ON THE FUNDAMENTAL GROUP OF
OPEN RICHARDSON VARIETIES

CHANGZHENG LI, FRANK SOTTILE, AND CHI ZHANG

ABSTRACT. We compute the fundamental group of an open Richardson variety in the manifold of complete flags that corresponds to a partial flag manifold. Rietsch showed that these log Calabi-Yau varieties underlie a Landau-Ginzburg mirror for the Langlands dual partial flag manifold, and our computation verifies a prediction of Hori for this mirror. It is log Calabi-Yau as it isomorphic to the complement of the Knutson–Lam–Speyer anti-canonical divisor for the partial flag manifold. We also determine explicit defining equations for this divisor.

1. INTRODUCTION

It is an old problem of Zariski [21] to compute the fundamental group of the complement of an algebraic curve in the complex projective plane. The fundamental group of the complement of a projective hypersurface reduces to the case of a plane curve by Zariski's Theorem of Lefschetz type [22]. More generally, one may ask about the fundamental group of the complement of a divisor in a projective variety. Examples of importance in mirror symmetry are log Calabi-Yau varieties [7, 8, 10], which are quasi-projective varieties that are the complement of an anti-canonical divisor in a smooth projective variety. We consider this case when the ambient projective variety is a flag variety.

Let G be a complex, simply-connected, simple Lie group with a Borel subgroup B. For an element u in the Weyl group W of G, the (opposite) Schubert cells $\mathring{X}_u$ and $\mathring{X}^u$ in G/B are affine spaces of codimension and dimension $\ell(u)$ respectively, where $\ell \colon W \to \mathbb{Z}_{\geq 0}$ is the length function. The open Richardson variety

$$\mathring{X}_v^w \; := \; \mathring{X}_v \cap \mathring{X}^w$$

is irreducible and has dimension $\ell(w) - \ell(v)$ if $v \leq w$ in Bruhat order and otherwise it is empty. It is a log Calabi-Yau variety [11]. We pose the following:

Problem 1.1. What is the fundamental group of $\mathring{X}_v^w$?

Fundamental groups of log Calabi-Yau varieties arise in mirror symmetry, which is about equivalences of two apparently completely different physical theories. For instance, one mirror symmetry statement asserts that the small quantum cohomology of a Fano manifold Y should be isomorphic to the Jacobi ring of a holomorphic function $f \colon Z \to \mathbb{C}$ defined on an open Calabi-Yau variety Z [2, 5, 6]. Such pair (Z, f) is a Landau-Ginzburg model mirror to Y. The Jacobi ring of f is the coordinate ring of the critical points of f, and therefore the mirror space Z is not uniquely determined. Nevertheless, physicists

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expect a mirror with certain optimal physical properties. According to Kentaro Hori[1], one of these properties is manifested in the fundamental group, $\pi_1(Z)$, of Z as follows.

Assertion 1.2. *Let Y be a Fano manifold, and D be a specified anti-canonical divisor on Y. If $\mathrm{Aut}(Y, D)$ contains a maximal compact torus $(S^1)^m$, then an optimal mirror Landau-Ginzburg model (Z, f) should have $\pi_1(Z) = \mathbb{Z}^m$.*

We consider this when Z is an open Richardson variety $\mathring{X}_{w_P}^{w_0}$. Here, $P \supset B$ is a parabolic subgroup of G and w_0 (resp. w_P) is the longest element in W (resp. the Weyl group W_P of the Levi subgroup of P). This is a log Calabi-Yau variety, as it is isomorphic to the complement of the Knutson-Lam-Speyer [11] anti-canonical divisor $-K_{G/P}$ in the flag manifold G/P. Let $G^\vee$ (resp. $P^\vee$) denote the Langlands dual Lie group of G (resp. P). Rietsch [20] constructed a Landau-Ginzburg model $(\mathring{X}_{w_P}^{w_0}, f)$ mirror to the flag manifold $G^\vee/P^\vee$, assuming unpublished work of Peterson [18]. This has been verified when $G^\vee/P^\vee$ is a flag manifold of Lie type A [19] and when it is either a minuscule or a cominuscule flag variety [12, 16, 17]. The automorphism group of $G^\vee/P^\vee$ is $G^\vee$ (except for three special types of Grassmannians of Lie type B, C, or G_2 which are homogeneous with respect to a larger simple Lie group) [1]. The subgroup of $\mathrm{Aut}(G^\vee/P^\vee)$ that preserves $-K_{G^\vee/P^\vee}$ is a complex torus $(\mathbb{C}^\times)^{n-1}$, where G has rank $n-1$. Following Assertion 1.2 and the belief that Rietsch's mirror is optimal, we expect that $\pi_1(\mathring{X}_{w_P}^{w_0}) = \mathbb{Z}^{n-1}$. Our main result verifies this prediction when $G^\vee/P^\vee$ has Lie type A.

Theorem 1.3. *Let P be a parabolic subgroup of $SL(n, \mathbb{C})$. Then $\pi_1(\mathring{X}_{w_P}^{w_0}) = \mathbb{Z}^{n-1}$.*

A flag variety of Lie type A is determined by a sequence $n_\bullet : 0 < n_1 < \cdots < n_r < n$ of integers. The corresponding flag variety $\mathbb{F}\ell(n_\bullet)$ is the set of all sequences of subspaces

$$F_{n_1} \subset F_{n_2} \subset \cdots \subset F_{n_r} \subset \mathbb{C}^n \quad \text{where} \quad \dim F_i = i \, .$$

This is a subvariety of the product of Grassmannians $G(n_1, n) \times \cdots \times G(n_r, n)$. Under the Plücker embedding of $G(n_i, n)$ into the projective space $\mathbb{P}^{\binom{n}{n_i}-1}$, the flag variety $\mathbb{F}\ell(n_\bullet)$ has a Plücker embedding into the product $\mathbb{P}^{\binom{n}{n_1}-1} \times \cdots \times \mathbb{P}^{\binom{n}{n_r}-1}$. Although $\mathbb{F}\ell(n_\bullet)$ is a compactification of $\mathring{X}_{w_P}^{w_0}$ in this Plücker embedding, we prove Theorem 1.3 by considering a different compactification of $\mathring{X}_{w_P}^{w_0}$ in a single projective space. This allows us to reduce Theorem 1.3 to Zariski's classical case of a plane curve complement. We do this by investigating the intersections of the different irreducible components of the Knutson-Lam-Speyer [11] anti-canonical divisor $-K_{\mathbb{F}\ell(n_\bullet)}$, whose defining equations we also determine.

A projected Richardson variety $pr_P(X_v^w)$ is the image of a Richardson variety $X_v^w = X_v \cap X^w$ under the natural projection $pr_P : G/B \to G/P$. This enjoys many geometric properties of Richardson varieties, such as being normal, Cohen-Macaulay, and having rational singularities [3, 4, 11]. The union of certain projected Richardson hypersurfaces forms an anti-canonical divisor $-K_{G/P}$ of G/P [11]. Another main result is explicit defining equations in Theorem 4.1 for these projected Richardson hypersurfaces in terms of the Plücker coordinates when $G = SL(n, \mathbb{C})$. Each is given either by a single Plücker variable or by a bilinear quadric. For instance, $\mathbb{F}\ell(1, 3; 4) \subset \mathbb{P}^3 \times \mathbb{P}^3$ is the hypersurface $\mathcal{V}(x_1 x_{234} - x_2 x_{134} + x_3 x_{124} - x_4 x_{123})$ and $-K_{\mathbb{F}\ell(1,3;4)}$ is the divisor

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[1] Personal communication and talks.

