfrak{n}_+$  ( $\mathfrak{n}_-$ ) are the strictly upper (lower) triangular matrices, and  $\mathfrak{h}$  consists of the diagonal matrices with zero trace. The universal enveloping algebra  $U\mathfrak{sl}_{n+1}$  of  $\mathfrak{sl}_{n+1}$  is the associative algebra generated by  $\mathfrak{sl}_{n+1}$  subject to the relations  $uv - vu = [u, v]$  for  $u, v \in \mathfrak{sl}_{n+1}$ , where  $[u, v]$  is the Lie bracket in  $\mathfrak{sl}_{n+1}$ .

We consider only finite-dimensional representations of  $\mathfrak{sl}_{n+1}$  (equivalently, of  $U\mathfrak{sl}_{n+1}$ ). For a more complete treatment, see [19]. Any representation  $V$  of  $\mathfrak{sl}_{n+1}$  decomposes into joint eigenspaces of  $\mathfrak{h}$ , called *weight spaces*,

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V[\mu],$$

where, for  $v \in V[\mu]$  and  $h \in \mathfrak{h}$ , we have  $h.v = \mu(h)v$ . The possible weights  $\mu$  of representations lie in the integral *weight lattice*. This has a distinguished basis of *fundamental weights*  $\omega_1, \dots, \omega_n$  that generate the cone of *dominant weights*.

An irreducible representation  $V$  has a unique one-dimensional weight space that is annihilated by the nilpotent subalgebra  $\mathfrak{n}_+$  of  $\mathfrak{sl}_{n+1}$ . The associated weight  $\mu$  is dominant, and it is called the *highest weight* of  $V$ . Any nonzero vector with this weight is a highest weight vector of  $V$ , and it generates  $V$ . Furthermore, any two irreducible modules with the same highest weight are isomorphic. Write  $V_\mu$  for the *highest weight module* with highest weight  $\mu$ . Lastly, there is one highest weight module for each dominant weight.

More generally, if  $V$  is any representation of  $\mathfrak{sl}_{n+1}$  and  $\mu$  is a weight, then the *singular vectors* in  $V$  of weight  $\mu$ , written  $\text{sing}(V[\mu])$ , are the vectors in  $V[\mu]$  annihilated by  $\mathfrak{n}_+$ . If  $v \in \text{sing}(V[\mu])$  is nonzero, then the submodule  $U\mathfrak{sl}_{n+1}.v$  it generates is isomorphic to the highest weight module  $V_\mu$ . Thus  $V$  decomposes as a direct sum of submodules generated by the singular vectors,

$$(3.1) \quad V = \bigoplus_{\mu} U\mathfrak{sl}_{n+1}.\text{sing}(V[\mu]),$$

so that the multiplicity of the highest weight module  $V_\mu$  in  $V$  is simply the dimension of its space of singular vectors of weight  $\mu$ .

When  $V$  is a tensor product of highest weight modules, the Littlewood-Richardson rule [18] gives formulas for the dimensions of the spaces of singular vectors. Since this is the same rule for the number of points in an intersection (1.5) of Schubert varieties from a Schubert problem, these geometric intersection numbers are equal to the dimensions of spaces of singular vectors. In particular, if  $V_{\omega_1} \simeq \mathbb{C}^{n+1}$  is the defining representation of  $\mathfrak{sl}_{n+1}$  and  $V_{\omega_n} = \bigwedge^n V_{\omega_1} = V_{\omega_1}^*$  (these are the first and last fundamental representations of  $\mathfrak{sl}_{n+1}$ ), then

$$(3.2) \quad \dim \text{sing}(V_{\omega_n}^{\otimes(n+1)(d-n)}[0]) = \delta(\mathbf{l}_{n,d}).$$

It is important to note that this equality of numbers is purely formal, in that the same formula governs both numbers. A direct connection remains to be found.

**3.2. The (periodic) Gaudin model.** The Bethe ansatz is a conjectural method to obtain a complete set of eigenvectors for the integrable system on  $V := V_{\omega_n}^{\otimes m}$  given by the Gaudin Hamiltonians (defined below). Since these Gaudin Hamiltonians commute with  $\mathfrak{sl}_{n+1}$ , the Bethe ansatz has the additional benefit of giving an explicit basis for  $\text{sing}(V[\mu])$ , thus explicitly giving the decomposition (3.1).

The Gaudin Hamiltonians act on  $V_{\omega_n}^{\otimes m}$  and depend upon  $m$  distinct complex numbers  $s_1, \dots, s_m$  and a complex variable  $t$ . Let  $\mathfrak{gl}_{n+1}$  be the Lie algebra of  $(n+1) \times (n+1)$  complex matrices. For each  $i, j = 1, \dots, n+1$ , let  $E_{i,j} \in \mathfrak{gl}_{n+1}$  be the matrix whose only nonzero entry is a 1 in row  $i$  and column  $j$ . For each pair  $(i, j)$  consider the differential operator  $X_{i,j}(t)$  acting on  $V_{\omega_n}^{\otimes m}$ -valued functions of  $t$ ,

$$X_{i,j}(t) := \delta_{i,j} \frac{d}{dt} - \sum_{k=1}^m \frac{E_{j,i}^{(k)}}{t - s_k},$$

where  $E_{j,i}^{(k)}$  acts on tensors in  $V_{\omega_n}^{\otimes m}$  by  $E_{j,i}$  in the  $k$ th factor and by the identity in other factors. Define a differential operator acting on  $V_{\omega_n}^{\otimes m}$ -valued functions of  $t$ ,

$$\mathbf{M} := \sum_{\sigma \in \mathcal{S}} \text{sgn}(\sigma) X_{1,\sigma(1)}(t) X_{2,\sigma(2)}(t) \cdots X_{n+1,\sigma(n+1)}(t),$$

where  $\mathcal{S}$  is the group of permutations of  $\{1, \dots, n+1\}$  and  $\text{sgn}(\sigma) = \pm 1$  is the sign of a permutation  $\sigma \in \mathcal{S}$ . Write  $\mathbf{M}$  in standard form

$$\mathbf{M} = \frac{d^{n+1}}{dt^{n+1}} + M_1(t) \frac{d^n}{dt^n} + \cdots + M_{n+1}(t).$$

These coefficients  $M_1(t), \dots, M_{n+1}(t)$  are called the (higher) *Gaudin Hamiltonians*. They are linear operators that depend rationally on  $t$  and act on  $V_{\omega_n}^{\otimes m}$ . We collect together some of their properties.

**Theorem 3.3.** *Suppose that  $s_1, \dots, s_m$  are distinct complex numbers. Then*

- (1) *The Gaudin Hamiltonians commute, that is,  $[M_i(u), M_j(v)] = 0$  for all  $i, j = 1, \dots, n+1$  and  $u, v \in \mathbb{C}$ .*
- (2) *The Gaudin Hamiltonians commute with the action of  $\mathfrak{sl}_{n+1}$  on  $V_{\omega_n}^{\otimes m}$ .*

Proofs are given in [38], as well as Propositions 7.2 and 8.3 in [41], and are based on results of Talalaev [65]. A consequence of the second assertion is that the Gaudin Hamiltonians preserve the weight space decomposition of the singular vectors of  $V_{\omega_n}^{\otimes m}$ . Since they commute, the singular vectors of  $V_{\omega_n}^{\otimes m}$  have a basis of common eigenvectors of the Gaudin Hamiltonians. The Bethe ansatz is a method to write down joint eigenvectors and their eigenvalues.

**3.3. The Bethe ansatz for the Gaudin model.** This begins with a rational function that takes values in a weight space  $V_{\omega_n}^{\otimes m}[\mu]$ ,

$$v : \mathbb{C}^l \times \mathbb{C}^m \mapsto V_{\omega_n}^{\otimes m}[\mu].$$

This *universal weight function* was introduced in [52] to solve the Knizhnik-Zamolodchikov equations with values in  $V_{\omega_n}^{\otimes m}[\mu]$ . When  $(\mathbf{x}, \mathbf{s})$  is a critical point of a master function, the vector  $v(\mathbf{x}, \mathbf{s})$  is both singular and an eigenvector of the Gaudin Hamiltonians. (This master function is a generalization of the one defined by (2.1).) The Bethe ansatz conjecture for the periodic Gaudin model asserts that the vectors  $v(\mathbf{x}, \mathbf{s})$  form a basis for the space of singular vectors.

Fix a highest weight vector  $v_{n+1} \in V_{\omega_n}[\omega_n]$ . Then  $v_{n+1}^{\otimes m}$  generates  $V_{\omega_n}^{\otimes m}$  as a  $U\mathfrak{sl}_{n+1}^{\otimes m}$ -module. In particular, any vector in  $V_{\omega_n}^{\otimes m}$  is a linear combination of vectors that are obtained from  $v_{n+1}^{\otimes m}$  by applying a sequence of operators  $E_{i+1,i}^{(k)}$ , for  $1 \leq k \leq m$  and  $1 \leq i \leq n$ . The universal weight function is a linear combination of such vectors of weight  $\mu$ .

When  $m = (n+1)(d-n)$ ,  $l = \binom{n+1}{2}(d-n)$ , and  $\mu = 0$ , the universal weight function is a map

$$v : \mathbb{C}^{\binom{n+1}{2}(d-n)} \times \mathbb{C}^{(n+1)(d-n)} \longrightarrow V_{\omega_n}^{\otimes (n+1)(d-n)}[0].$$

To describe it, note that a vector  $E_{a+1,a}E_{b+1,b} \cdots E_{c+1,c}.v_{n+1}$  is nonzero only if

$$(a, b, \dots, c) = (a, a+1, \dots, n-1, n).$$

Write  $v_a$  for this vector. The vectors  $v_1, \dots, v_{n+1}$  form a basis of  $V_{\omega_n}$ . Thus only some sequences of operators  $E_{i+1,i}^{(k)}$  applied to  $v_{n+1}^{\otimes (n+1)(d-n)}$  give a nonzero vector. These sequences are completely determined once we know the weight of the result. The operator  $E_{i+1,i}^{(k)}$  lowers the weight of a weight vector by the root  $\alpha_i$ . Since

$$(3.4) \quad (n+1)\omega_n = \alpha_1 + 2\alpha_2 + \cdots + n\alpha_n,$$

there are  $i(d-n)$  occurrences of  $E_{i+1,i}^{(k)}$ , which is the number of variables in  $\mathbf{x}^{(i)}$ .

Let  $\mathcal{B}$  be the set of all sequences  $(b_1, b_2, \dots, b_{(n+1)(d-n)})$ , where  $1 \leq b_k \leq n+1$  for each  $k$ , and we have

$$\#\{k \mid b_k \leq i\} = i(d-n).$$

Given a sequence  $B$  in  $\mathcal{B}$ , define

$$\begin{aligned} v_B &:= v_{b_1} \otimes v_{b_2} \otimes \cdots \otimes v_{b_{(n+1)(d-n)}} \\ &= \bigotimes_{k=1}^{(n+1)(d-n)} (E_{b_k+1,b_k}^{(k)} \cdots E_{n,n-1}^{(k)} \cdot E_{n+1,n}^{(k)}).v_{n+1}, \end{aligned}$$

where the operator  $E_{b_k+1,b_k}^{(k)} \cdots E_{n,n-1}^{(k)} \cdot E_{n+1,n}^{(k)}$  is the identity if  $b_k = n+1$ . Then  $v_B$  is a vector of weight 0, by (3.4). The universal weight function is a linear combination of these vectors  $v_B$ ,

$$v(\mathbf{x}; \mathbf{s}) = \sum_{B \in \mathcal{B}} w_B(\mathbf{x}; \mathbf{s}) \cdot v_B,$$

where  $w_B(\mathbf{x}, \mathbf{s})$  is separately symmetric in each set of variables  $\mathbf{x}^{(i)}$ .

To describe  $w_B(\mathbf{x}; \mathbf{s})$ , suppose that

$$\mathbf{z} = (\mathbf{z}^{(1)}, \mathbf{z}^{(2)}, \dots, \mathbf{z}^{((n+1)(d-n))})$$

is a partition of the variables  $\mathbf{x}$  into  $(n+1)(d-n)$  sets of variables where the  $k$ th set  $\mathbf{z}^{(k)}$  of variables has exactly one variable from each set  $\mathbf{x}^{(i)}$  with  $b_k \leq i$  (and is empty when  $b_k = n+1$ ). That is, if  $b_k \leq n$ , then

$$(3.5) \quad \mathbf{z}^{(k)} = (x_{c_{b_k}}^{(b_k)}, x_{c_{b_k+1}}^{(b_k+1)}, \dots, x_{c_n}^{(n)}),$$

for some indices  $c_{b_k}, \dots, c_n$ . If  $b_k = n+1$ , set  $w^{(k)}(\mathbf{z}) := 1$ , and otherwise

$$w^{(k)}(\mathbf{z}; \mathbf{s}) := \frac{1}{x_{c_{b_k}}^{(b_k)} - x_{c_{b_k+1}}^{(b_k+1)}} \cdots \frac{1}{x_{c_{n-1}}^{(n-1)} - x_{c_n}^{(n)}} \cdot \frac{1}{x_{c_n}^{(n)} - s_k},$$

in the notation (3.5). Then we set

$$w(\mathbf{z}; \mathbf{s}) := \prod_{k=1}^{(n+1)(d-n)} w^{(k)}(\mathbf{z}; \mathbf{s}).$$

Finally,  $w_B(\mathbf{x}; \mathbf{s})$  is the sum of the rational functions  $w(\mathbf{z}; \mathbf{s})$  over all such partitions  $\mathbf{z}$  of the variables  $\mathbf{x}$ . (Equivalently, the symmetrization of any single  $w(\mathbf{z}; \mathbf{s})$ .)

While  $v(\mathbf{x}, \mathbf{s})$  is a rational function of  $\mathbf{x}$  and hence not globally defined, if the coordinates of  $\mathbf{s}$  are distinct and  $\mathbf{x}$  is a critical point of the master function (2.1), then the vector  $v(\mathbf{x}, \mathbf{s}) \in V_{\omega_n}^{\otimes(n+1)(d-n)}[0]$  is well-defined, nonzero and it is in fact a singular vector (Lemma 2.1 of [48]). Such a vector  $v(\mathbf{x}, \mathbf{s})$  when  $\mathbf{x}$  is a critical point of the master function is called a *Bethe vector*. Mukhin and Varchenko also prove the following, which is the second part of Theorem 6.1 in [48].

**Theorem 3.6.** *When  $\mathbf{s} \in \mathbb{C}^{(n+1)(d-n)}$  is general, the Bethe vectors form a basis of the space  $\text{sing}(V_{\omega_n}^{\otimes(n+1)(d-n)}[0])$ .*

These Bethe vectors are the joint eigenvectors of the Gaudin Hamiltonians.

**Theorem 3.7** (Theorem 9.2 in [41]). *For any critical point  $\mathbf{x}$  of the master function (2.1), the Bethe vector  $v(\mathbf{x}, \mathbf{s})$  is a joint eigenvector of the Gaudin Hamiltonians  $M_1(t), \dots, M_{n+1}(t)$ . Its eigenvalues  $\mu_1(t), \dots, \mu_{n+1}(t)$  are given by the formula*

$$(3.8) \quad \frac{d^{n+1}}{dt^{n+1}} + \mu_1(t) \frac{d^n}{dt^n} + \dots + \mu_n(t) \frac{d}{dt} + \mu_{n+1}(t) = \left( \frac{d}{dt} + \ln'(p_1) \right) \left( \frac{d}{dt} + \ln' \left( \frac{p_2}{p_1} \right) \right) \dots \left( \frac{d}{dt} + \ln' \left( \frac{p_n}{p_{n-1}} \right) \right) \left( \frac{d}{dt} + \ln' \left( \frac{W}{p_n} \right) \right),$$

where  $p_1(t), \dots, p_n(t)$  are the polynomials (2.3) associated to the critical point  $\mathbf{x}$  and  $W(t)$  is the polynomial with roots  $\mathbf{s}$ .