$\mathcal{V}(x_1 x_{234}(x_1 x_{234} - x_2 x_{134}) x_4 x_{123})$. We expect these explicit defining equations to also be helpful in the study of the mirror symmetry for $\mathbb{F}\ell(n_\bullet)$, similar to the study of mirror symmetry for Grassmannians in [14].

The paper is organized as follows. We review basic facts on Richardson varieties in Section 2. We provide an expectation for the fundamental group $\pi_1(\mathring{X}_{\mathrm{id}}^v)$ in Section 3. For $G = SL(n, \mathbb{C})$, we derive the explicit defining equations of $-K_{G/P}$ in terms of the Plücker coordinates in Section 4, and then compute the fundamental group of the complement $-K_{G/P}$ in G/P in Section 5. Finally in Section 6, we provide the proof of Lemma 5.2.

2. OPEN RICHARDSON VARIETIES

Let G be a complex, simply-connected, simple Lie group of rank $n-1$, and $B \subset G$ be a Borel subgroup containing a maximal complex torus $\mathbb{T} \simeq (\mathbb{C}^\times)^{n-1}$. Let $\Delta = \{\alpha_1, \ldots, \alpha_{n-1}\}$ be a basis of simple roots in $(\mathrm{Lie}(\mathbb{T}))^*$. The Weyl group W of G is a Coxeter group generated by the simple reflections $\{s_\alpha \mid \alpha \in \Delta\}$, and is identified with the quotient $N_G(\mathbb{T})/\mathbb{T}$, where $N_G(\mathbb{T})$ is the normalizer of $\mathbb{T}$ in G. For each $u \in W$, choose a lift $\dot{u} \in N_G(\mathbb{T})$. The opposite Borel is $B^- := \dot{w}_0 B \dot{w}_0$, where w_0 is the longest element in W. The (opposite) Schubert cells

$$\mathring{X}_u := B^- \dot{u} B/B \cong \mathbb{C}^{\dim G/B - \ell(u)} \quad \text{and} \quad \mathring{X}^u := B \dot{u} B/B \cong \mathbb{C}^{\ell(u)}$$

are independent of choice of lift $\dot{u}$. Henceforth, we write u for $\dot{u}$.

The root system of (G, B) is $R := W \cdot \Delta = R^+ \sqcup (-R^+)$, where $R^+ := R \cap \bigoplus_{i=1}^{n-1} \mathbb{Z}_{\geq 0} \alpha_i$ is the set of positive roots. Each root $\gamma = w(\alpha_i) \in R$ gives a reflection $s_\gamma := w s_i w^{-1} \in W$, independent of the expressions for γ. The *Bruhat order* on W is the transitive closure of its covering relation, $u \lessdot v$ for $u, v \in W$ if $\ell(v) = \ell(u) + 1$ and $v = u s_\gamma$ for some $\gamma \in R$, where $\ell : W \to \mathbb{Z}_{\geq 0}$ is the length function. The *open Richardson variety*

$$\mathring{X}_v^u := \mathring{X}_v \cap \mathring{X}^u$$

is irreducible and of dimension $\ell(u) - \ell(v)$ if $v \leq u$, and otherwise it is empty. Its closure, a (closed) Richardson variety, is the intersection $X_v^u := X_v \cap X^u$ of (opposite) Schubert varieties X_v and X^u, which are closures of the corresponding Schubert cells. As $w_0^2 = \mathrm{id}$, we have the following identification of open Richardson varieties:

Proposition 2.1. *For any* $v \in W$, $\mathring{X}_v^{w_0} \cong \mathring{X}_{\mathrm{id}}^{w_0 v}$.

Proof. $\mathring{X}_v^{w_0} = w_0 B w_0 v B/B \cap B w_0 B/B \cong B w_0 v B/B \cap w_0 B w_0 B/B = X_{\mathrm{id}}^{w_0 v}$. $\square$

A proper parabolic subgroup $P \supsetneq B$ determines and is determined by a proper subset $\Delta_P \subsetneq \Delta$. The Weyl group W_P of (the Levi subgroup) of P is the subgroup of W generated by $\{s_\alpha \mid \alpha \in \Delta_P\}$. Let W^P be the set of minimal length coset representatives of W/W_P. We write pr_P for both the natural projection $G/B \to G/P$ and the map $W \to W^P$ determined by $w \in pr_P(w)W_P$. Then $pr_P(w_0) = w_0 w_P \in W^P$, where w_P is the longest element in W_P. Following [11], the *P-Bruhat order*, $\leq_P$, is the suborder of the Bruhat order whose covers are $u \lessdot_P v$ when $u \lessdot v$ and $pr_P(u) < pr_P(v)$. The varieties

$$\mathring{\Pi}_v^w := pr_P(\mathring{X}_v^w) \quad \text{and} \quad \Pi_v^w := pr_P(X_v^w)$$

are *open* and *closed projected Richardson varieties*, respectively. The next proposition is implicit in [11]. We explain how it follows from explicit results there.

Proposition 2.2. *The open Richardson variety $\mathring{X}^{w_0}_{w_P}$ is isomorphic to the complement in G/P of*

$$-K_{G/P} \; := \; \sum_{\mathrm{id}\,\lessdot\, u\leq_P w_0 w_P} pr_P(X^{w_0 w_P}_u) \; + \; \sum_{\mathrm{id}\,\leq_P v\,\lessdot\, w_0 w_P} pr_P(X^v_{\mathrm{id}})\,,$$

which is an anti-canonical divisor of G/P.

Proof. By Proposition 2.1, we have $\mathring{X}^{w_0}_{w_P} \cong \mathring{X}^{w_0 w_P}_{\mathrm{id}}$. As $\mathrm{id} \leq_P w_0 w_P$, we have $\dim \Pi^{w_0 w_P}_{\mathrm{id}} = \ell(w_0 w_P) = \dim G/P$ by [11, Corollary 3.2], and hence $\Pi^{w_0 w_P}_{\mathrm{id}} = G/P$. By [11, Lemma 3.1], $\mathring{X}^{w_0 w_P}_{\mathrm{id}} \cong \mathring{\Pi}^{w_0 w_P}_{\mathrm{id}}$. By [11, Proposition 3.6], we have $\Pi^{w_0 w_P}_{\mathrm{id}} \smallsetminus \mathring{\Pi}^{w_0 w_P}_{\mathrm{id}} = -K_{G/P}$. It follows again from [11, Proposition 3.6, Corollary 3.2] that $-K_{G/P}$ is the sum of all projected Richardson hypersurfaces in $\Pi^{w_0 w_P}_{\mathrm{id}}$, and hence it is an anti-canonical divisor of $\Pi^{w_0 w_P}_{\mathrm{id}}$ by [11, Lemma 5.4]. $\qquad\square$

3. Expectation for $\pi_1(\mathring{X}^v_{\mathrm{id}})$

The open Richardson variety $\mathring{X}^{w_0 w_P}_{\mathrm{id}}$ has the form $\mathring{X}^v_{\mathrm{id}}$ where $v \in W$. We begin with some well-known facts about fundamental groups.

Proposition 3.1 (Zariski Theorem of Lefschetz type [22]). *Let V be a hypersurface in $\mathbb{P}^N$. For almost every two-plane $\Lambda \subset \mathbb{P}^N$, the map*

$$\pi_1(\Lambda \smallsetminus V) \;\longrightarrow\; \pi_1(\mathbb{P}^N \smallsetminus V)$$

induced by the inclusion $(\Lambda \smallsetminus V) \hookrightarrow (\mathbb{P}^N \smallsetminus V)$ is an isomorphism.