Observe that (3.8) is similar to the formula (2.5) for the differential operator  $D_{\mathbf{x}}$  of the critical point  $\mathbf{x}$ . This similarity is made more precise if we replace the Gaudin Hamiltonians by a different set of operators. Consider the differential operator formally conjugate to  $(-1)^{n+1}M$ ,

$$\begin{aligned} K &= \frac{d^{n+1}}{dt^{n+1}} - \frac{d^n}{dt^n} M_1(t) + \dots + (-1)^n \frac{d}{dt} M_n(t) + (-1)^{n+1} M_{n+1}(t) \\ &= \frac{d^{n+1}}{dt^{n+1}} + K_1(t) \frac{d^n}{dt^n} + \dots + K_n(t) \frac{d}{dt} + K_{n+1}(t). \end{aligned}$$

These coefficients  $K_i(t)$  are operators on  $V_{\omega_n}^{\otimes(n+1)(d-n)}$  that depend rationally on  $t$ , and are also called the Gaudin Hamiltonians. Here are the first three,

$$\begin{aligned} K_1(t) &= -M_1(t), & K_2(t) &= M_2(t) - nM_1'(t), \\ K_3(t) &= -M_3(t) + (n-1)M_2''(t) - \binom{n}{2}M_1'''(t), \end{aligned}$$

and in general  $K_i(t)$  is a differential polynomial in  $M_1(t), \dots, M_i(t)$ .

Like the  $M_i(t)$ , these operators commute with each other and with  $\mathfrak{sl}_{n+1}$ , and the Bethe vector  $v(\mathbf{x}, \mathbf{s})$  is a joint eigenvector of these new Gaudin Hamiltonians  $K_i(t)$ . The corresponding eigenvalues  $\lambda_1(t), \dots, \lambda_{n+1}(t)$  are given by the formula

$$(3.9) \quad \frac{d^{n+1}}{dt^{n+1}} + \lambda_1(t) \frac{d^n}{dt^n} + \dots + \lambda_n(t) \frac{d}{dt} + \lambda_{n+1}(t) = \left( \frac{d}{dt} - \ln' \left( \frac{W}{p_n} \right) \right) \left( \frac{d}{dt} - \ln' \left( \frac{p_n}{p_{n-1}} \right) \right) \dots \left( \frac{d}{dt} - \ln' \left( \frac{p_2}{p_1} \right) \right) \left( \frac{d}{dt} - \ln'(p_0) \right),$$

which is (!) the fundamental differential operator  $D_{\mathbf{x}}$  of the critical point  $\mathbf{x}$ .

**Corollary 3.10.** *Suppose that  $\mathbf{s} \in \mathbb{C}^{(n+1)(d-n)}$  is generic.*



- (1) *The Bethe vectors form an eigenbasis of  $\text{sing}(V_{\omega_n}^{\otimes(n+1)(d-n)}[0])$  for the Gaudin Hamiltonians  $K_1(t), \dots, K_{n+1}(t)$ .*
- (2) *The Gaudin Hamiltonians  $K_1(t), \dots, K_{n+1}(t)$  have simple spectrum in that their eigenvalues separate the basis of eigenvectors.*

Statement (1) follows from Theorems 3.6 and 3.7. For Statement (2), suppose that two Bethe vectors  $v(\mathbf{x}, \mathbf{s})$  and  $v(\mathbf{x}', \mathbf{s})$  have the same eigenvalues. By (3.9), the corresponding fundamental differential operators would be equal,  $D_{\mathbf{x}} = D_{\mathbf{x}'}$ . But this implies that the fundamental spaces coincide,  $V_{\mathbf{x}} = V_{\mathbf{x}'}$ . By Theorem 2.7 the fundamental space determines the orbit of critical points, so the critical points  $\mathbf{x}$  and  $\mathbf{x}'$  lie in the same orbit, which implies that  $v(\mathbf{x}, \mathbf{s}) = v(\mathbf{x}', \mathbf{s})$ .

#### 4. SHAPOVALOV FORM AND THE PROOF OF THE SHAPIRO CONJECTURE

The last step in the proof of Theorem 1 is to show that if  $\mathbf{s} \in \mathbb{R}^{(n+1)(d-n)}$  is generic and  $\mathbf{x}$  is a critical point of the master function (2.1), then the fundamental space  $V_{\mathbf{x}}$  of the critical point  $\mathbf{x}$  has a basis of real polynomials. The reason for this reality is that the eigenvectors and eigenvalues of a symmetric matrix are real.

We begin with the Shapovalov form. The map  $\tau: E_{i,j} \mapsto E_{j,i}$  induces an antiautomorphism on  $\mathfrak{sl}_{n+1}$ . Given a highest weight module  $V_{\mu}$ , and a highest weight vector  $v \in V_{\mu}[\mu]$  the *Shapovalov form*  $\langle \cdot, \cdot \rangle$  on  $V_{\mu}$  is defined recursively by

$$\langle v, v \rangle = 1 \quad \text{and} \quad \langle g.u, v \rangle = \langle u, \tau(g).v \rangle,$$

for  $g \in \mathfrak{sl}_{n+1}$  and  $u, v \in V$ . In general, this Shapovalov form is nondegenerate on  $V_{\mu}$  and positive definite on the real part of  $V_{\mu}$ .

For example, the Shapovalov form on  $V_{\omega_n}$  is the standard Euclidean inner product,  $\langle v_i, v_j \rangle = \delta_{i,j}$ , in the basis  $v_1, \dots, v_{n+1}$  of Section 3.3. This induces the symmetric (tensor) Shapovalov form on the tensor product  $V_{\omega_n}^{\otimes(n+1)(d-n)}$ , which is positive definite on the real part of  $V_{\omega_n}^{\otimes(n+1)(d-n)}$ .


**Theorem 4.1** (Proposition 9.1 in [41]). *The Gaudin Hamiltonians are symmetric with respect to the tensor Shapovalov form,*

$$\langle K_i(t).u, v \rangle = \langle u, K_i(t).v \rangle,$$

for all  $i = 1, \dots, n+1$ ,  $t \in \mathbb{C}$ , and  $u, v \in V_{\omega_n}^{\otimes(n+1)(d-n)}$ .

We give the most important consequence of this result for our story.


**Corollary 4.2.** *When the parameters  $\mathbf{s}$  and variable  $t$  are real, the Gaudin Hamiltonians  $K_1(t), \dots, K_{n+1}(t)$  are real linear operators with real spectrum.*

*Proof.* The Gaudin Hamiltonians  $M_1(t), \dots, M_{n+1}(t)$  are real linear operators which act on the real part of  $V_{\omega_n}^{\otimes(n+1)(d-n)}$ , by their definition. The same is then also true of the Gaudin Hamiltonians  $K_1(t), \dots, K_{n+1}(t)$ . But these are symmetric with respect to the Shapovalov form and thus have real spectrum. 

*Proof of Theorem 1.* Suppose that  $\mathbf{s} \in \mathbb{R}^{(n+1)(d-n)}$  is general. By Corollary 4.2, the Gaudin Hamiltonians for  $t \in \mathbb{R}$  acting on  $\text{sing}(V_{\omega_n}^{\otimes(n+1)(d-n)}[0])$  are symmetric operators on a Euclidean space, and so have real eigenvalues. The Bethe vectors  $v(\mathbf{x}, \mathbf{s})$  for critical points  $\mathbf{x}$  of the master function with parameters  $\mathbf{s}$  form an eigenbasis for the

Gaudin Hamiltonians. As  $\mathbf{s}$  is general, the eigenvalues are distinct by Corollary 3.10 (2), and so the Bethe vectors must be real.

Given a critical point  $\mathbf{x}$ , the eigenvalues  $\lambda_1(t), \dots, \lambda_{n+1}(t)$  of the Bethe vectors are then real rational functions, and so the fundamental differential operator  $D_{\mathbf{x}}$  has real coefficients. But then the fundamental space  $V_{\mathbf{x}}$  of polynomials is real.

Thus each of the  $\delta(\mathbf{u}_{n,d})$  spaces of polynomials  $V_{\mathbf{x}}$  whose Wronskian has roots  $\mathbf{s}$  that were constructed in Section 2 is in fact real. This proves Theorem 1. 

## 5. OTHER PROOFS OF THE SHAPIRO CONJECTURE

The proofs of different Bethe ansätze for other models (other integrable systems) and other Lie algebras, which is ongoing work of Mukhin, Tarasov, and Varchenko, and others, can lead to generalizations of Theorem 1. One generalization is given in an appendix of [40], where it is conjectured that for real parameters  $\mathbf{s}$ , orbits of critical points of generalized master functions are real. For the Lie algebra  $\mathfrak{sl}_{n+1}$ , this holds as the polynomials  $p_i$  of Section 2.1 are real. This new conjecture also holds for the Lie algebras  $\mathfrak{sp}_{2n}$  and  $\mathfrak{so}_{2n+1}$ , by the results in Section 7 of [47].

We also discuss other proofs of the Shapiro conjecture.

**5.1. Discrete Wronskians to Calogero-Moser spaces.** The XXX model is another integrable system studied by Mukhin, Tarasov, and Varchenko [44]. This work is similar to their work on the periodic Gaudin model, including Wronskians, a Bethe ansatz, and symmetric operators. One difference is that  $U\mathfrak{gl}_{n+1}$  is replaced by the Yangians,  $Y\mathfrak{gl}_{n+1}$ , which are a deformation of the universal enveloping algebra of the current algebra  $\mathfrak{gl}_{n+1}[t]$ . (The current algebra  $\mathfrak{gl}_{n+1}[t]$  consists of polynomials in  $t$  with coefficients in  $\mathfrak{gl}_{n+1}$ .) Another is that the usual Wronskian is replaced by the *discrete Wronskian*, which depends upon a real number  $h$ ,

$$\mathrm{Wr}_h(f_0, \dots, f_n) := \det \begin{pmatrix} f_0(t) & f_0(t+h) & \cdots & f_0(t+nh) \\ f_1(t) & f_1(t+h) & \cdots & f_1(t+nh) \\ \vdots & \vdots & \ddots & \vdots \\ f_n(t) & f_n(t+h) & \cdots & f_n(t+nh) \end{pmatrix},$$

and the functions are *quasi-polynomials*,  $f_i(t) = e^{b_i t} g_i(t)$ , where  $g_i(t)$  is a polynomial. The linear span  $V$  of quasi-polynomials  $e^{b_0 t} g_0(t), \dots, e^{b_n t} g_n(t)$  is a space of quasi-polynomials. This discrete Wronskian has the form  $w(t)e^{\sum b_i t}$ , where  $w(t)$  is a polynomial that is well-defined up to a scalar.

**Theorem 5.1** (Theorem 2.1 of [44]). *Let  $V$  be a space of quasi-polynomials with discrete Wronskian  $\mathrm{Wr}_h(V) = \prod_{i=1}^N (t - s_i)e^{\sum b_i t}$  whose roots are real and satisfy*

$$|s_i - s_j| \geq |h| \quad \text{for all } i \neq j,$$

*then  $V$  has a basis of real quasi-polynomials.*

This condition on the separation of roots cannot be relaxed if the theorem is to hold for all exponents  $b_i$ . When  $n = 1$  and  $b_0 = b_1 = 0$ , this is a special case of the main theorem in Eremenko, *et al.* [15]. We will not discuss the proof of Theorem 5.1, except to remark that it depends upon the results of [41, 43].

In the limit as  $h \rightarrow 0$ , the discrete Wronskian becomes the usual Wronskian, which yields the following theorem.

**Theorem 5.2** (Theorem 4.1 of [44]). *Let  $V$  be a space of quasi-polynomials whose Wronskian has only real roots. Then  $V$  has a basis of real quasi-polynomials.*

When the exponents  $b_i$  are all zero, this reduces to Theorem 1, and therefore is a generalization of the Shapiro conjecture. It implies Theorem 2 from the Introduction. Suppose that  $b_0, \dots, b_n$  are distinct real numbers,  $\alpha_0, \dots, \alpha_n$  are complex numbers, and consider the matrix

$$(5.3) \quad Z := \begin{pmatrix} \alpha_0 & (b_0 - b_1)^{-1} & \cdots & (b_0 - b_n)^{-1} \\ (b_1 - b_0)^{-1} & \alpha_1 & \cdots & (b_1 - b_n)^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ (b_n - b_0)^{-1} & (b_n - b_1)^{-1} & \cdots & \alpha_n \end{pmatrix}.$$

**Theorem 2** (Theorem 5.4 of [44]). *If  $Z$  has only real eigenvalues, then the numbers  $\alpha_0, \dots, \alpha_n$  are real.*

*Proof.* We follow [44], deducing this from Theorem 5.2 and some matrix identities. Since

$$(5.4) \quad \frac{d^m}{dt^m}(t-a)e^{bt} = b^m(t-a)e^{bt} + mb^{m-1}e^{bt},$$

if  $A$  is the diagonal matrix  $\text{diag}(a_0, \dots, a_n)$ ,  $E := \text{diag}(e^{b_0t}, \dots, e^{b_nt})$ ,  $V$  is the Vandermonde matrix  $(b_j^i)_{i,j=0}^n$ , and  $W := (ib_j^{i-1})_{i,j=0}^n$ , then (5.4) implies that

$$\left( \frac{d^i}{dt^i}(t-a_j)e^{b_jt} \right)_{i,j=0}^n = [V(It - A) + W]E,$$

and therefore

$$(5.5) \quad \begin{aligned} \text{Wr}((t-a_0)e^{b_0t}, \dots, (t-a_n)e^{b_nt}) \\ = e^{\sum b_i t} \cdot \prod_{i < j} (b_j - b_i) \cdot \det [It - (A - V^{-1}W)]. \end{aligned}$$

We deduce a formula for  $V^{-1}W$ . The inverse of the Vandermonde matrix comes from Lagrange's interpolation formula. For each  $i = 0, \dots, n$ , set

$$\ell_i(u) := \frac{\prod_{k \neq i} (u - b_k)}{\prod_{k \neq i} (b_i - b_k)} = \sum_{j=0}^n \ell_{i,j} u^j.$$

Since  $\ell_i(b_j) = \delta_{i,j}$ , we see that  $V^{-1} = (\ell_{i,j})_{i,j=0}^n$ . But then

$$V^{-1}W = (\ell'_i(b_j))_{i,j=0}^n.$$

This gives formulas for the entries of  $V^{-1}W = (m_{i,j})_{i,j=0}^n$ ,

$$\begin{aligned} m_{i,j} &= \frac{\prod_{k \neq i,j} (b_j - b_k)}{\prod_{k \neq i} (b_i - b_k)} \quad \text{if } i \neq j \quad \text{and} \\ m_{i,i} &= \sum_{k \neq i} \frac{1}{b_i - b_k}. \end{aligned}$$


Let  $B$  be the diagonal matrix  $\text{diag}(\prod_{k \neq i} (b_i - b_k), i = 0, \dots, n)$  and  $M$  be the diagonal of  $V^{-1}W$ . We leave the following calculation to the reader,

$$B^{-1}ZB = \text{diag}(\alpha_0, \dots, \alpha_n) + M - V^{-1}W.$$

Combining this with (5.5), we see that if

$$(5.6) \quad a_i = \alpha_i + m_{i,i} \quad i = 0, \dots, n,$$

then the eigenvalues of  $Z$  are exactly the roots of the Wronskian (5.5).

Since the matrix  $Z$  has only real eigenvalues, Theorem 5.2 implies that the span of  $(t - a_0)e^{b_0 t}, \dots, (t - a_n)e^{b_n t}$  has a basis of real quasi-polynomials. Since the exponents  $b_i$  are real and distinct, the numbers  $a_0, \dots, a_n$  are real as are the entries of  $V^{-1}W$ , and so (5.6) implies that the entries  $\alpha_i$  of  $Z$  are real. 

Theorem 2 has an interesting consequence.

**Corollary 5.7.** *Suppose that  $X$  and  $Z$  are square complex matrices such that*

$$(5.8) \quad [X, Z] = I - K,$$


where  $K$  has rank 1. If both  $X$  and  $Z$  have real eigenvalues, then they may be simultaneously conjugated to real matrices.

*Proof.* It suffices to show this for a dense open subset of such pairs  $(X, Z)$  of matrices. Suppose that  $X$  is diagonalizable with eigenvalues  $b_0, \dots, b_n$  and that we have conjugated  $(X, Z)$  so that  $X$  is diagonal. If we write  $Z = (z_{i,j})_{i,j=0}^n$ , then

$$(5.9) \quad [X, Z] = (z_{i,j}(b_j - b_i))_{i,j=0}^n.$$

The rank 1 matrix  $K$  has the form  $(\beta_i \gamma_j)_{i,j=0}^n$ , where  $\beta, \gamma$  are complex vectors. By (5.8) and (5.9), the diagonal entries of  $K$  are all 1, so that  $\beta_i \gamma_i = 1$ , so in fact  $\beta, \gamma \in (\mathbb{C}^\times)^{n+1}$  with  $\gamma = \beta^{-1}$ . Conjugating (5.8) by  $\beta$  (considered as a diagonal matrix), we may assume that  $K$  is the matrix whose every entry is 1, and so

$$z_{i,j}(b_j - b_i) = \delta_{i,j} - 1,$$

or, if  $i \neq j$ ,  $b_i \neq b_j$ , and  $z_{i,j} = (b_i - b_j)^{-1}$ . But then  $Z$  has the form (5.3) (where  $\alpha_i = z_{i,i}$ ), and Theorem 2 implies that all of its entries are real. 

Mukhin, Tarasov, and Varchenko noted that Theorem 5.2 follows from Theorem 2 by the duality studied in [42], and that the Shapiro conjecture for Grassmannians is the case of Corollary 5.7 when  $Z$  is nilpotent. We close this section with an interesting circle of ideas related to Corollary 5.7.

Let  $\overline{C}_n$  be the set of all pairs  $(X, Z)$  of  $(n+1) \times (n+1)$  complex matrices such that  $[X, Z] - I$  has rank 1. The group  $Gl_{n+1}(\mathbb{C})$  acts on  $\overline{C}_n$  by simultaneous conjugation and Wilson [70] defines the *Calogero-Moser space*  $C_n$  to be the quotient of  $\overline{C}_n$  by  $Gl_{n+1}(\mathbb{C})$ . He shows this is a smooth affine variety of dimension  $2n$ . It has many incarnations. It is the phase space of the (complex) Calogero-Moser integrable system [31], Etingof and Ginzburg [16] showed that  $C_n$  parametrizes irreducible representations of a certain rational Cherednik algebra, and Wilson's adelic Grassmannian [70] is naturally the union of all the spaces  $C_n$ .

Let  $C_n(\mathbb{R})$  be the real points of  $C_n$ . This turns out to be image of the real points of  $\overline{C}_n$  under the quotient map  $\pi_n: \overline{C}_n \rightarrow C_n$ . The map that takes a matrix to its eigenvalues induces a map

$$\Upsilon: C_n \longrightarrow \mathbb{C}^{(n+1)} \times \mathbb{C}^{(n+1)},$$

where  $\mathbb{C}^{(n+1)} := \mathbb{C}^{n+1}/\mathcal{S}_{n+1}$ . Etingof and Ginzburg showed that  $\Upsilon$  is a finite map of degree  $(n+1)!$  We restate Corollary 5.7.

**Corollary 5.7.**  $\Upsilon^{-1}(\mathbb{R}^{(n+1)} \times \mathbb{R}^{(n+1)}) \subset C_n(\mathbb{R})$ .

This in turn implies the Shapiro conjecture for Grassmannians, which is the case of Corollary 5.7 when  $Z$  is nilpotent,

$$\Upsilon^{-1}(\mathbb{R}^{(n+1)} \times \{0\}) \subset C_n(\mathbb{R}).$$

The *rational Cherednik algebra*  $H_n$  [16] is generated by the polynomial subalgebras  $\mathbb{C}[x_0, \dots, x_n]$  and  $\mathbb{C}[z_0, \dots, z_n]$  and the group algebra  $\mathbb{C}\mathcal{S}_{n+1}$  subject to

$$(5.10) \quad \begin{aligned} \sigma_{ij}x_i &= x_j\sigma_{ij} & [x_i, z_j] &= \sigma_{ij} & \text{if } i \neq j, \\ \sigma_{ij}z_i &= z_j\sigma_{ij} & [x_i, z_i] &= -\sum_{j \neq i} \sigma_{ij}, \end{aligned}$$

where  $\sigma_{ij} \in \mathcal{S}_{n+1}$  is the transposition  $(i, j)$ . The symmetrizing idempotent is

$$e := \frac{1}{(n+1)!} \sum_{\sigma \in \mathcal{S}_{n+1}} \sigma.$$

For  $p \in C_n$ , write  $\mathbb{C}_p$  for the 1-dimensional representation of the coordinate ring of  $C_n$  in which a function  $f$  acts by the scalar  $f(p)$ .

**Theorem 5.11** ([16], Theorems 1.23 and 1.24).

- (1)  $eH_n e$  is isomorphic to the coordinate ring of  $C_n$ .
- (2) Irreducible representations of  $H_n$  are parametrized by the points  $p$  of  $C_n$ , where the corresponding representation is

$$M_p := H_n e \otimes_{eH_n e} \mathbb{C}_p.$$

- (3)  $M_p$  is isomorphic to  $\mathbb{C}\mathcal{S}_{n+1}$  as an  $\mathcal{S}_{n+1}$ -module.

Etingof and Ginzburg connect the structure of the representations  $M_p$  to the Calogero-Moser space. Let  $\mathcal{S}_n$  act on the indices  $\{0, \dots, n-1\}$ . Then  $x_n$  and  $z_n$  both stabilize the subspace  $M_p^{\mathcal{S}_n}$  of invariants, which has dimension  $n+1$ .

**Theorem 5.12** ([16], Theorem 11.16). *In any basis of  $M_p^{\mathcal{S}_n}$ ,  $x_n, z_n$  act by a pair of matrices  $(X, Z) \in \overline{C}_n$  such that  $\pi_n(X, Z) = p$ .*

**5.2. Transversality in the Shapiro conjecture.** While Mukhin, Tarasov, and Varchenko prove Theorem 1.11 in [45], their actual result is much deeper. Each ramification sequence  $\mathbf{a}$  for  $\mathbb{G}(n, d)$  corresponds to a dominant weight  $\mu(\mathbf{a})$  for  $\mathfrak{sl}_{n+1}$  such that, given a Schubert problem  $\mathbf{A} := (\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(m)})$ , the intersection number  $\delta(\mathbf{A})$  is equal to the dimension of the space of singular vectors

$$(V_{\mu(\mathbf{a}^{(1)})} \otimes V_{\mu(\mathbf{a}^{(2)})} \otimes \cdots \otimes V_{\mu(\mathbf{a}^{(m)})}) [0],$$

as both numbers are determined by the same formula based on the Littlewood-Richardson rule. The result of [45] links more subtle scheme-theoretic information about the intersection of Schubert varieties to algebraic information about the action of commuting operators on the singular vectors.

The coordinate ring  $R_{\mathbf{A}}(\mathbf{s})$  of an intersection of Schubert varieties (1.12) is an Artin algebra of dimension  $\delta(\mathbf{A})$ , because the Plücker formula (1.3) forces the intersection to be zero dimensional. It is semisimple exactly when the intersection is transverse. Because of the Plücker formula, the intersection lies in the big Schubert cell  $\mathbb{G}(n, d)^\circ$ , and so  $R_{\mathbf{A}}(\mathbf{s})$  is a quotient of the coordinate ring  $\mathcal{R}$  of  $\mathbb{G}(n, d)^\circ$ . Then the coregular

representation of  $R_{\mathbf{A}}(\mathbf{s})$  on its dual  $R_{\mathbf{A}}(\mathbf{s})^*$  induces an action of  $\mathcal{R}$  on  $R_{\mathbf{A}}(\mathbf{s})^*$ . This is the scheme-theoretic information.

Given a finite-dimensional representation  $V$  of  $\mathfrak{gl}_{n+1}$  and a complex number  $s$ , requiring  $t$  to act on  $V$  via scalar multiplication by  $s$ , gives the *evaluation module*  $V(s)$  of the current algebra  $\mathfrak{gl}_{n+1}[t]$ . If  $\boldsymbol{\mu} = (\mu^1, \dots, \mu^m)$  are dominant weights and  $\mathbf{s} = (s_1, \dots, s_m)$  are distinct complex numbers, then the evaluation module

$$V_{\boldsymbol{\mu}}(\mathbf{s}) := V_{\mu^1}(s_1) \otimes V_{\mu^2}(s_2) \otimes \cdots \otimes V_{\mu^m}(s_m)$$

is an irreducible  $\mathfrak{gl}_{n+1}[t]$ -module [7].

The universal enveloping algebra  $U\mathfrak{gl}_{n+1}[t]$  has a commutative subalgebra  $\mathcal{B}$ , called the *Bethe algebra* [21, 65]. As  $\mathcal{B}$  commutes with  $\mathfrak{gl}_{n+1}$ , it acts on spaces of singular vectors in the evaluation module  $V_{\boldsymbol{\mu}}(\mathbf{s})$ . The action of the Bethe algebra on the singular vectors  $V_{\boldsymbol{\mu}}(\mathbf{s})[0]$  is the algebraic information from representation theory. Let  $\mathcal{B}_{\boldsymbol{\mu}}(\mathbf{s})$  be the image of  $\mathcal{B}$  in the endomorphism algebra of  $V_{\boldsymbol{\mu}}(\mathbf{s})[0]$ .

A main result in [45] is that these two actions,  $\mathcal{R}$  on  $R_{\mathbf{A}}(\mathbf{s})$  and  $\mathcal{B}$  on  $V_{\boldsymbol{\mu}}(\mathbf{s})[0]$ , are isomorphic when  $\boldsymbol{\mu} = (\mu(\mathbf{a}^1), \dots, \mu(\mathbf{a}^m))$ , which we write as  $\boldsymbol{\mu}(\mathbf{A})$ . This requires that we identify  $\mathcal{R}$  with  $\mathcal{B}$  in some way. For that, the big cell  $\mathbb{G}(n, d)^\circ$  can be identified with  $(n+1) \times (d-n)$  matrices  $(y_{i,j})$ , whose entries identify  $\mathcal{R}$  with  $\mathbb{C}[y_{i,j}]$ . The Bethe algebra has generators  $B_{i,j}$ , where  $1 \leq i \leq n+1$  and  $1 \leq j$ . Define the map  $\tau: \mathcal{R} \hookrightarrow \mathcal{B}$  by

$$\tau(y_{i,j}) = B_{i,j}.$$

Mukhin, Tarasov, and Varchenko also give a linear bijection  $\phi: R_{\mathbf{A}}(\mathbf{s})^* \rightarrow V_{\boldsymbol{\mu}}(\mathbf{s})[0]$ .

**Theorem 5.13.** *Let  $\mathbf{A} = (\mathbf{a}^1, \dots, \mathbf{a}^m)$  be a Schubert problem for  $\mathbb{G}(n, d)$  and  $\mathbf{s} = (s_1, \dots, s_m)$  be distinct complex numbers. Then the map  $\tau$  induces an isomorphism of algebras  $\tau: R_{\mathbf{A}}(\mathbf{s}) \xrightarrow{\sim} \mathcal{B}_{\boldsymbol{\mu}(\mathbf{A})}(\mathbf{s})$  so that, for  $f \in R_{\mathbf{A}}(\mathbf{s})$  and  $g \in R_{\mathbf{A}}(\mathbf{s})^*$ ,  $\mu(f^*g) = \tau(f)\mu(g)$ . That is,  $(\tau, \mu)$  gives an isomorphism between the coregular representation of  $R_{\mathbf{A}}(\mathbf{s})$  on its linear dual  $R_{\mathbf{A}}(\mathbf{s})^*$  and the action of the Bethe algebra  $\mathcal{B}_{\boldsymbol{\mu}(\mathbf{A})}(\mathbf{s})$  on the singular vectors  $V_{\boldsymbol{\mu}}(\mathbf{s})[0]$ .*

Theorem 1.11 now follows, as the image of the Bethe algebra on the singular vectors  $V_{\boldsymbol{\mu}}(\mathbf{s})[0]$  is generated by the Gaudin Hamiltonians, which are diagonalizable when the parameters  $s_i$  are real. Thus  $\mathcal{B}_{\boldsymbol{\mu}(\mathbf{A})}(\mathbf{s})$  and hence  $R_{\mathbf{A}}(\mathbf{s})$  are semisimple, which implies that the intersection of Schubert varieties (1.12) was transverse.

We remark that this uses the general form of the results in [40] which we did not describe in these notes. Also, the coincidence of numbers,  $\delta(\mathbf{A}) = \dim(V_{\boldsymbol{\mu}(\mathbf{A})}(\mathbf{s})[0])$ , is an important ingredient in the proof that  $\mu$  is a bijection.

Very recently, Mukhin, Tarasov, and Varchenko related this Bethe algebra to the center of the rational Cherednik algebra of Section 5.1, and to the algebra of regular functions on the Calogero-Moser space [46].

## 6. APPLICATIONS OF THE SHAPIRO CONJECTURE

Theorem 1 and its stronger version, Theorem 1.11, have other applications in mathematics. Some are straightforward, such as linear series on  $\mathbb{P}^1$  with real ramification. Here, we discuss this application in the form of maximally inflected curves and rational functions with real critical points, as well as Purbhoo's considerably deeper application in which he recovers the basic combinatorial algorithms of Young tableaux from the



monodromy of the Wronski map. We close with Eremenko and Gabrielov's computation of the degree of the real Wronski map, which implies that when  $d$  is even and  $W$  is a generic real polynomial, there are many spaces of real polynomials with Wronskian  $W$ .

**6.1. Maximally inflected curves.** One of the earliest occurrences of the central mathematical object of these notes, spaces of polynomials with prescribed ramification, was in algebraic geometry, as these spaces of polynomials are linear series  $P \subset H^0(\mathbb{P}^1, \mathcal{O}(d))$  on  $\mathbb{P}^1$  with prescribed ramification. Their connection to Schubert calculus originated in work of Castelnuovo in 1889 [6] on  $g$ -nodal rational curves, and this was important in Brill-Noether theory (see Ch. 5 of [25]) and the Eisenbud-Harris theory of limit linear series [9, 10].