Proposition 3.2 ([15]). *Let C_1 and C_2 be algebraic curves in $\mathbb{C}^2$. Assume that the intersection $C_1 \cap C_2$ consists of $d_1 d_2$ points where d_i is the degree of C_i. Then the fundamental group $\pi_1(\mathbb{C}^2 \smallsetminus C_1 \cup C_2)$ is isomorphic to the product $\pi_1(\mathbb{C}^2 \smallsetminus C_1) \times \pi_1(\mathbb{C}^2 \smallsetminus C_2)$.*

Subvarieties X and Y of projective or affine space meet *transversally* at a point $p \in X \cap Y$ if p is a smooth point of each and the defining equations for the tangent spaces T_pX and T_pY are in direct sum. They meet transversally if they are transverse at every point of their intersection, which implies that $X \cap Y$ is smooth and of the expected dimension. They meet *generically transversally* if the subset of points of $X \cap Y$ where they meet transversally is dense in every irreducible component of $X \cap Y$. The conditions in Proposition 3.2 on the curves C_1 and C_2 is that they meet transversally. Indeed, by Bézout's Theorem, their projective completions meet in $d_1 d_2$ isolated points, counted with multiplicity. As their intersection consists of $d_1 d_2$ points, they are transverse at every point of their intersection.

Proposition 3.3 (see e.g. Remark 2.13 (1) of [13]). *If C is a smooth algebraic curve in $\mathbb{C}^2$ whose projective completion is transverse to the line at infinity, then $\pi_1(\mathbb{C}^2 \smallsetminus C) = \mathbb{Z}$.*

Since $X_{\mathrm{id}} = G/B$, the Schubert cell $\mathring{X}_{\mathrm{id}}$ is the complement of the union of Schubert hypersurfaces X_{s_α} for $\alpha \in \Delta$. For $v \in W$, the Schubert cell $\mathring{X}^v \cong \mathbb{C}^{\ell(v)}$. Therefore,

$$\mathring{X}^v_{\mathrm{id}} \;=\; \mathring{X}^v \cap \mathring{X}_{\mathrm{id}} \;=\; \mathring{X}^v \cap X_{\mathrm{id}} \smallsetminus X^v \cap \partial X_{\mathrm{id}}$$

$$=\; \mathbb{C}^{\ell(v)} \smallsetminus X^v \cap \cup_{\alpha \in \Delta} X_{s_\alpha} \;=\; \mathbb{C}^{\ell(v)} \smallsetminus \bigcup_{\alpha \in \Delta} X^v_{s_\alpha}\,.$$

The Richardson variety $X_{s_\alpha}^v$ has dimension $\ell(v)-1$ (and contains $\mathring{X}_{s_\alpha}^v$ as a Zariski open dense subset) if $s_\alpha \le v$, and otherwise it is empty. A Richardson variety is reduced and normal, and thus its singular set has codimension at least two. Therefore, if $\Lambda \subset \mathbb{C}^{\ell(v)} = \mathring{X}^v$ is a general affine two-plane, then $C_\alpha := X_{s_\alpha}^v \cap \Lambda$ is a smooth curve in Λ, whenever $s_\alpha \le v$. If these curves satisfy the hypotheses of Propositions 3.2 and 3.3, we are led to the following expectation. For any $v \in W$, define $\Gamma(v) := \{\alpha \in \Delta \mid s_\alpha \le v\}$.

Conjecture 3.4. We have $\pi_1(\mathring{X}_{\mathrm{id}}^v) = \mathbb{Z}^{|\Gamma(v)|}$.

Example 3.5. The flag manifold $SL(3,\mathbb{C})/B = \{F_1 \subset F_2 \subset \mathbb{C}^3 \mid \dim F_i = i\}$ is the hypersurface $\mathcal{V}(x_1 x_{23} - x_2 x_{13} + x_3 x_{12})$ in $\mathbb{P}^2 \times \mathbb{P}^2$, where $[x_1, x_2, x_3]$ are coordinates for the first $\mathbb{P}^2$ and $[x_{12}, x_{13}, x_{23}]$ are coordinates for the second. The Schubert cell $\mathring{X}_{\mathrm{id}}$ (resp. $\mathring{X}^{w_0}$) is the subset of this hypersurface where $x_1 x_{12} \ne 0$ (resp. $x_3 x_{23} \ne 0$). Dehomogenizing by setting $x_1 = x_{12} = 1$, writing the remaining coordinates as $(z_2, z_3, z_{13}, z_{23}) \in \mathbb{C}^4$, and using the equation $0 = z_{23} - z_2 z_{13} + z_3$ to solve for z_{23}, we obtain

$$\mathring{X}_{\mathrm{id}}^{w_0} = \{(z_2, z_3, z_{13}) \in \mathbb{C}^3 \mid z_3 \ne 0, z_2 z_{13} - z_3 \ne 0\}.$$

This is the complement in $\mathbb{C}^3$ of two smooth hypersurfaces whose intersection is transverse away from $(0,0,0)$. Intersecting with a general two-plane Λ gives two smooth curves in Λ that satisfy the hypotheses of Propositions 3.2 and 3.3. Thus $\pi_1(\mathring{X}_{\mathrm{id}}^{w_0}) = \mathbb{Z}^2$.

The Schubert subvariety $X^{s_1 s_2}$ of $SL(3,\mathbb{C})/B$ is $\mathcal{V}(x_3, x_1 x_{23} - x_2 x_{13})$. The Schubert cell $\mathring{X}^{s_1 s_2}$ is the subset where $x_2 x_{23} \ne 0$. Setting $x_2 = x_{23} = 1$ and using z_* for the remaining coordinates, gives $\mathring{X}^{s_1 s_2} = \{(z_1, z_{12}, z_{13}) \in \mathbb{C}^3 \mid z_1 - z_{13} = 0\}$. Solving for z_{13}, we obtain $\mathring{X}_{\mathrm{id}}^{s_1 s_2} = \{(z_1, z_{12}) \in \mathbb{C}^2 \mid z_1 z_{12} \ne 0\}$, which shows that $\pi_1(\mathring{X}_{\mathrm{id}}^{s_1 s_2}) = \mathbb{Z}^2$. Fundamental groups of the remaining open Richardson varieties in $SL(3,\mathbb{C})/B$ are as follows.

v	id	s_1	s_2	$s_1 s_2$	$s_2 s_1$	w_0
$\Gamma(v)$	$\emptyset$	$\{\alpha_1\}$	$\{\alpha_2\}$	$\{\alpha_1, \alpha_2\}$	$\{\alpha_1, \alpha_2\}$	$\{\alpha_1, \alpha_2\}$
$\pi_1(\mathring{X}_{\mathrm{id}}^v)$	$\{\mathrm{id}\}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$

$\diamond$

We establish some lemmas that will help to rewrite the expression for $-K_{G/P}$ from Proposition 2.2. They use basic facts about reflection groups as could be found in, for example [9, §1].

Lemma 3.6. (1) If $w = s_{i_1} \cdots s_{i_m} \in W^P$ is a reduced expression of w, then $s_{i_j} \cdots s_{i_m}$ is also in W^P and is again a reduced expression (of length $(m - j + 1)$).
(2) If $\beta \in \Delta_P$ and $v \in W^P$ satisfy both $s_\beta \not\le v$ and $s_\beta v \ne v s_\beta$, then $\ell(s_\beta v) = \ell(v) + 1$ and $s_\beta v \in W^P$.

Proof. (1) $w \in W^P$ if and only if $\ell(w s_\alpha) = \ell(w) + 1$ for all $\alpha \in \Delta_P$. Since the given expression of w is reduced, we have $\ell(s_{i_j} \cdots s_{i_m} s_\alpha) = (m - j + 1) + 1 = \ell(s_{i_j} \cdots s_{i_m}) + 1$ for any $\alpha \in \Delta_P$. Hence, $s_{i_j} \cdots s_{i_m} \in W^P$ and it is a reduced expression.

(2) Since $s_\beta \not\le v$, any reduced expression of v^{-1} does not contain s_β, and hence $v^{-1}(\alpha) \in R^+$. Thus $\ell(s_\beta v) = \ell(v^{-1} s_\beta) = \ell(v^{-1}) + 1 = \ell(v) + 1$.

For any $\alpha \in \Delta_P$, we have $v(\alpha) \in R^+$ as $v \in W^P$; we claim $s_\beta v(\alpha) \in R^+$ for all such α and hence $s_\beta v \in W^P$. Indeed, if $\alpha \ne \beta$, then we have $v(\alpha) \ne \beta$, as any reduced expression of v does not contain s_β. Moreover, $v(\beta) \ne \beta$ (otherwise $v s_\beta v^{-1} = s_\beta$, contradicting to the hypothesis). Therefore the claim holds by noting $s_\beta(R^+ \setminus \{\beta\}) = R^+ \setminus \{\beta\}$. $\square$

Lemma 3.7. *For any parabolic subgroup P, we have $\Gamma(w_0 w_P) = \Delta$.*

Proof. For any $\alpha \in \Delta \smallsetminus \Delta_P$, we have $w_P(\alpha) > 0$ and thus $w_0 w_P(\alpha) < 0$. Consequently, $w_0 w_P$ has a reduced expression ending with s_α (by [9, §1.7 Exchange Condition]). Thus $w_0 w_P \geq s_\alpha$ and $\alpha \in \Gamma(w_0 w_P)$. It remains to show $\Delta_P \subset \Gamma(w_0 w_P)$.

If $\Delta_P \not\subset \Gamma(w_0 w_P)$, then there exists $\alpha \in \Delta_P$ such that $s_\alpha \not\leq w_0 w_P$. Since the Dynkin diagram of Δ is a tree, there exist $\{\beta_1, \ldots, \beta_m\}$ satisfying both (1) $\beta_1 = \alpha$, $\{\beta_1, \ldots, \beta_{m-1}\} \subset \Delta_P$, $\beta_m \in \Delta \setminus \Delta_P$, and (2) β_i is adjacent to β_{i+1} for $i = 1, \ldots, m-1$. Then for $\gamma := s_{\beta_1} \cdots s_{\beta_{m-1}}(\beta_m) = \sum_{j=1}^{m} a_j \beta_j$ with $a_j > 0$ for all j, we have $w_P(\gamma) = w_P(\sum_{j=1}^{m-1} a_j \beta_j) + w_P(a_m \beta_m) > 0$ and consequently $w_0 w_P(\gamma) < 0$. However, $w_0 w_P$ is in the Weyl subgroup generated by $\{s_\beta \mid \beta \in \Delta \smallsetminus \{\alpha\}\}$, by the hypothesis $s_\alpha \not\leq w_0 w_P$. Thus we deduce a contradiction by noting $w_0 w_P(\gamma) = w_0 w_P(a_1 \alpha) + w_0 w_P(\sum_{j=2}^{m} a_j \beta_j) > 0$. $\square$

For general G/P, the expectation $\pi_1(\mathring{X}_{\mathrm{id}}^{w_0 w_P}) \simeq \mathbb{Z}^{|\Delta|}$ would follow from Conjecture 3.4 and Lemma 3.7. We refine the description of $-K_{G/P}$ of Proposition 2.2. Moreover, we have the following.

Lemma 3.8. *Let $u \in W$. Then we have*
  (1) *$\mathrm{id} \lessdot u \leq_P w_0 w_P$ if and only if $u = s_\alpha$ for some $\alpha \in \Delta$.*
  (2) *$\mathrm{id} \leq_P u \lessdot w_0 w_P$ if and only if $u = pr_P(w_0 s_\alpha) = w_0 s_\alpha w_P$ for $\alpha \in \Delta \smallsetminus \Delta_P$.*

Proof. If $\mathrm{id} \lessdot u$, then $u = s_\alpha$ for some $\alpha \in \Delta$. If $\alpha \in \Delta \smallsetminus \Delta_P$, then as in the beginning of the proof of Lemma 3.7, $w_0 w_P$ admits a reduced expression $w_0 w_P = s_{i_1} \cdots s_{i_l}$ where $l = \ell(w_0 w_P)$ and $s_{i_l} = s_\alpha$. For $j = 1, \ldots, l$, set $v_j := s_{i_{l-j+1}} \cdots s_{i_{l-1}} s_{i_l}$. Then we have $s_\alpha = v_1 \lessdot \cdots \lessdot v_l = w_0 w_P$ with $v_j \in W^P$ for any j by Lemma 3.6 (1), which implies $pr_P(v_j) = v_j$. Hence, $s_\alpha \leq_P w_0 w_P$. If $\beta \in \Delta_P$, then we still have $s_\beta \leq w_0 w_P$ by Lemma 3.7, so s_β must occur in the aforementioned reduced expression of $w_0 w_P$. Let $m := \max\{j \mid s_{i_j} = s_\beta\}$. Set $u_j = s_{i_{l-j+1}} \cdots s_{i_{l-1}} s_{i_l}$ if $l - m + 1 \leq j \leq l$, $u_j = s_{i_m} s_{i_{l-j+2}} \cdots s_{i_{l-1}} s_{i_l}$ if $2 \leq j \leq l - m$ (in which case $s_\beta u_j = s_{i_{l-j+2}} \cdots s_{i_l} \in W^P$ by Lemma 3.6 (1), and we will discuss whether it commutes with s_β), and $u_1 = s_{i_m}$. Then we have $s_\beta = u_1 \lessdot \cdots \lessdot u_l = w_0 w_P$. For $j \leq l - m$, we notice that if $u_j = s_{i_m} u_j s_{i_m}$, then $u_r = s_{i_m} u_r s_{i_m}$ and $s_{i_m} u_r \in W^P$ hold for any $r \leq j$, and if $u_j \neq s_{i_m} u_j s_{i_m}$ then $u_j \in W^P$ by Lemma 3.6 (2). It follows that $s_\beta \leq_P w_0 w_P$ by definition.

If $\mathrm{id} \leq_P u$, then by definition we have $\ell(pr_P(u)) - \ell(pr_P(\mathrm{id})) \geq \ell(u) - \ell(\mathrm{id})$, implying that $\ell(pr_P(u)) \geq \ell(u)$ and hence $u \in W^P$. Together with $u \lessdot w_0 w_P$, it follows that $\ell(u) = \ell(w_0 w_P) - 1 = \ell(w_0) - \ell(w_P) - 1$, so that $\ell(u w_P) = \ell(w_0) - 1$. Hence, $u w_P = w_0 s_\alpha$ for some $\alpha \in \Delta$. This further implies $u = w_0 s_\alpha w_P$, and hence $\ell(w_0) - \ell(w_P) - 1 = \ell(u) = \ell(w_0 s_\alpha w_P) = \ell(w_0) - \ell(s_\alpha w_P)$. Therefore $\ell(s_\alpha w_P) = \ell(w_P) + 1$, implying $\alpha \notin \Delta_P$. On the other hand, for $\alpha \in \Delta \smallsetminus \Delta_P$, for any $\gamma \in R_P^+$, we have $w_P(\gamma) \in R_P^-$, implying $s_\alpha w_P(\gamma) \in R^-$ and hence $w_0 s_\alpha w_P(\gamma) \in R^+$. Therefore $pr_P(w s_\alpha) = w s_\alpha w_P \in W^P$ and $\mathrm{id} \leq_P w s_\alpha w_P$ for any such α. $\square$