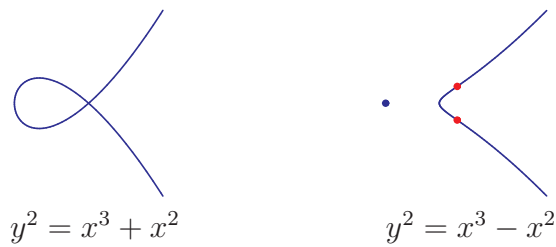
A linear series  $P$  on  $\mathbb{P}^1$  of degree  $d$  and dimension  $n+1$  (a point in  $\mathbb{G}(n, d)$ ) gives rise to a degree  $d$  map

$$(6.1) \quad \varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^n = \mathbb{P}(P^*)$$

of  $\mathbb{P}^1$  to projective space. We will call this map a curve. The linear series is *ramified* at points  $s \in \mathbb{P}^1$  where  $\varphi$  is not convex (the derivatives  $\varphi(s), \varphi'(s), \dots, \varphi^{(n)}(s)$  do not span  $\mathbb{P}^n$ ). Call such a point  $s$  a *flex* of the curve (6.1).

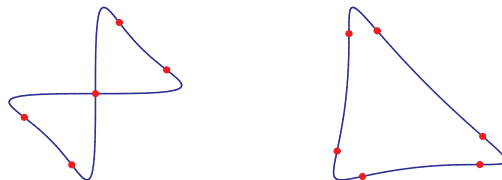
A curve is real when  $P$  is real. It is *maximally inflected* if it is real and all of its flexes are real. The study of these curves was initiated in [32], where restrictions on the topology of plane maximally inflected curves were established.

Let us look at some examples. There are two types of real rational cubic curves, which are distinguished by their singular points. The singular point of the curve on the left below is a node and it is connected to the rest of the curve, while the singular point on the other curve is isolated from the rest of the curve, and is called a *solitary point*.



While both curves have one of their three flexes at infinity, only the curve on the right has its other two flexes real (the dots) and is therefore maximally inflected. A nodal cubic cannot be maximally inflected.

Similarly, a maximally inflected quartic with six flexes has either 1 or 0 of its (necessarily 3) singular points a node, and necessarily 2 or 3 solitary points. We draw the two types of maximally inflected quartics having six flexes, omitting their solitary points. Due to the symmetry in the placement of the flexes, the first quartic has two flexes on its node—one for each branch through the node.





When  $d \geq 3$ , the image of a rational curve  $\gamma: \mathbb{P}^1 \rightarrow \mathbb{P}^2$  of degree  $d$  is a singular curve of arithmetic genus  $\binom{d-1}{2}$ . In general, the singularities consist of  $\binom{d-1}{2}$  ordinary double points, which are where two smooth branches of the curve meet transversally. A real rational curve has three types of such double points. We have already seen nodes and solitary points. The third kind of real double point consists of a pair of complex conjugate double points, and is invisible in  $\mathbb{R}\mathbb{P}^2$ .

The examples we gave had few nodes. This is always the case.

**Theorem 6.2** (Corollary 3.3 and Theorem 4.1 of [32]). *Given a maximally inflected plane curve of degree  $d$ , let  $\delta, \eta, c$  be its numbers of solitary points, nodes, and pairs of complex conjugate double points. Then we have*

$$d - 2 \leq \delta \leq \binom{n-1}{2} \quad \text{and} \quad 0 \leq \eta + 2c \leq \binom{n-2}{2}.$$

Furthermore, there exist maximally inflected curves of degree  $d$  with  $\binom{n-1}{2}$  solitary points (and hence no other singularities), and there exist curves with  $\binom{n-2}{2}$  nodes and  $d-2$  solitary points.

While many constructions of maximally inflected curves were known, Theorem 1, and in particular Theorem 1.11, show that there are many maximally inflected curves: Any curve  $\varphi: \mathbb{P}^1 \rightarrow \mathbb{P}^n$  whose ramification lies in  $\mathbb{R}\mathbb{P}^1$  must be real and is therefore maximally inflected.

Theorem 6.2 is proven using the Plücker formula (1.3) and the Klein formula from topology. This result and some intriguing calculations in Section 6 of [32] suggest that the number of solitary points is a deformation invariant. That is, if the points of inflection move, then the number of solitary points is constant. Examples show that the number of nodes may change under a continuous deformation of the inflection points, with a pair of nodes colliding to become a pair of complex conjugate double points, but we have not observed collisions of solitary points.

**6.2. Rational functions with real critical points.** A special case of Theorem 1, proved earlier by Eremenko and Gabrielov, serves to illustrate the breadth of mathematical areas touched by this Shapiro conjecture. When  $n = 1$ , we may associate a rational function  $\varphi_P := f_1(t)/f_2(t)$  to a basis  $\{f_1(t), f_2(t)\}$  of a vector space  $P \in \mathbb{G}(1, d)$  of polynomials. Different bases give different rational functions, but they all differ from each other by a fractional linear transformation of the image  $\mathbb{P}^1$ . We say that such rational functions are *equivalent*.

The critical points of a rational function are the points of the domain  $\mathbb{P}^1$  where the derivative of  $\varphi_P$ ,

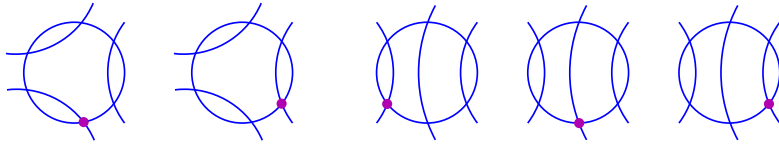
$$d\varphi_P := \frac{f_1'f_2 - f_1f_2'}{f_2^2} = \frac{1}{f_2^2} \cdot \det \begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix},$$

vanishes. That is, at the roots of the Wronskian. These critical points only depend upon the equivalence class of the rational function. Eremenko and Gabrielov [13] prove the following result about the critical points of rational functions.

**Theorem 6.3.** *A rational function  $\varphi$  whose critical points lie on a circle in  $\mathbb{P}^1$  maps that circle to a circle.*

To see that this is equivalent to Theorem 1 when  $n = 1$ , note that we may apply a change of variables to  $\varphi$  so that its critical points lie on the circle  $\mathbb{RP}^1 \subset \mathbb{P}^1$ . Similarly, the image circle may be assumed to be  $\mathbb{RP}^1$ . Reversing these coordinate changes establishes the equivalence.

The proof used methods specific to rational functions. Goldberg showed [22] that there are at most  $c_d := \frac{1}{d} \binom{2d-2}{d-1}$  rational functions of degree  $d$  with a given collection of  $2d - 2$  simple critical points. If the critical points of a rational function  $\varphi$  of degree  $d$  lie on a circle  $C \subset \mathbb{CP}^1$  and if  $\varphi$  maps  $C$  to  $C$ , then  $\varphi^{-1}(C)$  forms a graph on the Riemann sphere with nodes the  $2d-2$  critical points, each of degree 4, and each having two edges along  $C$  and one edge on each side of  $C$ . We mark one of the critical points to fix the ordered list of the critical points. It turns out that there are also  $c_d$  such abstract graphs with a distinguished vertex. Call these graphs *nets*. (In fact,  $c_d$  is Catalan number, which counts many objects in combinatorics [64, Exer. 6.19, p. 219].) Here are the  $c_4 = 5$  nets for  $d = 4$ :



Eremenko and Gabrielov used the uniformization theorem in complex analysis to construct such a rational function  $\varphi$  for each net and choice of critical points on  $C$ . Since  $c_d$  is the upper bound for the number of such rational functions, this gave all rational functions with a given set of critical points and thus proved Theorem 6.3.

More recently, Eremenko and Gabrielov [14] found an elementary proof of Theorem 6.3 which is based upon analytic continuation and a very refined version of the construction underlying Theorem 1.10. This has unfortunately never been published.

By Theorem 1.10, there exists a Wronski polynomial  $W_0(t)$  of degree  $2d-2$  with distinct real roots for which there are  $c_d$  spaces of real polynomials with Wronskian  $W_0(t)$ . Suppose that  $W_0$  is a member of a continuous family  $W_\tau$  for  $\tau \in [0, 1]$  of polynomials of degree  $2d-2$  with distinct real roots. Since there are  $c_d$  distinct spaces of polynomials with Wronskian  $W_0$ , there are  $c_d$  distinct lifts of the path  $W_\tau$  to paths of spaces of polynomials with Wronskian  $W_\tau$ , at least for  $\tau$  near zero. The obstruction to analytically continuing these  $c_d$  lifts occurs at critical points of the Wronski map  $\text{Wr}: \mathbb{G}(1, d)_{\mathbb{R}} \rightarrow \mathbb{RP}^{2d-2}$ . Since this map is at most  $c_d$  to 1, the first critical point in a fiber is a point where two of the lifted paths collide.

Eremenko and Gabrielov show that such a collision cannot occur. The reason is simple: nets are constant along paths of spaces of polynomials in  $\mathbb{G}(1, d)$  whose Wronskian has  $2d - 2$  distinct roots, and each of the spaces of polynomials above  $W_0(t)$  has a different net. Thus each lifted path has a different net, and no collision is possible. They show that nets are constant along paths by a simple set-theoretic/topological argument. A similar elementary argument applied to the construction of the spaces of polynomials in Theorem 1.10 shows that each space has a distinct net. The proof is completed by observing that any Wronski polynomial may be joined to  $W_0$  along some path  $W_\tau$ .

The elementary and constructive nature of this proof suggests that the Shapiro conjecture for Grassmannians may have an elementary proof, if a suitable substitute for nets can be found when  $n > 1$ .

**6.3. Tableaux combinatorics.** Starting from Theorems 1 and 1.11 for the Schubert problem  $\iota_{n,d}$ , Purbhoo [49] shows that the basic combinatorial properties and algorithms for Young tableaux are realized geometrically via the monodromy groupoid of the Wronski map,  $\text{Wr}: \mathbb{G}(n, d) \rightarrow \mathbb{P}(\mathbb{C}_{(n+1)(d-n)}[t])$ . In particular, Schützenberger slides, evacuation, Knuth equivalence and dual equivalence all arise geometrically. Purbhoo uses his analysis of the monodromy groupoid to get a new proof of the Littlewood-Richardson rule.

A *partition* is a weakly decreasing sequence of nonnegative integers  $\lambda: \lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_n \geq 0$ . We impose the restriction that  $n-d \geq \lambda_0$ . Partitions are ramification sequences in disguise, with  $\mathbf{a}: 0 \leq a_0 < a_1 < \dots < a_n \leq d$  corresponding to

$$(6.4) \quad \lambda(\mathbf{a}) : a_n - n \geq \dots \geq a_1 - 1 \geq a_0.$$

We identify a partition with its diagram, which is a left-justified array of boxes with  $\lambda_i$  boxes in the  $i$ th row. For example,

$$\lambda = 5322 \longleftrightarrow \begin{array}{cccc} \square & \square & \square & \square \\ \square & \square & \square & \\ \square & \square & \square & \\ \square & \square & \square & \end{array}.$$

Write  $|\lambda|$  for the number of boxes in  $\lambda$ . By (6.4),  $|\mathbf{a}| = |\lambda(\mathbf{a})|$ .

The partial order on ramification sequences induces the partial order of component-wise comparison on partitions, which is the inclusion of their diagrams. The minimal partition is  $\emptyset$  and the maximal partition (for us) is  $(d-n, \dots, d-n)$ , which has  $d-n$  repeated  $n+1$  times. Write this as  $\square$ . Given  $\mu \leq \lambda$ , the *skew partition*  $\lambda/\mu$  is the difference of their diagrams. We set  $|\lambda/\mu| := |\lambda| - |\mu|$ . For example,

$$5322/21 = \begin{array}{ccc} & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{array} \quad \text{and} \quad |5322/21| = 9.$$

A *standard Young tableau* of *shape*  $\lambda/\mu$  is a filling of the boxes of  $\lambda/\mu$  with the integers  $1, 2, \dots, |\lambda/\mu|$  so that the entries increase across each row and down each column. Here are three fillings of the shape  $331/1$ ; only the first two are tableaux.

$$\begin{array}{|c|c|c|} \hline & 1 & 3 \\ \hline 2 & 5 & 6 \\ \hline 4 & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline & 2 & 4 \\ \hline 1 & 3 & 5 \\ \hline 6 & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline & 2 & 6 \\ \hline 5 & 4 & 3 \\ \hline 1 & & \\ \hline \end{array}$$

Let  $\text{SYT}(\lambda/\mu)$  be the set of all standard Young tableaux of shape  $\lambda/\mu$ .

The degree  $\delta(\iota_{n,d})$  of the Wronski map equals the cardinality of  $\text{SYT}(\square)$ . By Theorem 1.11, the Wronski map is unramified over the locus of polynomials with distinct real roots, and so the points in each fiber are in bijection with the set  $\text{SYT}(\square)$ . This identification is almost canonical because in the region where the roots of the Wronskian are clustered (1.9) the identification is canonical, by the work of Eremenko and Gabrielov [11], and the Wronski map is unramified over the locus of polynomials with distinct roots. Since nets are in natural bijection with tableaux, this identification for  $n = 1$  was done by Eremenko and Gabrielov in [14].

This identification can be extended to skew tableaux. Given a partition  $\lambda$ , its dual is  $\lambda^\vee: d-n-\lambda_n \geq \dots \geq d-n-\lambda_0$ . For partitions  $\mu \leq \lambda$ , set

$$\mathbb{G}(\lambda/\mu) := \Omega_{\mathbf{a}(\mu)} F_\bullet(0) \cap \Omega_{\mathbf{a}(\lambda^\vee)} F_\bullet(\infty).$$

The Wronskian of a space of polynomials  $P \in \mathbb{G}(\boldsymbol{\lambda}/\boldsymbol{\mu})$  has degree at most  $|\boldsymbol{\lambda}|$  and vanishes to order least  $|\boldsymbol{\mu}|$  at zero. Let  $\mathbb{P}(\boldsymbol{\lambda}/\boldsymbol{\mu})$  be the projective space of such polynomials. This has dimension  $|\boldsymbol{\lambda}/\boldsymbol{\mu}|$ , which is equal to the dimension of  $\mathbb{G}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ . The restriction of the Wronski map to  $\mathbb{G}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ ,

$$\text{Wr} : \mathbb{G}(\boldsymbol{\lambda}/\boldsymbol{\mu}) \longrightarrow \mathbb{P}(\boldsymbol{\lambda}/\boldsymbol{\mu}),$$

is finite, flat, and has degree equal to the cardinality of  $\text{SYT}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ . Lastly, the Wronski map is unramified over the locus of polynomials in  $\mathbb{P}(\boldsymbol{\lambda}/\boldsymbol{\mu})$  with  $|\boldsymbol{\lambda}/\boldsymbol{\mu}|$  distinct nonzero real roots, and there is an identification of the fibers with  $\text{SYT}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ .

Purbhoo gives an explicit identification of the fibers of the Wronski map by extending the notion of standard tableaux. Let  $\mathbf{s} = \{s_1, \dots, s_{|\boldsymbol{\lambda}/\boldsymbol{\mu}|}\} \subset \mathbb{R}\mathbb{P}^1$  be a set of  $|\boldsymbol{\lambda}/\boldsymbol{\mu}|$  real numbers, possibly including  $\infty$ , that satisfy

- (I) If  $i \neq j$ , then  $|s_i| \neq |s_j|$ .
- (II)  $0 \in \mathbf{s}$  only if  $\boldsymbol{\mu} = \emptyset$  and  $\infty \in \mathbf{s}$  only if  $\boldsymbol{\lambda} = \square$ .