Proposition 3.9. $-K_{G/P} = \sum_{\alpha \in \Delta} pr_P(X_{s_\alpha}^{w_0 w_P}) + \sum_{\alpha \in \Delta \smallsetminus \Delta_P} pr_P(X_{\mathrm{id}}^{w_0 s_\alpha w_P}).$

Proof. This is a direct consequence of Proposition 2.2 and Lemma 3.8. $\square$

4. Defining equations of $-K_{SL(n,\mathbb{C})/P}$

Henceforth, we assume that $G = SL(n,\mathbb{C})$. Then $SL(n,\mathbb{C})/P = \mathbb{F}\ell(n_\bullet)$ is the manifold of partial flags $F_\bullet : F_{n_1} \subset \cdots \subset F_{n_r} \subset \mathbb{C}^n$ of type $n_\bullet$ ($\dim F_{n_i} = n_i$). Here $n_\bullet := 1 \leq n_1 < \cdots < n_r < n$ is an increasing sequence of integers and P is the parabolic subgroup corresponding to the roots not in $n_\bullet$, so that $\Delta_P = \{\alpha_i \mid i \notin n_\bullet\}$. Also, $W = S_n$ is the symmetric group generated by simple transpositions $\{s_i \mid 1 \leq i \leq n-1\}$.

The natural embedding of $\mathbb{F}\ell(n_\bullet)$ into the product

$$G(n_1, n) \ \times \ G(n_2, n) \ \times \ \cdots \ \times \ G(n_r, n)$$

of Grassmannians and then into the product $\mathbb{P}(\wedge^{n_1}\mathbb{C}^n) \times \cdots \times \mathbb{P}(\wedge^{n_r}\mathbb{C}^n)$ of Plücker spaces gives Plücker coordinates x_J for $\mathbb{F}\ell(n_\bullet)$. We describe their indexing. For any positive integer m, set $[m] := \{1, \ldots, m\}$ and write $\binom{[m]}{j}$ for the set of subsets J of $[m]$ of cardinality j, which we always write as increasing sequences. There is a Plücker coordinate x_J for $\mathbb{F}\ell(n_\bullet)$ for every $J \in \binom{[n]}{n_j}$, for each $j = 1, \ldots, r$.

Let us explain x_J concretely in terms of local coordinates for $\mathbb{F}\ell(n_\bullet)$. A point $F_\bullet \in \mathbb{F}\ell(n_\bullet)$ is represented by a $n_r \times n$ matrix $A_\bullet$ of full rank n_r, where F_{n_j} is the row space of the first n_j rows of $A_\bullet$. For $J \in \binom{[n]}{n_j}$, the Plücker coordinate x_J of $F_\bullet$ is the determinant of the $n_j \times n_j$ submatrix of $A_\bullet$ formed by the first n_j rows and the columns from J. This is the Jth minor of the matrix formed by the first n_j rows of $A_\bullet$.

For $a < b \leq n$, let us write $(a, b]$ for the set $\{a+1, \ldots, b\}$ and $[a, b)$ for $\{a, \ldots, b-1\}$. Note that $(0, i] = [i]$. If $J \subset [a]$ and $J' \subset (a, n]$, then J, J' is the index $J \cup J' \subset [n]$.

Elements of W^P index Schubert varieties in $\mathbb{F}\ell(n_\bullet)$, while elements of S_n index Schubert varieties in $\mathbb{F}\ell(n) = G/B$. The Richardson variety $X_{\mathrm{id}}^{w_0 w_P}$ projects birationally onto $\mathbb{F}\ell(n_\bullet)$, under the map $pr_P \colon \mathbb{F}\ell(n) \to \mathbb{F}\ell(n_\bullet)$. We describe explicit equations for the irreducible components of $-K_{\mathbb{F}\ell(n_\bullet)}$, which were identified in Proposition 3.9.

Theorem 4.1. *Let* $i \in [n-1]$.

(1) *For* $i \in n_\bullet$, $pr_P(X_{\mathrm{id}}^{w_0 s_i w_P})$ *is the Schubert divisor of* $\mathbb{F}\ell(n_\bullet)$ *defined by the Plücker coordinate hyperplane* $x_{(n-i,n]} = 0$.

(2) *When* $i \in n_\bullet$, $pr_P(X_{s_i}^{w_0 w_P})$ *is the Schubert divisor of* $\mathbb{F}\ell(n_\bullet)$ *defined by the Plücker coordinate hyperplane* $x_{[i]} = 0$.

(3) *When* $i < n_1$, $pr_P(X_{s_i}^{w_0 w_P})$ *is given by* $x_{[i],(n-n_1+i,n]} = 0$.

(4) *When* $i > n_r$, $pr_P(X_{s_i}^{w_0 w_P})$ *is given by* $x_{(i-n_r,i]} = 0$.

(5) *When* $n_j < i < n_{j+1}$ *with* $j \in [r-1]$, *set* $k := i - n_j$ *and* $l := \min\{i, n-n_{j+1}+k\}$. *The projected Richardson hypersurface* $pr_P(X_{s_i}^{w_0 w_P})$ *is given by*

$$\text{(4.1)} \qquad \sum_{J \in \binom{[l]}{k}} (-1)^{|J|} x_{[i] \smallsetminus J} \cdot x_{J,(n-n_{j+1}+k,n]} \ = \ 0 \ ,$$

where $|J|$ *is the sum of the elements in* J.

Proof. We start with the most involved case (5). As a first check, note that in (4.1) the first Plücker coordinate $x_{[i] \smallsetminus J}$ has n_j indices, while $x_{J,(n-n_{j+1}+k,n]}$ has n_{j+1} indices. To prove Statement (5), set $a_1 := n_1$ and $a_i := n_i - n_{i-1}$ for $i = 2, \ldots, r$. We start with a

structured matrix parameterizing the Schubert cell $\mathring{X}^{w_0 w_P}$, which has the block form

$$
(4.2) \qquad
\begin{pmatrix}
* & * & \cdots & * & * & I_{a_1} \\
* & * & \cdots & * & I_{a_2} & 0 \\
* & * & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & 0 & 0 \\
\vdots & \vdots & \cdot^{\cdot^{\cdot}} & 0 & 0 & 0 \\
* & I_{a_r} & 0 & \cdots & 0 & 0
\end{pmatrix}_{n_r \times n}
\qquad .
$$

Here, I_a is the $a \times a$ identity matrix. Observe that the first column block has $n - n_r$ columns. The hypersurface Schubert variety X_{s_i} in G/B is defined by the single Plücker coordinate $x_{[i]}$, which is not a Plücker coordinate on G/P when $i \notin n_\bullet$. Our equation for $pr_P(X_{s_i}^{w_0 w_P})$ is obtained by evaluating $x_{[i]}$ on the coordinates (4.2) for $\mathring{X}^{w_0 w_P}$ and expressing it in terms of the Plücker coordinates for G/P.

To that end, suppose that $i \notin n_\bullet$, and for now that $n_j < i < n_{j+1}$ as above. Consider the first i rows of (4.2),

$$
\begin{pmatrix}
* & * & * & \cdots & * & * & I_{a_1} \\
* & * & * & \cdots & * & I_{a_2} & 0 \\
\vdots & \vdots & \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & 0 & 0 \\
* & * & * & I_{a_j} & 0 & 0 & 0 \\
* & I_k & 0_{k,a_{j+1}-k} & 0 & \cdots & 0 & 0
\end{pmatrix}_{i \times n}
\qquad .
$$

Here, $0_{k,a_{j+1}-k}$ is the zero matrix with k rows and $a_{j+1}-k$ columns, and the first column block has size $n - n_{j+1}$. The Plücker coordinate $x_{[i]}$ is the determinant of the first i columns of this matrix. Use Laplace expansion on the last k rows to get

$$
x_{[i]} \;=\; \sum_{J \in \binom{[i]}{k}} x_{[i] \smallsetminus J} \cdot z_J \;,
$$

where z_J is the Jth minor of the last k rows, $\begin{pmatrix} * & I_k & 0_{k,a_{j+1}-k} & 0 & \cdots \end{pmatrix}$. Its last nonzero column is in position $n - n_{j+1} + k$, so we may assume that $J \subset [l]$, as otherwise $z_J = 0$.

If we consider the form of the matrix (4.2) (specifically, its first n_{j+1} rows), then we see that $z_J = \pm x_{J,(n-n_{j+1}+k,n]}$, as the columns in the final positions in $(n - n_{j+1} + k, n]$ all end with a 1 in rows $1, \ldots, n_j, n_j + k + 1, \ldots, n_{j+1}$. Rather than compute the sign, we note that the sign does not depend upon J, but only on $n_\bullet$ and i. Hence, $pr_P(X_{s_i}^{w_0 w_P})$ satisfies the formula (4.1), and then this completes the proof, by noting that the hypersurface of G/P defined by (4.1) is irreducible.

The arguments for cases (2), (3), (4) are similar and much simpler. Case (1) follows from case (2) by noting $pr_P(X_{\mathrm{id}}^{w_0 s_i w_P}) = w_0 pr_P(X_{s_i}^{w_0 w_P})$ for $i \in n_\bullet$. $\qquad \square$

Example 4.2. For $\mathbb{F}\ell(3,6;7)$, $pr_P(X_{s_4}^{w_0 w_P})$ is given by $x_{234}x_{134567} - x_{134}x_{234567} = 0$, and $pr_P(X_{s_5}^{w_0 w_P})$ is given by $x_{145}x_{234567} - x_{245}x_{134567} + x_{345}x_{124567} = 0$. $\qquad \diamond$

5. Fundamental group of the complement of $-K_{\mathbb{F}\ell(n_\bullet)}$ in $\mathbb{F}\ell(n_\bullet)$

To study $\mathbb{F}\ell(n_\bullet) \smallsetminus (-K_{\mathbb{F}\ell(n_\bullet)})$, we first remove the Schubert divisors (1) in Theorem 4.1. These are given by the Plücker coordinates $x_{(n-n_j,n]}$ for $n_j \in n_\bullet$, and correspond to the second sum in Proposition 3.9. This leaves the dense Schubert cell of $\mathbb{F}\ell(n_\bullet)$, which is

identified with $\mathring{X}^{w_0 w_P}$, and is parameterized by the coordinates (4.2). Let $N := \ell(w_0 w_P)$ so that $\mathring{X}^{w_0 w_P} \simeq \mathbb{C}^N$, and let $\mathbb{P}^N := \mathbb{C}^N \sqcup \mathbb{P}(\mathbb{C}^N)$ be its projective completion.

For any subvariety D of $\mathbb{F}\ell(n_\bullet)$ which meets the cell $\mathring{X}^{w_0 w_P}$, we also write D for its closure in $\mathbb{P}^N$. Write D_0 for the hyperplane $\mathbb{P}(\mathbb{C}^N)$ at infinity and for $i = 1, \ldots, n-1$, let $D_i := pr_P(X_{s_i}^{w_0 w_P}) \cap \mathring{X}^{w_0 w_P}$ be the image of a projected Richardson variety that meets $\mathring{X}^{w_0 w_P}$. Set $D := D_0 \cup D_1 \cup \cdots \cup D_{n-1}$, a divisor in $\mathbb{P}^N$.

Theorem 5.1. *The fundamental group of $\mathbb{F}\ell(n_\bullet) \smallsetminus (-K_{\mathbb{F}\ell(n_\bullet)})$ is $\mathbb{Z}^{n-1}$.*

Proof. Since $\mathbb{P}^N \smallsetminus D \simeq \mathbb{F}\ell(n_\bullet) \smallsetminus (-K_{\mathbb{F}\ell(n_\bullet)})$, we study the fundamental group of the hypersurface complement $\mathbb{P}^N \smallsetminus D$. Let $\Lambda \subset \mathbb{P}^N$ be a general two-plane. For $i = 0, \ldots, n-1$, set $C_i := \Lambda \cap D_i$ and set $C := \Lambda \cap D$, which are curves, as Λ is general. We claim that:
    (1) Each curve C_i is smooth.
    (2) For $i \neq j$, then intersection $C_i \cap C_j$ is transverse.
    (3) For i, j, k distinct $C_i \cap C_j \cap C_k = \emptyset$.
Given these claims, Propositions 3.2 and 3.3 imply that $\pi_1(\Lambda \smallsetminus C) = \mathbb{Z}^{n-1}$, and Proposition 3.1 implies $\pi_1(\mathbb{P}^N \smallsetminus D) = \mathbb{Z}^{n-1}$, which implies the theorem.

By Bertini's Theorem and the genericity of Λ, these three properties of the curves C_i are consequences of the following three properties of the divisors D_i.
    (1) Each D_i is smooth in codimension 1.
    (2) For $i \neq j$, the intersection $D_i \cap D_j$ is generically transverse.
    (3) For i, j, k distinct, the intersection $D_i \cap D_j \cap D_k$ has codimension three.

The hyperplane D_0 at infinity in $\mathbb{P}^N$ is smooth. For $0 < i$ the intersection $D_i \cap \mathbb{C}^N$ with the complement of D_0 is isomorphic to an open part of the projected Richardson variety $pr_P(X_{s_i}^{w_0 w_P})$ in $\mathbb{F}\ell(n_\bullet)$. Projected Richardson varieties are normal [3, 4, 11], and thus smooth in codimension 1. Therefore, the first property is satisfied.

For the second, we notice that for $0 < i < j$, the intersection $X_{s_i} \cap X_{s_j}$ is given by $X_{s_i s_j}$ if $j > i+1$, or $X_{s_i s_j} \cup X_{s_j s_i}$ if $j = i+1$, and in either case the intersection is reduced. It follows from the defining equations that $pr_P(X_{s_i}^{w_0 w_P}) \cap pr_P(X_{s_j}^{w_0 w_P}) = pr_P(X_{s_i}^{w_0 w_P} \cap X_{s_j}^{w_0 w_P})$, and hence the intersection is given by $pr_P(X_{s_i s_j}^{w_0 w_P})$ if $j > i+1$, or $pr_P(X_{s_i s_j}^{w_0 w_P}) \cup pr_P(X_{s_j s_i}^{w_0 w_P})$ if $j = i+1$. Thus the intersection $D_i \cap D_j$ is generically transverse for $0 < i < j$.