We identify such a subset  $\mathbf{s}$  with the polynomial  $W_{\mathbf{s}} := t^{|\boldsymbol{\mu}|} \prod_{s \in \mathbf{s}} (t - s)$  in  $\mathbb{P}(\boldsymbol{\lambda}/\boldsymbol{\mu})$  vanishing at  $\mathbf{s}$ . A *standard Young tableau of shape  $\boldsymbol{\lambda}/\boldsymbol{\mu}$*  with entries in  $\mathbf{s}$  is a filling of the boxes of  $\boldsymbol{\lambda}/\boldsymbol{\mu}$  with elements of  $\mathbf{s}$  such that if we replace each entry  $s_i$  with its absolute value  $|s_i|$ , then the entries increase across each row and down each column.

Let  $\text{SYT}(\boldsymbol{\lambda}/\boldsymbol{\mu}; \mathbf{s})$  be the set of all standard Young tableaux of shape  $\boldsymbol{\lambda}/\boldsymbol{\mu}$  with entries in  $\mathbf{s}$ . Replacing each entry  $s_i$  in a tableau by  $k$  if  $s_i$  has the  $k$ th smallest absolute value in  $\mathbf{s}$  defines the map  $\text{ord} : \text{SYT}(\boldsymbol{\lambda}/\boldsymbol{\mu}; \mathbf{s}) \rightarrow \text{SYT}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ . For example,

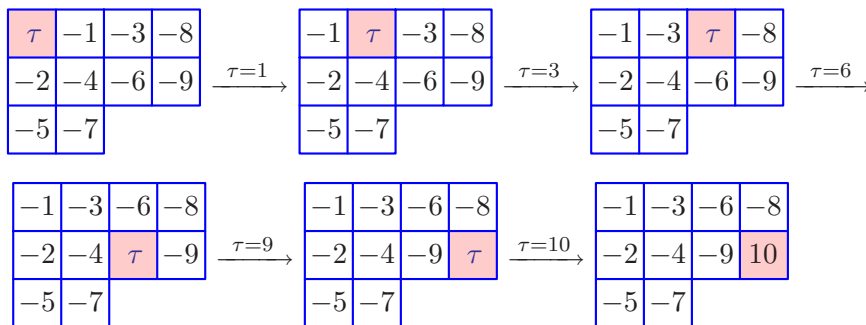
$$\begin{array}{|c|c|c|} \hline & \sqrt{2} & 4 \\ \hline e & -8 & \pi^2 \\ \hline -6 & & \\ \hline \end{array} \xrightarrow{\text{ord}} \begin{array}{|c|c|c|} \hline & 1 & 3 \\ \hline 2 & 5 & 6 \\ \hline 4 & & \\ \hline \end{array}$$

Let  $\mathbf{s}(\tau)$  for  $\tau \in [a, b]$  be a continuous path of subsets of  $\mathbb{R}\mathbb{P}^1$  where  $\mathbf{s}(\tau)$  satisfies Conditions (I) and (II), except for finitely many points  $\tau \in (a, b)$  at which Condition (I) is violated exactly once in that  $s_i(\tau) = -s_j(\tau)$  for some  $i \neq j$ . A path that is transverse to the locus where  $s_i = -s_j$  for all  $i \neq j$  is *generic*. Given a standard Young tableau  $T_a$  of shape  $\boldsymbol{\lambda}/\boldsymbol{\mu}$  and filling  $\mathbf{s}(a)$ , we can try to lift  $T_a$  to a family  $T_\tau$  of standard tableaux for all  $\tau \in [a, b]$ . We do this by requiring that the entries in the boxes of  $T_\tau$  vary continuously, unless the condition that  $T_\tau$  forms a tableau is violated.

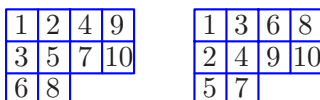
In any interval where  $\mathbf{s}(\tau)$  satisfies Condition (I), the entries of  $T_\tau$  vary continuously and  $\text{ord}(T_\tau)$  is constant. Suppose that  $\tau_0$  is a point of the path where Condition (I) is violated, and that  $s_i(\tau_0) = -s_j(\tau_0)$  is the pair witnessing this violation. If  $s_i$  and  $s_j$  are in different rows and columns, they remain in their respective boxes as  $\tau$  passes  $\tau_0$  and  $\text{ord}(T_\tau)$  changes as  $\tau$  passes  $\tau_0$ . If  $s_i$  and  $s_j$  are in the same row or column, then they are adjacent and leaving them in their respective boxes violates the condition that  $T_\tau$  is a tableau, so we require them to switch places and  $\text{ord}(T_\tau)$  does not change as  $\tau$  passes  $\tau_0$ .

Given a generic path  $\mathbf{s}(\tau)$  for  $\tau \in [a, b]$  and a tableau  $T_a \in \text{SYT}(\boldsymbol{\lambda}/\boldsymbol{\mu}; \mathbf{s}(a))$ , define  $\text{slide}_{\mathbf{s}(\tau)}(T_a)$  to be the result of this process applied to  $T_a$ . This gives a bijection between  $\text{SYT}(\boldsymbol{\lambda}/\boldsymbol{\mu}; \mathbf{s}(a))$  and  $\text{SYT}(\boldsymbol{\lambda}/\boldsymbol{\mu}; \mathbf{s}(b))$ .

**Example 6.5.** We show this on a tableau of shape  $(4, 4, 2)$ , for the path  $\mathbf{s}(\tau) = \{\tau, -1, \dots, -9\}$  for  $\tau \in [0, 10]$ . We only display when the tableau  $T_\tau$  changes.



The combinatorial enthusiast will note that the box containing  $\tau$  has just performed a Schützenberger slide through the subtableau formed by the negative entries. For comparison, we show the tableaux  $\text{ord}(T_0)$  and  $\text{ord}(T_{10})$ .



We give Purbhoo’s main theorem about the monodromy groupoid of the Wronski map  $\text{Wr}: \mathbb{G}(\boldsymbol{\lambda}/\boldsymbol{\mu}) \rightarrow \mathbb{P}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ .

**Theorem 6.6** ([49], Theorem 3.5). *For each  $\mathbf{s} = \{s_1, \dots, s_{|\boldsymbol{\lambda}/\boldsymbol{\mu}|}\} \subset \mathbb{RP}^1$  satisfying Conditions (I) and (II), there is a correspondence  $P \leftrightarrow T(P)$  between points  $P \in \mathbb{G}(\boldsymbol{\lambda}/\boldsymbol{\mu})$  with Wronskian  $W_{\mathbf{s}}$  and tableaux  $T(P) \in \text{SYT}(\boldsymbol{\lambda}/\boldsymbol{\mu}; \mathbf{s})$ . Under this correspondence, if  $\mathbf{s}(\tau) \subset \mathbb{RP}^1$  is a generic path for  $\tau \in [a, b]$  and  $P_\tau$  is any lifting of that path to  $\mathbb{G}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ , then*

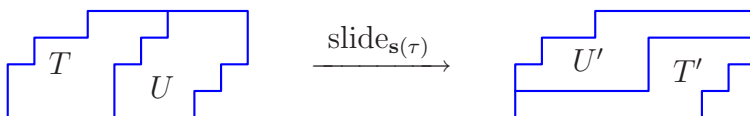
$$T(P_b) = \text{slide}_{\mathbf{s}(\tau)} T(P_a) .$$

Thus the combinatorial operation of sliding a tableau along a generic path  $\mathbf{s}(\tau)$  exactly describes analytic continuation in the fibers of the Wronski map above that path. This sliding operation contains Schützenberger’s *jeu de taquin* [55], and much of tableaux combinatorics [18, 51, 64] may be recovered from the geometry of the Wronski map.

Suppose  $\mathbf{s} = \{s_1, \dots, s_{|\boldsymbol{\lambda}/\boldsymbol{\mu}|}\}$  with  $|s_1| < \dots < |s_{|\boldsymbol{\lambda}/\boldsymbol{\mu}|}|$ . If  $T \in \text{SYT}(\boldsymbol{\lambda}/\boldsymbol{\mu}; \mathbf{s})$  and  $\mathbf{t} = \{s_i, \dots, s_j\}$  with  $i < j$ , then the entries of  $T$  in the set  $\mathbf{t}$  form a subtableau  $T|_{\mathbf{t}}$ . Now suppose that  $\mathbf{s} = \mathbf{t} \cup \mathbf{u}$  where the elements of  $\mathbf{t}$  are positive, those of  $\mathbf{u}$  are negative, and we additionally have that  $|t| < |u|$  for  $t \in \mathbf{t}$  and  $u \in \mathbf{u}$ . Write  $|\mathbf{t}| < |\mathbf{u}|$  when this occurs. Let  $\mathbf{t}'$  be a set of  $|\boldsymbol{\lambda}/\boldsymbol{\mu}|$  positive numbers with  $|\mathbf{u}| < |\mathbf{t}'|$  and suppose that  $\mathbf{s}(\tau)$  for  $\tau \in [0, 1]$  is a generic path from  $\mathbf{s} = \mathbf{t} \cup \mathbf{u}$  to  $\mathbf{s}' = \mathbf{u} \cup \mathbf{t}'$ . Given a tableau  $S \in \text{SYT}(\boldsymbol{\lambda}/\boldsymbol{\mu}; \mathbf{s})$ , let  $S' := \text{slide}_{\mathbf{s}(\tau)} S \in \text{SYT}(\boldsymbol{\lambda}/\boldsymbol{\mu}; \mathbf{s}')$ , and define the subtableaux,

$$T := S|_{\mathbf{t}}, \quad U := S|_{\mathbf{u}}, \quad T' := S'|_{\mathbf{t}'}, \quad \text{and} \quad U' := S'|_{\mathbf{u}} .$$

Because  $|\mathbf{t}| < |\mathbf{u}| < |\mathbf{t}'|$ ,  $T$  is inside of  $U$  and during the slide  $T$  and  $U$  move through each other to obtain the tableaux  $U'$  and  $T'$  with  $U'$  inside of  $T'$ . Schematically,



We write

$$U' = \text{slide}_T U \quad \text{and} \quad T' = \text{slide}_{U'} T.$$

Reversing the path  $\mathbf{s}(\tau)$  enables the definition of  $U = \text{slide}_{T'} U'$  and  $T = \text{slide}_{U'} T'$ . These notions are independent of the choice of path  $\mathbf{s}(\tau)$ , by Theorem 6.6. In fact  $\text{slide}_T U$  does not depend upon the set  $\mathbf{t}'$ . This geometrically defined operation was studied from a combinatorial perspective [3], where it was called tableaux switching, and its independence from choices was Theorem 2.2(4) *ibid*.

**Definition 6.7.** Let  $\mathbf{u}$  be a set of negative numbers. Two tableaux  $U_1, U_2 \in \text{SYT}(\boldsymbol{\lambda}/\boldsymbol{\mu}; \mathbf{u})$  are *equivalent* if, for any set  $\mathbf{t}$  of  $|\boldsymbol{\mu}|$  positive numbers with  $|\mathbf{t}| < |\mathbf{u}|$  and any  $T_1, T_2 \in \text{SYT}(\boldsymbol{\mu}; \mathbf{t})$ , we have

$$\text{slide}_{T_1} U_1 = \text{slide}_{T_2} U_2.$$

Two tableaux  $U_1, U_2 \in \text{SYT}(\boldsymbol{\lambda}/\boldsymbol{\mu}; \mathbf{u})$  are *dual equivalent* if, for any sets  $\mathbf{t}, \mathbf{t}'$  of positive numbers with  $|\mathbf{t}| < |\mathbf{u}| < |\mathbf{t}'|$ , shapes  $\boldsymbol{\mu}/\boldsymbol{\nu}, \boldsymbol{\kappa}/\boldsymbol{\lambda}$ , and tableaux  $T \in \text{SYT}(\boldsymbol{\mu}/\boldsymbol{\nu}; \mathbf{t})$  and  $T' \in \text{SYT}(\boldsymbol{\kappa}/\boldsymbol{\lambda}; \mathbf{t}')$ , each pair

$$(\text{slide}_T U_1, \text{slide}_T U_2) \quad \text{and} \quad (\text{slide}_{T'} U_1, \text{slide}_{T'} U_2)$$

has the same shape. Replacing numbers by their negatives extends these definitions to tableaux with positive entries.

These are the fundamental equivalence relations on tableaux of Knuth-equivalence and of Haiman's dual equivalence [24]. Purbhoo shows that these combinatorial equivalence relations coincide with geometrically-defined relations that come from nonreduced fibers of the Wronski map.

Suppose that  $\mathbf{s} = \{s_1, \dots, s_{(n+1)(d-n)}\}$  is a subset of  $\mathbb{R}\mathbb{P}^1$  satisfying Condition (I) with  $|s_1| < \dots < |s_{(n+1)(d-n)}|$ , and suppose that  $\mathbf{t} = \{s_i, s_{i+1}, \dots, s_j\}$  are the positive elements of  $\mathbf{s}$  and let  $\mathbf{u} = \mathbf{s} - \mathbf{t}$  be its nonpositive elements. Pick a positive number  $a \in [s_i, s_j]$  and consider any path  $\mathbf{s}(\tau)$  for  $\tau \in [0, 1]$  that satisfies Condition (I) for  $\tau \in [0, 1)$  with  $\mathbf{s}(0) = \mathbf{s}$ , has constant nonpositive elements  $\mathbf{u}$ , but whose positive elements all approach  $a$  as  $t \rightarrow 1$  so that  $\mathbf{s}(1) = \{s_1, \dots, s_{i-1}, a, \dots, a, s_{j+1}, \dots, s_{(n+1)(d-n)}\}$ .

Given a tableau  $T \in \text{SYT}(\square; \mathbf{s})$  corresponding to a point  $P_T \in \mathbb{G}(n, d)$  with Wronskian  $W_{\mathbf{s}}$ , we may analytically continue  $P_T$  in the fibers of the Wronski map over the path  $W_{\mathbf{s}(\tau)}$ . When  $\tau \neq 1$ , this continuation will be  $P_{T_\tau}$ , but when  $\tau = 1$  it will be  $\lim_{\tau \rightarrow 1} P_{T_\tau}$ . Write  $P_T(\tau)$  for these points. For each  $\tau < 1$  the points  $P_T(\tau)$  are distinct for different  $T \in \text{SYT}(\square; \mathbf{s}(\tau))$ , but in the limit as  $\tau \rightarrow 1$  some paths may coalesce, as the fiber of the Wronskian is nonreduced at  $\mathbf{s}(1)$ .

**Theorem 6.8** ([49]). *Let  $T, T' \in \text{SYT}(\square; \mathbf{s})$ . Then  $T_{\mathbf{t}}$  is equivalent to  $T'|_{\mathbf{t}}$  if and only if  $P_T(1) = P_{T'}(1)$ .*

Let  $\mathbf{s}'(\tau)$  for  $\tau \in [0, 1]$  be another generic path with  $\mathbf{s}'(0) = \mathbf{s}$  in which the positive elements are constant, but the others converge to some fixed negative number  $a$ . We define  $P'_T(\tau)$  to be the analytic continuation of  $P_T$  over the path  $\mathbf{s}'(\tau)$ .

**Theorem 6.9** ([49]). *Let  $T, T' \in \text{SYT}(\square; \mathbf{s})$ . Then  $T_{\mathbf{t}}$  is dual equivalent to  $T'|_{\mathbf{t}}$  if and only if  $P'_T(1) = P'_{T'}(1)$ .*



**6.4. Degree of the real Wronski map.** Recall that the complex Wronski map  $\text{Wr}: \mathbb{G}(n, d) \rightarrow \mathbb{P}^{(n+1)(d-n)}$  has degree (1.6)

$$\delta(\mathfrak{L}_{n,d}) = [(n+1)(d-n)]! \frac{1!2! \cdots n!}{(d-n)!(d-n+1)! \cdots d!}.$$

If we restrict the domain to the real Grassmannian, we get the real Wronski map  $\text{Wr}_{\mathbb{R}}: \mathbb{G}(n, d)_{\mathbb{R}} \rightarrow \mathbb{R}\mathbb{P}^{(n+1)(d-n)}$ . By Theorem 1.11, over the locus of polynomials with  $(n+1)(d-n)$  distinct real roots, this is a  $\delta(\mathfrak{L}_{n,d})$ -to-one cover. Eremenko and Gabrielov [11] studied this real Wronski map, computing its topological degree.