To show that $D_0 \cap D_i$ is generically transverse, we study the equations for D_i. The divisor D_i is defined by the determinant $f^{(i)}$ of the upper left $i \times i$ submatrix of the local coordinates (4.2). Write $f^{(i)}$ as a sum of homogeneous pieces,

$$f^{(i)} = f_{d_i}^{(i)} + f_{d_i-1}^{(i)} + \cdots + f_0^{(i)},$$

where $\deg f_j^{(i)} = j$ and $\deg f^{(i)} = d_i$. If z is a new homogenizing variable, so that $z = 0$ defines the hyperplane D_0 at infinity in $\mathbb{P}^N$, then D_i is defined in $\mathbb{P}^N$ by

$$(5.1) \qquad f_{d_i}^{(i)} + z f_{d_i-1}^{(i)} + z^2 f_{d_i-2}^{(i)} + \cdots + z^{d_i} f_0^{(i)}.$$

Lemma 5.2. *With these definitions, we have the following.*
    (1) *For each $i \in [n-1]$, the top homogeneous component $f_{d_i}^{(i)}$ of D_i is square-free. If $f^{(i)}$ is inhomogeneous, then its second highest homogeneous component $f_{d_i-1}^{(i)}$ is nonzero and coprime to $f_{d_i}^{(i)}$.*

(2) *For $i, j \in [n{-}1]$ with $i \neq j$, the top homogeneous components of $f^{(i)}$ and $f^{(j)}$ are coprime.*

We will prove this in Section 6 and assume it for now. Then $D_0 \cap D_i = \mathcal{V}(z, f^{(i)})$ is defined in D_0 by the top homogeneous component $f_{d_i}^{(i)}$ of $f^{(i)}$. Since $f_{d_i}^{(i)}$ is square-free, $D_0 \cap D_i$ is reduced in the plane D_0. When $f^{(i)}$ is homogeneous, this shows that the intersection is generically transverse. When $f^{(i)}$ is inhomogeneous, the intersection will be nontransverse where $\mathcal{V}(f_{d_i}^{(i)})$ is singular, and at points of $\mathcal{V}(f_{d_i}^{(i)}, f_{d_i-1}^{(i)})$. Since $f_{d_i}^{(i)}$ and $f_{d_i-1}^{(i)}$ are coprime, we see again that the intersection is generically transverse.

Consider now the final point, that for $i < j < k$, $D_i \cap D_j \cap D_k$ has codimension three. If $i \neq 0$, then this follows from the same fact about the Richardson divisors. We may also see this from the defining equations, which give the following four cases.

i, j, k	$D_i \cap D_j \cap D_k$
$i < j - 1 < k - 2$	$pr_P(X_{s_i s_j s_k}^{w_0 w_P})$
$i = j - 1 < k - 2$	$pr_P(X_{s_i s_j s_k}^{w_0 w_P}) \cup pr_P(X_{s_j s_i s_k}^{w_0 w_P})$
$i < j - 1 = k - 2$	$pr_P(X_{s_i s_j s_k}^{w_0 w_P}) \cup pr_P(X_{s_i s_k s_j}^{w_0 w_P})$
$i = j - 1 = k - 2$	$pr_P(X_{s_i s_j s_k}^{w_0 w_P}) \cup pr_P(X_{s_j s_i s_k}^{w_0 w_P}) \cup pr_P(X_{s_k s_i s_j}^{w_0 w_P}) \cup pr_P(X_{s_k s_j s_i}^{w_0 w_P})$

If $i = 0$, then this holds as $f_{d_j}^{(j)}$ and $f_{d_k}^{(k)}$ are coprime. $\qquad\square$

6. Proof of Lemma 5.2

Let M be a principal $a \times a$ submatrix of (4.2). We will later show that its determinant equals the determinant of a matrix with a block form (6.1) described below. Consequently, we first investigate the factorization of the top homogeneous component of the determinant of such a matrix, and use that to deduce Lemma 5.2. Until we deduce Lemma 5.2 at the end of this section, all symbols, N, r, etc. will have different meanings than in Sections 4 and 5. We start with a well-known fact, as we will use similar arguments later.

Lemma 6.1. *The determinant $\det\big(x_{ij}\big)_{a \times a}$ of a matrix of indeterminates is irreducible.*

Proof. Let g be this determinant, and note that it has degree one in every variable x_{ij}. Suppose that $g = pq$. We may assume that x_{11} appears in p, so that p is of degree one in x_{11}. Then x_{1j} appears in p for all j, for otherwise x_{1j} appears in q, which implies that $x_{11}x_{1j}$ appears in g, which is a contradiction. Similarly, x_{j1} appears in p, and then similar arguments show that each x_{jk} appears in p. Consequently, q is constant. $\qquad\square$

For sequences $i_\bullet := (i_1, i_2, \ldots, i_r)$ and $j_\bullet := (j_1, j_2, \ldots, j_r)$ of positive integers with $|i_\bullet| = |j_\bullet| = N$, consider a matrix of the following form,

$$(6.1) \qquad M(i_\bullet; j_\bullet) := \begin{pmatrix} * & \cdots & \cdots & * & * & A^1 \\ * & \cdots & \cdots & * & A^2 & I_{i_2, j_1} \\ \vdots & \cdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & 0 \\ * & \cdots & A^s & I_{i_s, j_{s-1}} & 0 & 0 \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & 0 & 0 \\ A^r & I_{i_r, j_{r-1}} & 0 & \cdots & 0 & 0 \end{pmatrix}_{N \times N} .$$

Here, $I_{c,d}$ is a $c \times d$ matrix with 1s on its diagonal and 0s elsewhere, the blocks A^r are $i_r \times j_r$ matrices of indeterminates, and every $*$ denotes another matrix of indeterminates. As the entries of $M(i_\bullet; j_\bullet)$ that are not specified to be 0 or 1 are different indeterminates, all properties of its determinant $g = g(i_\bullet; j_\bullet)$ depend only upon the sequences $i_\bullet$ and $j_\bullet$. This includes whether or not $g = 0$, its degree, its irreducibility and factorization, as well as the same properties of its top degree homogeneous component.

We need not determine whether $g = 0$, or if it is irreducible, or its degree. We do study the factorization of its top degree homogeneous component. For this, we set

$$\Upsilon(i_\bullet; j_\bullet) := \{s \in [r{-}1] \mid i_1 + \cdots + i_s = j_1 + \cdots + j_s\}.$$

We show that this set controls the factorization of the top homogeneous component of the determinant g of M. For a polynomial f, let $\mathrm{top}(f)$ be the top homogeneous component of f and $\mathrm{snd}(f)$ be the homogeneous component of f of degree $\deg(f){-}1$.

Lemma 6.2. *Let g be the determinant of the matrix $M(i_\bullet; j_\bullet)$ (6.1). Assume that g is irreducible and nonzero. Then $\mathrm{top}(g)$ is reducible if and only if $\Upsilon(i_\bullet; j_\bullet) \neq \emptyset$.*

Proof. Suppose that $\Upsilon(i_\bullet; j_\bullet) \neq \emptyset$. Let $s \in \Upsilon(i_\bullet; j_\bullet)$ and observe that removing I_{i_{s+1}, j_s} from (6.1) gives a block upper left triangular matrix, $\left(\begin{smallmatrix} * & * \\ * & 0 \end{smallmatrix}\right)$. Using Laplace expansion of g along the first $i_1 + \cdots + i_s$ rows of $M(i_\bullet; j_\bullet)$, gives

$$g = \pm \begin{vmatrix} \cdots & * & * & A^{s+1} \\ * & * & A^{s+2} & I_{i_{s+2}, j_{s+1}} \\ \vdots & \ddots & \ddots & 0 \\ A^r & I_{i_r, j_{r-1}} & 0 & 0 \end{vmatrix} \cdot \begin{vmatrix} \cdots & * & * & A^1 \\ * & * & A^2 & I_{i_2, j_1} \\ \vdots & \ddots & \ddots & 0 \\ A^s & I_{i_s, j_{s-1}} & 0 & 0 \end{vmatrix}$$

$$+ \ (\text{other terms}).$$

In the other terms, at least one column of the first minor is from the lower right submatrix

$$\begin{pmatrix} I_{i_{s+1}, j_s} & 0 \\ 0 & 0 \end{pmatrix},$$

and thus its degree is strictly less than that of the first minor in the first term. Indeed, the minor is zero if any column is zero, and if not, then expanding that minor along the rows containing 1s from I_{i_{s+1}, j_s} shows that its degree drops by the number of such rows/columns. However, the second minor has the same degree as the second minor in the first term (as they have the same format $M(i_1, \ldots, i_s; j_1, \ldots, j_s)$). Since we assumed that $g \neq 0$, these second minors are all nonzero, and we conclude that the degree of the other terms is strictly less than that of the first term. Therefore, $\mathrm{top}(g)$ is given by the product of top homogeneous components of the two minors in the first term of g, neither of which is a constant (we see this by Laplace expansion along their first rows of indeterminates). Thus $\Upsilon(i_\bullet; j_\bullet) \neq \emptyset$ is sufficient for the reducibility of $\mathrm{top}(g)$.