This requires some explanation, for real projective spaces and Grassmannians are not always orientable, and hence maps between them do not necessarily have a degree. However, the Wronski map can be lifted to their orienting double covers, after which its degree is well-defined up to a sign. By the Plücker formula, the Wronski map restricted to the big Schubert cell  $\mathbb{G}(n, d)_{\mathbb{R}}^{\circ}$  of the Grassmannian is a finite, proper map to  $\mathbb{R}^{(n+1)(d-n)}$ , realized as the space of monic real polynomials of degree  $(n+1)(d-n)$ . To compute the degree of the Wronski map over this big cell.

Fix a standard tableau  $T_0 \in \text{SYT}(\square)$ . Given any tableau  $T \in \text{SYT}(\square)$ , let  $\sigma_T$  be the permutation in  $\mathcal{S}_{(n+1)(d-n)}$  with  $\sigma_T(i) = j$  if the entries  $i$  in  $T_0$  and  $j$  in  $T$  occupy the same cell of  $\square$ . Define

$$\delta(\mathfrak{L}_{n,d})_{\mathbb{R}} := \sum_{T \in \text{SYT}(\square)} |\sigma_T|,$$

where  $|\sigma| = \pm 1$  is the sign of the permutation  $\sigma$ .

**Theorem 6.10** (Theorem 2 of [11]).  $\deg \text{Wr}_{\mathbb{R}} = \delta(\mathfrak{L}_{n,d})_{\mathbb{R}}$ .

This statistic,  $\delta(\mathfrak{L}_{n,d})_{\mathbb{R}}$ , was computed by White [69], who showed that it vanishes unless  $d$  is even, and in that case it equals

$$\frac{1!2! \cdots (p-1)!(m-1)!(m-2)! \cdots (m-p+1)! \binom{mp}{2}}{(m-p+2)!(m-p+4)! \cdots (m+p-2)! \binom{m-p+1}{2}! \binom{m-p+3}{2}! \cdots \binom{m+p-1}{2}!},$$

where  $m := \max\{n+1, d-n\}$  and  $p := \min\{n+1, d-n\}$ .

The significance of these results is that  $\delta(\mathfrak{L}_{n,d})_{\mathbb{R}}$  is a lower bound for the number of real spaces of polynomials with given real Wronskian. This gave the first example of a geometric problem possessing a nontrivial lower bound on its number of real solutions. In the 1990's, Kontsevich [36] determined the number  $N_d$  of complex rational curves of degree  $d$  interpolating  $3d-1$  general points in the plane. Work of Welschinger [68], Mikhalkin [39], and Itenberg, *et al.* [28, 29] established a nontrivial lower bound  $W_d$  on the number of real curves interpolating real points. Not only is  $W_d > 0$ , but

$$\lim_{d \rightarrow \infty} \frac{\log W_d}{\log N_d} = 1 \quad (!)$$

More recently, Solomon [57] realized this number  $W_d$  as the degree of a map.

Such lower bounds, if they were widespread, could have significant value for applications of mathematics, as they are existence proofs for real solutions. (One application of the nontriviality of  $W_3 = 8$  is given in [17].) Initial steps in this direction were made in [58, 30], which established lower bounds for certain systems of sparse polynomials.



## 7. EXTENSIONS OF THE SHAPIRO CONJECTURE

The Shapiro conjecture for Grassmannians makes sense for other flag manifolds. In this more general setting, it is known to fail, but in very interesting ways. In some cases, we have been able to modify it to give a conjecture that holds under scrutiny. The Shapiro conjecture also admits some appealing generalizations, but its strongest and most subtle form remains open for Grassmannians.

**7.1. Lagrangian and Orthogonal Grassmannians.** The Lagrangian and orthogonal Grassmannians are closely related to the classical Grassmannian. For each of these, the Shapiro conjecture is particularly easy to state.

The (odd) orthogonal Grassmannian requires a nondegenerate symmetric bilinear form  $\langle \cdot, \cdot \rangle$  on  $\mathbb{C}^{2n+1}$ . This vector space has a basis  $e_1, \dots, e_{2n+1}$  such that

$$\langle e_i, e_{2n+2-j} \rangle = \delta_{i,j}.$$

A subspace  $V$  of  $\mathbb{C}^{2n+1}$  is *isotropic* if  $\langle V, V \rangle = 0$ . Isotropic subspaces have dimension at most  $n$ . The (*odd orthogonal Grassmannian*  $OG(n)$ ) is the set of all maximal ( $n$ -dimensional) isotropic subspaces  $V$  of  $\mathbb{C}^{2n+1}$ . This variety has dimension  $\binom{n+1}{2}$ .

The Shapiro conjecture for  $OG(n)$  begins with a particular rational normal curve  $\gamma$  having parametrization

$$\begin{aligned} t \longmapsto e_1 + te_2 + \frac{t^2}{2}e_3 + \cdots + \frac{t^n}{n!}e_{n+1} \\ - \frac{t^{n+1}}{(n+1)!}e_{n+2} + \frac{t^{n+2}}{(n+2)!}e_{n+3} - \cdots + (-1)^n \frac{t^{2n}}{(2n)!}e_{2n+1}. \end{aligned}$$

This has special properties with respect to the form  $\langle \cdot, \cdot \rangle$ . For  $t \in \mathbb{C}$ , define the flag  $F_\bullet(t)$  in  $\mathbb{C}^{2n+1}$  by

$$F_i(t) := \text{Span}\{\gamma(t), \gamma'(t), \dots, \gamma^{(i-1)}(t)\}.$$

Then  $F_\bullet(t)$  is *isotropic* in that

$$\langle F_i(t), F_{2n+1-i}(t) \rangle = 0.$$

In general, an isotropic flag  $F_\bullet$  of  $\mathbb{C}^{2n+1}$  is a flag such that  $\langle F_i, F_{2n+1-i} \rangle = 0$ .

Schubert varieties of  $OG(n)$  are defined with respect to an isotropic flag,  $F_\bullet$ , and are the restriction of Schubert varieties of  $\mathbb{G}(n-1, 2n)$ —the Grassmannian of  $n$  dimensional subspaces of  $\mathbb{C}^{2n+1}$ —under the inclusion  $OG(n) \hookrightarrow \mathbb{G}(n-1, 2n)$ . Schubert varieties for  $OG(n)$  are indexed by *strict partitions*, which are integer sequences

$$\boldsymbol{\kappa} : n \geq \kappa^1 > \kappa^2 > \cdots > \kappa^k > 0.$$

Set  $\|\boldsymbol{\kappa}\| = \kappa^1 + \cdots + \kappa^k$ . We do not give the precise relation between these indices and the ramification sequences  $\mathbf{a}$  of Section 1, but this may be done using the descriptions given in [20, § 6.1] or [60]. Write  $X_{\boldsymbol{\kappa}}F_\bullet$  for the Schubert variety of  $OG(n)$  defined by the Schubert index  $\boldsymbol{\kappa}$  and an isotropic flag  $F_\bullet$ . It has codimension  $\|\boldsymbol{\kappa}\|$ . A Schubert problem is a list  $(\boldsymbol{\kappa}_1, \dots, \boldsymbol{\kappa}_m)$  of Schubert indices such that

$$\|\boldsymbol{\kappa}_1\| + \|\boldsymbol{\kappa}_2\| + \cdots + \|\boldsymbol{\kappa}_m\| = \dim OG(n) = \binom{n+1}{2}.$$

We state the Shapiro conjecture for  $OG(n)$ .

**Conjecture 7.1.** *If  $(\kappa_1, \dots, \kappa_m)$  is a Schubert problem for  $OG(n)$  and  $s_1, \dots, s_m$  are distinct real numbers, then the intersection*

$$X_{\kappa_1} F_{\bullet}(s_1) \cap X_{\kappa_2} F_{\bullet}(s_2) \cap \cdots \cap X_{\kappa_m} F_{\bullet}(s_m)$$

*is transverse with all points real.*

Besides optimism based upon the validity of the Shapiro conjecture for Grassmannians, the evidence for Conjecture 7.1 comes in two forms. Several tens of thousands of instances have been checked with a computer and when each  $\|\kappa_i\| = 1$  and the points  $s_i$  are clustered (1.9), the intersection is transverse with all points real [63].

There is a similar story but with a different outcome for the Lagrangian Grassmannian. Let  $\langle \cdot, \cdot \rangle$  be a nondegenerate skew symmetric bilinear form on  $\mathbb{C}^{2n}$ . This vector space has a basis  $e_1, \dots, e_{2n}$  such that

$$\langle e_i, e_{2n+1-j} \rangle = \begin{cases} \delta_{i,j} & \text{if } i \leq n \\ -\delta_{i,j} & \text{if } i > n \end{cases}.$$

Isotropic subspaces in  $\mathbb{C}^{2n}$  may have any dimension up to  $n$ , and those of maximal dimension are called *Lagrangian subspaces*. The *Lagrangian Grassmannian*  $LG(n)$  is the set of all Lagrangian subspaces  $V$  of  $\mathbb{C}^{2n}$ . This variety has dimension  $\binom{n+1}{2}$ .

For the Shapiro conjecture for  $LG(n)$ , we have the rational normal curve  $\gamma$  with parametrization

$$\begin{aligned} t \longmapsto & e_1 + te_2 + \frac{t^2}{2}e_3 + \cdots + \frac{t^n}{n!}e_{n+1} \\ & - \frac{t^{n+1}}{(n+1)!}e_{n+2} + \frac{t^{n+2}}{(n+2)!}e_{n+3} - \cdots + (-1)^{n-1} \frac{t^{2n-1}}{(2n-1)!}e_{2n}. \end{aligned}$$

For  $t \in \mathbb{C}$ , define the flag  $F_{\bullet}(t)$  in  $\mathbb{C}^{2n+1}$  by

$$F_i(t) := \text{Span}\{\gamma(t), \gamma'(t), \dots, \gamma^{(i-1)}(t)\}.$$

The flag  $F_{\bullet}(t)$  is *isotropic* in that

$$\langle F_i(t), F_{2n-i}(t) \rangle = 0.$$

More generally, an isotropic flag  $F_{\bullet}$  of  $\mathbb{C}^{2n}$  is a flag such that  $\langle F_i, F_{2n-i} \rangle = 0$ .

As with  $OG(n)$ , given an isotropic flag, Schubert varieties for  $LG(n)$  are induced from Schubert varieties of  $\mathbb{G}(n-1, 2n-1)$  by the inclusion  $LG(n) \hookrightarrow \mathbb{G}(n-1, 2n-1)$ . Schubert varieties  $X_{\kappa} F_{\bullet}$  of  $LG(n)$  are also indexed by strict partitions  $\kappa$  and  $\|\kappa\|$  is the codimension of  $X_{\kappa} F_{\bullet}$ . We give the relation between strict partitions for  $LG(n)$  and ramification sequences for  $\mathbb{G}(n-1, 2n-1)$ . Given a strict partition  $\kappa: n \geq \kappa^1 > \cdots > \kappa^k$ , let  $\mu: 0 < \mu_1 < \cdots < \mu_{n-k}$  be the complement of the set  $\{\kappa^1, \dots, \kappa^k\}$  in  $\{1, 2, \dots, n\}$ . Call  $k$  the *length* of the strict partition  $\kappa$ . For example, if  $n = 6$  and  $\kappa = 4, 2$ , then  $k = 2$  and  $\mu = 1, 3, 5, 6$ . If we define  $\mathbf{a}(\kappa) = (a_0, \dots, a_{n-1})$  to be the sequence

$$0 \leq n - \kappa^1 < \cdots < n - \kappa^k < n - 1 + \mu_1 < \cdots < n - 1 + \mu_{n-k} \leq 2n - 1,$$

then  $X_{\kappa} F_{\bullet} = \Omega_{\mathbf{a}(\kappa)} F_{\bullet} \cap LG(n)$ , so that

$$X_{\kappa} F_{\bullet} = \{V \in LG(n) \mid F_{2n-a_j} \geq n - j, \text{ for } j = 0, 1, \dots, n-1\}.$$

A Schubert problem is a list  $(\kappa_1, \dots, \kappa_m)$  such that

$$\|\kappa_1\| + \|\kappa_2\| + \dots + \|\kappa_m\| = \dim LG(n) = \binom{n+1}{2}.$$

The obvious generalization of Theorem 1 and Conjecture 7.1 to  $LG(n)$  turns out to be false. We offer a modification that we believe is true. Belkale and Kumar [2] define a notion they call Levi movability. A Schubert problem  $(\kappa_1, \dots, \kappa_m)$  for  $LG(n)$  is *Levi movable* if the corresponding Schubert indices,  $(\mathbf{a}(\kappa_1), \dots, \mathbf{a}(\kappa_m))$  also form a Schubert problem for  $\mathbb{G}(n-1, 2n-1)$ . Unraveling the definitions shows that this is equivalent to having the lengths of the strict partitions  $(\kappa_1, \dots, \kappa_m)$  sum to  $n$ .

**Conjecture 7.2.** *If  $(\kappa_1, \dots, \kappa_m)$  is a Schubert problem for  $LG(n)$  and  $s_1, \dots, s_m$  are distinct real numbers, then the intersection*

$$X_{\kappa_1} F_{\bullet}(s_1) \cap X_{\kappa_2} F_{\bullet}(s_2) \cap \dots \cap X_{\kappa_m} F_{\bullet}(s_m)$$

*is transverse. If  $(\kappa_1, \dots, \kappa_m)$  is Levi movable, then all points of intersection are real, but if it is not Levi movable, then no point in the intersection is real.*

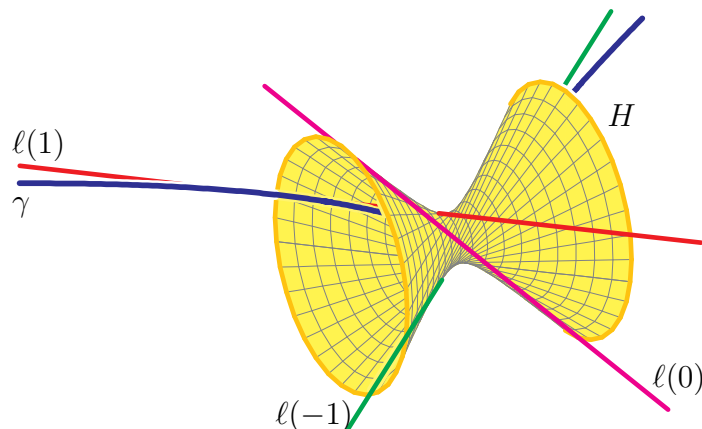
The strongest evidence in favor of Conjecture 7.2 is that it is true when the Schubert problem  $(\kappa_1, \dots, \kappa_m)$  is Levi movable. This follows from the definition of Levi movability and the Shapiro conjecture for Grassmannians. Further evidence is that if each  $\kappa_i$  is simple in that  $\|\kappa_i\| = 1$ , then a local version, similar to Theorem 1.10 but without transversality, is true [63]. That is, if the  $s_i$  are clustered (1.9), then no point in the intersection is real. Lastly, several tens of thousands of instances have been checked with a computer.

**7.2. Monotone conjecture for flag manifolds.** The Shapiro conjecture was originally made for the classical (type- $A$ ) flag manifold, where it fails spectacularly. It is false for the first nontrivial Schubert problem on a flag variety that is not a Grassmannian. Namely, the geometric problem of partial flags consisting of a line  $\ell$  lying on a plane  $\Lambda$  in 3-dimensional space where  $\ell$  meets three fixed lines and  $\Lambda$  contains two fixed points.

This is just the problem of four lines in disguise. Suppose that  $p$  and  $q$  are the two fixed points that  $\Lambda$  is required to contain. Then  $\Lambda$  contains the line  $\overline{p, q}$  they span. Since  $\ell \subset \Lambda$ , it must meet  $\overline{p, q}$ . As  $\ell$  must also meet three lines, this problem reduces to the problem of four lines. In this way, there are two solutions to this Schubert problem.

Now let us investigate the Shapiro conjecture for this Schubert problem, which posits that both flags  $\ell \subset \Lambda$  will be real, if we require that  $\ell$  meets three fixed tangent lines to a rational curve and  $\Lambda$  contains two fixed points of the rational curve. Let  $\gamma$  be the rational normal curve (1) from the Introduction and suppose that the three fixed lines of our problem are its tangent lines  $\ell(-1)$ ,  $\ell(0)$ , and  $\ell(1)$ . These lines lie on the hyperboloid  $H$  of one sheet (2). Here is another view of these lines, the curve  $\gamma$ , and

the hyperboloid.



If we require  $\ell$  to meet the three tangent lines  $\ell(-1)$ ,  $\ell(0)$ , and  $\ell(1)$  and  $\Lambda$  to contain the two points  $\gamma(v)$  and  $\gamma(w)$  of  $\gamma$ , then  $\ell$  also meets the line  $\lambda(v, w)$  spanned by these two points. As in the Introduction, the lines  $\ell$  that we seek will come from points where the secant line  $\lambda(v, w)$  meets  $H$ .

Figure 3 shows an expanded view down the throat of the hyperboloid, with a secant line  $\lambda(v, w)$  that meets the hyperboloid in two points. For these points  $\gamma(v)$  and  $\gamma(w)$

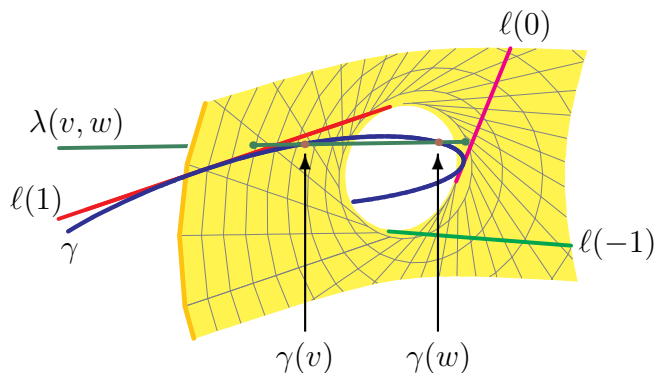


FIGURE 3. A secant line meeting  $H$ .

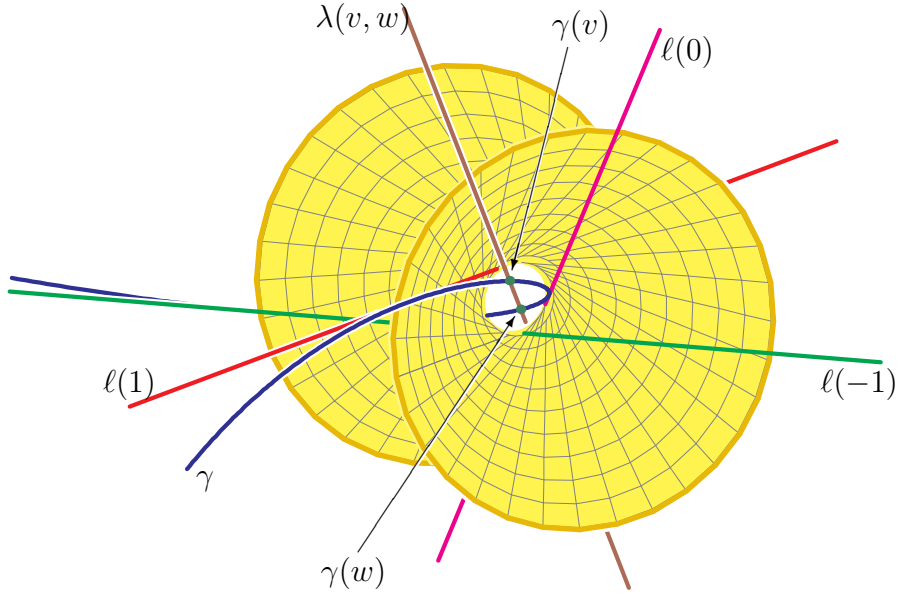
there will be two real flags  $\ell \subset \Lambda$  satisfying our conditions. This is consistent with the Shapiro conjecture.

In contrast, Figure 4 shows a secant line  $\lambda(v, w)$  that does not meet the hyperboloid in any real points. For these points  $\gamma(v)$  and  $\gamma(w)$ , neither flag  $\ell \subset \Lambda$  satisfying our conditions is real. This is a counterexample to the Shapiro conjecture.

This failure of the Shapiro conjecture is however quite interesting. If we label the points  $-1, 0, 1$  with 1 (conditions on the line) and  $v, w$  by 2 (conditions on the plane), then along  $\gamma$  they occur in order

$$(7.3) \quad 11122 \text{ in Figure 3} \quad \text{and} \quad 11212 \text{ in Figure 4.}$$

The sequence for Figure 3 is *monotone* increasing and in this case both solutions are always real, but the sequence for Figure 4 is not monotone. This example suggests a way to correct the Shapiro conjecture, that we call the *monotone conjecture*.

FIGURE 4. A secant line not meeting  $H$ .

Specifically, let  $\mathbf{n}: 0 \leq n_1 < \cdots < n_k < d$  be a sequence of integers. The manifold  $\mathbb{F}\ell_{\mathbf{n},d}$  of flags of type  $\mathbf{n}$  is the set of all sequences of subspaces

$$E_{\bullet} : E_{n_1} \subset E_{n_2} \subset \cdots \subset E_{n_k} \subset \mathbb{C}_d[t]$$

with  $\dim E_{n_i} = n_i + 1$ . The forgetful map  $E_{\bullet} \mapsto E_{n_i}$  induces a projection

$$\pi_i : \mathbb{F}\ell_{\mathbf{n},d} \longrightarrow \mathbb{G}(n_i, d)$$

to a Grassmannian. A *Grassmannian Schubert variety* is a subvariety of  $\mathbb{F}\ell_{\mathbf{n},d}$  of the form  $\pi_i^{-1}\Omega_{\mathbf{a}}F_{\bullet}$ . That is, it is the inverse image of a Schubert variety in a Grassmannian projection. Write  $X_{(\mathbf{a},n_i)}F_{\bullet}$  for this Grassmannian Schubert variety and call  $(\mathbf{a}, n_i)$  a Grassmannian Schubert condition.

A *Grassmannian Schubert problem* is a list

$$(7.4) \quad (\mathbf{a}^{(1)}, n^{(1)}), (\mathbf{a}^{(2)}, n^{(2)}), \dots, (\mathbf{a}^{(m)}, n^{(m)}),$$

of Grassmannian Schubert conditions satisfying  $|\mathbf{a}^{(1)}| + \cdots + |\mathbf{a}^{(m)}| = \dim \mathbb{F}\ell_{\mathbf{n},d}$ . We state the monotone conjecture.

**Conjecture 7.5.** *Let  $((\mathbf{a}^{(1)}, n^{(1)}), \dots, (\mathbf{a}^{(m)}, n^{(m)}))$  be a Grassmannian Schubert problem for the flag variety  $\mathbb{F}\ell_{\mathbf{n},d}$  with  $n^{(1)} \leq n^{(2)} \leq \cdots \leq n^{(m)}$ . Whenever  $s_1 < s_2 < \cdots < s_m$  are real numbers, the intersection*

$$X_{(\mathbf{a}^{(1)}, n^{(1)})}F_{\bullet}(s_1) \cap X_{(\mathbf{a}^{(2)}, n^{(2)})}F_{\bullet}(s_2) \cap \cdots \cap X_{(\mathbf{a}^{(m)}, n^{(m)})}F_{\bullet}(s_m),$$

*is transverse with all points of intersection real (when it is nonempty).*

There is significant evidence for this monotone conjecture. First, the Shapiro conjecture for Grassmannians is the special case case when  $m = 1$  so then  $\mathbf{n} = n_1$  and  $\mathbb{F}\ell_{\mathbf{n},d} = \mathbb{G}(n_1, d)$ : the monotonicity condition  $s_1 < \cdots < s_m$  is empty as any reordering of the Schubert conditions remains sorted.

	Number of Real Solutions						
	0	2	4	6	8	10	12
11112222							400000
11211222			118	65425	132241	117504	84712
11122122			104	65461	134417	117535	82483
11221122			1618	57236	188393	92580	60173
11212212			25398	90784	143394	107108	33316
11221212		2085	79317	111448	121589	60333	25228
11121222		7818	34389	58098	101334	81724	116637
12121212	15923	41929	131054	86894	81823	30578	11799

TABLE 1. The Schubert problem (7.6) on  $\mathbb{F}\ell_{1<2,5}$ .

This conjecture was formulated in [50]. That project was based upon computer experimentation using 15.76 gigaHertz-years of computing to study over 520 million instances of 1126 different Schubert problems on 29 flag manifolds. Some of this computation studied intersections of Schubert varieties that were not necessarily monotone. For example, consider the Schubert problem on  $\mathbb{F}\ell_{1<2,5}$ ,

$$(7.6) \quad (0<2, 1)^4, (0<1<3, 2)^4,$$

where the exponent indicates a repeated condition. Table 1 displays the computation on this Schubert problem. The rows are labeled by different orderings of the conditions along the rational normal curve  $\gamma$  in the notation of (7.3). Each cell contains the number of computed instances with a given ordering and number of real solutions. The empty cells indicate no observed instances. Only the first row tests the monotone conjecture: Each of the 400,000 computed instances had all 12 solutions real. The other rows reveal a very interesting pattern; for nonmonotone orderings of the conditions along  $\gamma$ , not all solutions are always real and there seems to be a lower bound on the number of real solutions. Only in the last row, which represents the maximal possible intertwining of the conditions, were no real solutions observed.

A third piece of evidence for the monotone conjecture was provided by Eremenko, *et al.* [15], who showed that it is true for two-step flag manifolds, when  $\mathbf{n} = d-2 < d-1$ . This is a special case of their main theorem, which asserts the reality of a rational function  $\varphi$  with prescribed critical points on  $\mathbb{R}\mathbb{P}^1$  and certain prescribed coincidences  $\varphi(v) = \varphi(w)$ , when  $v, w$  are real.

This result of Eremenko, *et al.* can be described in terms of  $\mathbb{G}(d-2, d)$ , where it becomes a statement about real points in an intersection of Schubert varieties given by flags that are secant to a rational normal curve in a particular way. This condition on secant flags makes sense for any Grassmannian, and the resulting secant conjecture is also a generalization of the Shapiro conjecture.

A flag  $F_\bullet$  is *secant along an arc*  $I$  of a rational normal curve  $\gamma$  if every subspace in the flag is spanned by its intersections with  $I$ . A collection of flags that are secant to  $\gamma$  is *disjoint* if they are secant along disjoint arcs of  $\gamma$ . The *secant conjecture* asserts that a Schubert problem given by disjoint secant flags has all solutions real. We give a more precise statement.

**Conjecture 7.7.** *If  $(\mathbf{a}_1, \dots, \mathbf{a}_m)$  is a Schubert problem for  $\mathbb{G}(n, d)$  and  $F_\bullet^1, \dots, F_\bullet^m$  are disjoint secant flags, then the intersection*

$$\Omega_{\mathbf{a}_1} F_\bullet^1 \cap \Omega_{\mathbf{a}_2} F_\bullet^2 \cap \dots \cap \Omega_{\mathbf{a}_m} F_\bullet^m$$

*is transverse with all points real.*

The main result of [13] is that an intersection of Schubert varieties in  $\mathbb{G}(d-2, d)$  given by disjoint secant flags is transverse with all points real. The Shapiro conjecture is a limiting case of the secant conjecture, as the flag osculating  $\gamma$  at a point  $s$  is the limit of flags that are secant along arcs that shrink to the point  $s$ .

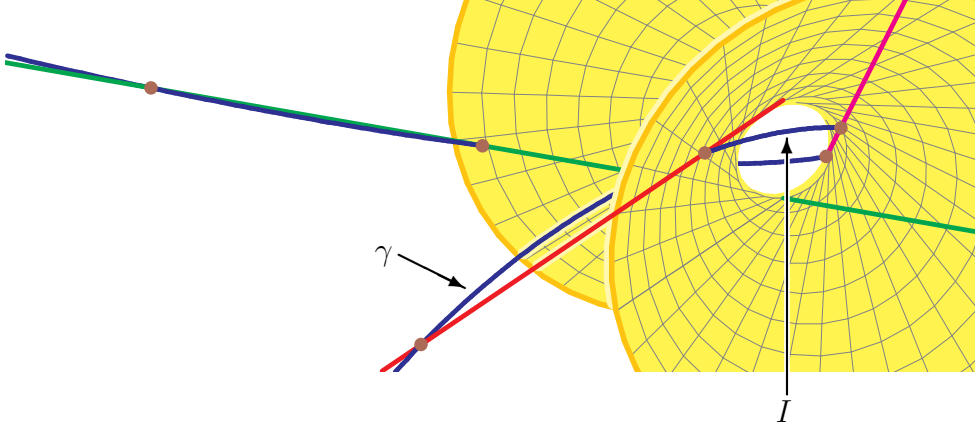


FIGURE 5. The problem of four secant lines.

Consider this secant conjecture for the problem of four lines. The hyperboloid in Figure 5 contains three lines that are secant to  $\gamma$  along disjoint arcs. Any line secant along the arc  $I$  (which is disjoint from the other three arcs) meets the hyperboloid in two points, giving two real solutions to this instance of the secant conjecture.

This secant conjecture is currently being studied on a supercomputer whose day job is calculus instruction. For each of hundreds of Schubert problems, thousands to millions of instances of the secant conjecture are being tested, and much more. The *overlap number* measures how far a collection of secant flags is from being disjoint, and it is zero if and only if the flags are disjoint. This experiment tests instances of the secant conjecture *and* near misses when the flags have low overlap number. The results (number of real solutions vs. overlap number) are stored in a publicly accessible database available from the webpage [56]. In the first nine months of operation, this has studied over 1.3 billion instances of Schubert problems and consumed over 600 gigaHertz-years of computing.

Table 2 shows the results for a Schubert problem with 16 solutions on  $\mathbb{G}(2, 5)$ . Computing the 20 million instances of this problem used 4.473 gigaHertz-years. The rows are labeled with the even integers from 0 to 16 as the number of real solutions has the same parity as the number of complex solutions. The column with overlap number 0 represents tests of the secant conjecture. Since its only entries are in the row for 16 real solutions, the secant conjecture was verified in 4,568,553 instances. The column labeled 1 is empty because flags for this problem cannot have overlap number 1. The most interesting feature is that for overlap number 2, all solutions were still real, while for overlap numbers 3, 4, and 5, at least 4 of the 16 solutions were real, and only with



# real	Overlap Number								
	0	1	2	3	4	5	6	...	Total
solns0							20	...	7977
2							116	...	88578
4				6154	23561	526	3011	...	542521
6				25526	63265	2040	9460	...	1571582
8				33736	78559	2995	13650	...	2834459
10				25953	39252	2540	11179	...	3351159
12				35578	44840	3271	14160	...	2944091
14				17367	17180	1705	7821	...	1602251
16	4568553		182668	583007	468506	36983	83169	...	7057382
Total	4568553		182668	727321	735163	50060	142586	...	20000000

TABLE 2. Number of real solutions vs. overlap number.

overlap number 6 and greater does the Schubert problem have no real solutions. This inner border, which indicates that the secant conjecture does not completely fail when there is small overlap, is found on many of the other problems that we investigated and is a new phenomenon that we do not understand. A description of the technical aspects of this running experiment is given in [27].

**7.3. Discriminant conjecture.** Despite the proofs of Theorems 1 and 1.11 (weak and strong form of the Shapiro for Grassmannians), the strongest and most subtle form of that conjecture remains open.

The discriminant of a polynomial  $W = \prod_i(t - s_i)$  is  $\prod_{i < j}(s_i - s_j)^2$ , the symmetric function of its roots having lowest degree that vanishes when  $W$  has a double root. More generally, suppose that we have a family of polynomial systems in a space  $X$  that are parametrized by a space  $S$ . (For example, the intersection in Theorem 1.11,

$$(1.12) \quad \Omega_{\mathbf{a}(1)}F_{\bullet}(s_1) \cap \Omega_{\mathbf{a}(2)}F_{\bullet}(s_2) \cap \cdots \cap \Omega_{\mathbf{a}(m)}F_{\bullet}(s_m)$$

in which  $X = \mathbb{G}(n, d)$  and  $S$  is  $\mathbb{C}^m$  or  $(\mathbb{P}^1)^m$ .) Then the discriminant variety of this system is the subvariety  $\Sigma \subset S$  where the system is not transverse. This is expected to be a hypersurface, and the *discriminant* of the system is the function that defines  $\Sigma$ .

By Theorem 1.11 this discriminant does not vanish when the parameters  $s_i$  are real and distinct. However, in the few cases when it has been computed much more is true, it is a sum of squares [62] and therefore nonnegative. For example, for the Schubert problem  $\iota_{1,4}$  with 5 solutions, if we fix  $s_5 = 0$  and  $s_6 = \infty$ , then the discriminant is a homogeneous polynomial of degree 20 in the four variables  $s_1, \dots, s_4$  with 711 terms, which turns out to be a sum of squares. This is remarkable because Hilbert [26] showed that, except for  $m = 3$  and  $\deg = 4$ , not all nonnegative homogeneous polynomials in  $m > 2$  variables of degree more than 2 are sums of squares. Work of Blekherman [4] suggests that it is extremely rare for a nonnegative polynomial to be a sum of squares.

**Conjecture 7.8** (Question 4 of [62]). *The discriminant of an intersection (1.12) of a Schubert problem on a Grassmannian given by osculating flags is sum of squares in the parameters  $s_1, \dots, s_m$ .*

We conjecture that this remains true for any cominuscule flag variety, which includes the Lagrangian Grassmannian, the orthogonal Grassmannian, quadrics, as well as the two exceptional cases  $E_6/D_5$  and  $E_7/E_6$ . There is also a form of this conjecture, Conjecture 2.10 of [50], involving preorders for the semialgebraic set of monotone parameters  $s_1 < s_2 < \cdots < s_m$ .

We close with the remark that we have not yet investigated the Shapiro conjecture for other flag manifolds and do not yet know when it fails or how to repair the failures. Also, the methods of Mukhin, Tarasov, and Varchenko only work for the Grassmannian, and it is completely unclear how to even approach a proof of these generalizations.

## REFERENCES

- [1] D. J. Bates, F. Bihan, and F. Sottile, *Bounds on the number of real solutions to polynomial equations*, Int. Math. Res. Not. IMRN (2007), no. 23, Art. ID rnm114, 7.
- [2] P. Belkale and S. Kumar, *Eigenvalue problem and a new product in cohomology of flag varieties*, Invent. Math. **166** (2006), no. 1, 185–228.
- [3] G. Benkart, F. Sottile, and J. Stroomer, *Tableau switching: algorithms and applications*, J. Combin. Theory Ser. A **76** (1996), no. 1, 11–43.
- [4] G. Blekherman, *There are significantly more nonnegative polynomials than sums of squares*, Israel J. Math. **153** (2006), 355–380.
- [5] C. I. Byrnes, *Pole assignment by output feedback*, Three decades of mathematical system theory, Lecture Notes in Control and Inform. Sci., vol. 135, Springer, Berlin, 1989, pp. 31–78.
- [6] G. Castelnuovo, *Numero delle involuzioni razionali gaicenti sopra una curva di dato genere*, Rend. R. Accad. Lincei **4** (1889), no. 5, 130–133.
- [7] V. Chari and A. Pressley, *Weyl modules for classical and quantum affine algebras*, Represent. Theory **5** (2001), 191–223 (electronic).
- [8] D. Cox, J. Little, and D. O’Shea, *Ideals, varieties, and algorithms*, third ed., Undergraduate Texts in Mathematics, Springer, New York, 2007.
- [9] D. Eisenbud and J. Harris, *Divisors on general curves and cuspidal rational curves*, Invent. Math. **74** (1983), no. 3, 371–418.
- [10] D. Eisenbud and J. Harris, *When ramification points meet*, Invent. Math. **87** (1987), 485–493.
- [11] A. Eremenko and A. Gabrielov, *Degrees of real Wronski maps*, Discrete Comput. Geom. **28** (2002), no. 3, 331–347.
- [12] ———, *Pole placement static output feedback for generic linear systems*, SIAM J. Control Optim. **41** (2002), no. 1, 303–312 (electronic).
- [13] ———, *Rational functions with real critical points and the B. and M. Shapiro conjecture in real enumerative geometry*, Ann. of Math. (2) **155** (2002), no. 1, 105–129.
- [14] ———, *Elementary proof of the B. and M. Shapiro conjecture for rational functions*, 2005, arXiv:math/0512370.
- [15] A. Eremenko, A. Gabrielov, M. Shapiro, and A. Vainshtein, *Rational functions and real Schubert calculus*, Proc. Amer. Math. Soc. **134** (2006), no. 4, 949–957 (electronic).
- [16] P. Etingof and V. Ginzburg, *Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism*, Invent. Math. **147** (2002), no. 2, 243–348.
- [17] S. Fiedler-Le Touzé, *Pencils of cubics as tools to solve an interpolation problem*, Appl. Algebra Engng. Comm. Comput. **18** (2007), no. 1-2, 53–70.
- [18] Wm. Fulton, *Young tableaux*, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, Cambridge, 1997.
- [19] Wm. Fulton and J. Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991.
- [20] Wm. Fulton and P. Pragacz, *Schubert varieties and degeneracy loci*, Lecture Notes in Mathematics, vol. 1689, Springer-Verlag, Berlin, 1998.
- [21] M. Gaudin, *La fonction d’onde de Bethe*, Collection du Commissariat à l’Énergie Atomique: Série Scientifique., Masson, Paris, 1983.

- [22] L. R. Goldberg, *Catalan numbers and branched coverings by the Riemann sphere*, Adv. Math. **85** (1991), no. 2, 129–144.
- [23] P. Griffiths and J. Harris, *Principles of algebraic geometry*, J. Wiley and Sons, 1978.
- [24] Mark D. Haiman, *Dual equivalence with applications, including a conjecture of Proctor*, Discrete Math. **99** (1992), no. 1-3, 79–113.
- [25] J. Harris and I. Morrison, *Moduli of curves*, Graduate Texts in Mathematics 187, Springer-Verlag, 1998.
- [26] D. Hilbert, *Über die Darstellung definiter Formen als Summe von Formen-quadrate*, Math. Ann. **32** (1888), 342–350.
- [27] C. Hillar, L. García-Puente, A. Martín del Campo, J. Ruffo, Z. Teitler, Stephen L. Johnson, and F. Sottile, *Experimentation at the frontiers of reality in Schubert calculus*, 2009, arXiv:0906.2497.
- [28] I. V. Itenberg, V. M. Kharlamov, and E. I. Shustin, *Welschinger invariant and enumeration of real rational curves*, Int. Math. Res. Not. (2003), no. 49, 2639–2653.
- [29] ———, *Logarithmic equivalence of the Welschinger and the Gromov-Witten invariants*, Uspekhi Mat. Nauk **59** (2004), no. 6(360), 85–110.
- [30] M. Joswig and N. Witte, *Products of foldable triangulations*, Adv. Math. **210** (2007), no. 2, 769–796.
- [31] D. Kazhdan, B. Kostant, and S. Sternberg, *Hamiltonian group actions and dynamical systems of Calogero type*, Comm. Pure Appl. Math. **31** (1978), no. 4, 481–507.
- [32] V. Kharlamov and F. Sottile, *Maximally inflected real rational curves*, Mosc. Math. J. **3** (2003), no. 3, 947–987, 1199–1200.
- [33] A. G. Khovanskii, *Fewnomials*, Translations of Mathematical Monographs, vol. 88, American Mathematical Society, Providence, RI, 1991.
- [34] S. L. Kleiman, *The transversality of a general translate*, Compositio Math. **28** (1974), 287–297.
- [35] S. L. Kleiman and D. Laksov, *Schubert calculus*, Amer. Math. Monthly **79** (1972), 1061–1082.
- [36] M. Kontsevich and Yu. Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry*, Comm. Math. Phys. **164** (1994), no. 3, 525–562.
- [37] E. Kostlan, *On the distribution of roots of random polynomials*, From Topology to Computation: Proc. Smalefest (Berkeley, CA, 1990), Springer, New York, 1993, pp. 419–431.
- [38] P. P. Kulish and E. K. Sklyanin, *Quantum spectral transform method. Recent developments*, Integrable quantum field theories: proceedings of the symposium held at Tvarminne, Finland, 23-27 March, 1981 (J. Hietarinta and C. Montonen, eds.), Lecture Notes in Phys., vol. 151, Springer, Berlin, 1982, pp. 61–119.
- [39] G. Mikhalkin, *Enumerative tropical algebraic geometry in  $\mathbb{R}^2$* , J. Amer. Math. Soc. **18** (2005), no. 2, 313–377.
- [40] E. Mukhin, V. Tarasov, and A. Varchenko, *The B. and M. Shapiro conjecture in real algebraic geometry and the Bethe ansatz*, 2005, Annals of Mathematics, to appear.
- [41] ———, *Bethe eigenvectors of higher transfer matrices*, J. Stat. Mech. Theory Exp. (2006), no. 8, P08002, 44 pp. (electronic).
- [42] ———, *Bispectral and  $(\mathfrak{gl}_N, \mathfrak{gl}_M)$  dualities*, Funct. Anal. Other Math. **1** (2006), no. 1, 47–69.
- [43] ———, *Generating operator of XXX or Gaudin transfer matrices has quasi-exponential kernel*, SIGMA Symmetry Integrability Geom. Methods Appl. **3** (2007), Paper 060, 31 pp.
- [44] ———, *On reality property of Wronski maps*, 2007, Confluentes Mathematici, to appear.
- [45] ———, *Schubert calculus and representations of general linear group*, 2007, Journal of the AMS, to appear.
- [46] ———, *Bethe algebra of Gaudin model, Calogero-Moser space and Cherednik algebra*, 2009, arXiv:0906.5185.
- [47] E. Mukhin and A. Varchenko, *Critical points of master functions and flag varieties*, Commun. Contemp. Math. **6** (2004), no. 1, 111–163.
- [48] ———, *Norm of a Bethe vector and the Hessian of the master function*, Compos. Math. **141** (2005), no. 4, 1012–1028.
- [49] K. Purbhoo, *Jeu de taquin and a monodromy problem for Wronskians of polynomials*, 2009, arXiv:0902.1321.

- [50] J. Ruffo, Y. Sivan, E. Soprunova, and F. Sottile, *Experimentation and conjectures in the real Schubert calculus for flag manifolds*, Experiment. Math. **15** (2006), no. 2, 199–221.
- [51] B. Sagan, *The symmetric group*, second ed., Graduate Texts in Mathematics, vol. 203, Springer-Verlag, New York, 2001.
- [52] V. Schechtman and A. Varchenko, *Arrangements of hyperplanes and Lie algebra homology*, Invent. Math. **106** (1991), no. 1, 139–194.
- [53] I. Scherbak and A. Varchenko, *Critical points of functions,  $\mathfrak{sl}_2$  representations, and Fuchsian differential equations with only univalued solutions*, Mosc. Math. J. **3** (2003), 621–645, 745.
- [54] H. Schubert, *Anzahl-Bestimmungen für lineare Räume beliebiger Dimension*, Acta. Math. **8** (1886), 97–118.
- [55] M. P. Schützenberger, *Quelques remarques sur une construction de Schensted*, Math. Scand. **12** (1963), 117–128.
- [56] Secant Team, *Frontiers of Reality in Schubert Calculus*, [www.math.tamu.edu/~secant](http://www.math.tamu.edu/~secant).
- [57] J. Solomon, *Intersection theory on the moduli space of holomorphic curves with Lagrangian boundary conditions*, [math.SG/0606429](http://math.SG/0606429).
- [58] E. Soprunova and F. Sottile, *Lower bounds for real solutions to sparse polynomial systems*, Adv. Math. **204** (2006), no. 1, 116–151.
- [59] F. Sottile, *Enumerative geometry for the real Grassmannian of lines in projective space*, Duke Math. J. **87** (1997), no. 1, 59–85.
- [60] ———, *Pieri-type formulas for maximal isotropic Grassmannians via triple intersections*, Colloq. Math. **82** (1999), no. 1, 49–63.
- [61] ———, *The special Schubert calculus is real*, Electron. Res. Announc. Amer. Math. Soc. **5** (1999), 35–39 (electronic).
- [62] ———, *Real Schubert calculus: polynomial systems and a conjecture of Shapiro and Shapiro*, Experiment. Math. **9** (2000), no. 2, 161–182.
- [63] ———, *Some real and unreal enumerative geometry for flag manifolds*, Michigan Math. J. **48** (2000), 573–592, Dedicated to William Fulton on the occasion of his 60th birthday.
- [64] R. P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999.
- [65] D. Talalaev, *Quantization of the Gaudin system*, 2004, [arXiv:hep-th/0404153](http://arXiv:hep-th/0404153).
- [66] R. Vakil, *Schubert induction*, Ann. of Math. (2) **164** (2006), no. 2, 489–512.
- [67] J. Verschelde, *Numerical evidence for a conjecture in real algebraic geometry*, Experiment. Math. **9** (2000), no. 2, 183–196.
- [68] J.-Y. Welschinger, *Invariants of real rational symplectic 4-manifolds and lower bounds in real enumerative geometry*, C. R. Math. Acad. Sci. Paris **336** (2003), no. 4, 341–344.
- [69] D. White, *Sign-balanced posets*, J. Combin. Theory Ser. A **95** (2001), no. 1, 1–38.
- [70] G. Wilson, *Collisions of Calogero-Moser particles and an adelic Grassmannian*, Invent. Math. **133** (1998), no. 1, 1–41, With an appendix by I. G. Macdonald.

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX 77843, USA

*E-mail address:* [sottile@math.tamu.edu](mailto:sottile@math.tamu.edu)

*URL:* [www.math.tamu.edu/~sottile](http://www.math.tamu.edu/~sottile)