We use induction on r for necessity. If $r = 1$, then we are done by Lemma 6.1. Suppose that for any sequences $i_\bullet$ and $j_\bullet$ of length $s < r$ with $i_1 + \cdots + i_s = j_1 + \cdots + j_s$, if $g(i_\bullet; j_\bullet)$ is irreducible and $i_1 + \cdots + i_t \neq j_1 + \cdots + j_t$ for all $1 \leq t < s$, then $\mathrm{top}(g(i_\bullet; j_\bullet))$ is irreducible.

Let $i_\bullet$ and $j_\bullet$ be sequences of length r such that $i_1 + \cdots + i_s \neq j_1 + \cdots + j_s$ for any $1 \leq s < r$, but $i_1 + \cdots + i_r = j_1 + \cdots + j_r = N$. Note that this implies that $i_r \neq j_r$.

Assume $i_r < j_r$. Consider the Laplace expansion of g along the last i_r rows of $M(i_\bullet; j_\bullet)$. For each $L \in \binom{[N]}{i_r}$, write C_L for the determinant of the square submatrix formed by the columns from L and the last i_r rows, and let $\hat{C}_L$ be its cofactor (determinant of the square submatrix formed by the columns from $[N] \smallsetminus L$ and the first $N - i_r$ rows, with the appropriate sign). If $b := \min\{i_r, j_{r-1}\}$, then

$$(6.2) \qquad g \;=\; \sum_{L \in \binom{[j_r+b]}{i_r}} C_L \hat{C}_L \;=\; \sum_{L \in \binom{[j_r]}{i_r}} C_L \hat{C}_L \;+\; \text{(other terms)}.$$

(The first sum is restricted as these are the only nonzero columns in the last i_r rows.) In the second expression, the degree of each of the (other terms) is strictly less than the degree of the terms in the sum over $L \in \binom{[j_r]}{i_r}$. Indeed, in each, the minor C_L has degree $|L \cap [j_r]| < i_r$ as L includes at least one column beyond the j_rth. Thus these minors have smaller degree than those in the sum over $\binom{[j_r]}{i_r}$. Also, each cofactor $\hat{C}_L$ in either expression is, up to a sign, the determinant of a matrix of the form (6.1) with indices

$$(6.3) \qquad M(i_1, \ldots, i_{r-2}, i_{r-1}\,;\, j_1, \ldots, j_{r-2}, j_{r-1}+j_r-i_r)\,.$$

Thus they are either all zero or all nonzero. As $g \neq 0$, we have $\hat{C}_L \neq 0$ for all L and they all have the same degree and are irreducible. Indeed, suppose that for some L, $\hat{C}_L = pq$ factors with neither p nor q a constant. Since $\hat{C}_L \neq 0$, every entry in the lower left $i_{r-1} \times (j_{r-1} + j_r - i_r)$ submatrix of the matrix for $\hat{C}_L$ appears in $\hat{C}_L$, which we may see by Laplace expansion along its last i_{r-1} rows. If one entry occurs in p, then the argument used in the proof of Lemma 6.1 implies that they all do, and no such entry occurs in q. But then q depends only on the last $(j_1 + \cdots + j_{r-2})$ columns of the matrix for $\hat{C}_L$. Since all the $\hat{C}_L$ have the same form (6.3), they are all reducible with the same factor q. But this implies that q divides g, contradicting the irreducibility of g.

As each C_L for $L \in \binom{[j_r]}{i_r}$ is homogeneous of degree i_r, we have

$$\mathrm{top}(g) \;=\; \sum_{L \in \binom{[j_r]}{i_r}} C_L \cdot \mathrm{top}(\hat{C}_L)\,.$$

Each term of some minor C_L occurs only in that minor, and therefore appears in $\mathrm{top}(g)$. In particular, every indeterminate entry x_{st} of the matrix A^r occurs in $\mathrm{top}(g)$. We note that for each L, $\mathrm{top}(\hat{C}_L)$ is irreducible, by our induction hypothesis, as $\hat{C}_L$ is irreducible and the corresponding sequences in (6.3) have length $r-1 < r$ and unequal partial sums.

Suppose that $\mathrm{top}(g) = pq$ factors as a product of polynomials. We may assume that x_{11} appears in p. Arguing as in the proof of Lemma 6.1 shows that each entry of A^r appears in p, and none appears in q.

For $L \in \binom{[j_r]}{i_r}$, let y_L be the specialization obtained by replacing A^r by a matrix whose only nonzero entries form the identity matrix in the columns of L. Since $A_K(y_L) = \delta_{K,L}$, the Kronecker delta, if we evaluate $\mathrm{top}(g)$ at this specialization, we obtain

$$\mathrm{top}(\hat{C}_L) \;=\; \mathrm{top}(g)(y_L) \;=\; p(y_L) \cdot q(y_L) \;=\; p(y_L) \cdot q\,.$$

Since $\mathrm{top}(\hat{C}_L)$ is irreducible, if q is nonconstant, then $p(y_L)$ is a nonzero constant. Thus for $K, L \in \binom{[j_r]}{i_r}$ with $K \neq L$, we have

$$p(y_L) \cdot \mathrm{top}(\hat{C}_K) \;=\; p(y_K) \cdot \mathrm{top}(\hat{C}_L)\,,$$

which is a contradiction, as $\mathrm{top}(\hat{C}_K)$ and $\mathrm{top}(\hat{C}_L)$ have different indeterminates. (Expand $\hat{C}_L$ along a column of $K \smallsetminus L$, whose indeterminates do not appear in $\hat{C}_K$.)

Suppose that $i_r > j_r$. We prove that $\mathrm{top}(g)$ is irreducible by modifying the argument for the case $i_r < j_r$. Since g is nonzero and irreducible, the matrix $M(i_\bullet; j_\bullet)$ does not contain a $l \times (N - l)$ submatrix of zeroes, for any l (containing such a submatrix implies that $M(i_\bullet; j_\bullet)$ is upper left triangular so that g factors, and a larger submatrix forces g to be zero). Considering the last i_r rows of $M(i_\bullet; j_\bullet)$, this implies that $i_r < j_r + j_{r-1}$.

The lower left $i_r \times (j_r + j_{r-1})$-corner of $M(i_\bullet; j_\bullet)$ is $(A^r \;\; I_{i_r, j_{r-1}})$. This has $j_r + b$ nonzero columns where $b = \min\{i_r, j_{r-1}\}$. Let us reconsider the expansion (6.2) of g,

$$g \;=\; \sum_{L \in \binom{[j_r + b]}{i_r}} C_L \hat{C}_L \;=\; \sum_{L \in \binom{[j_r + b]}{i_r},\ [j_r] \subset L} C_L \hat{C}_L \;+\; (\text{other terms})\,.$$

The cofactors $\hat{C}_L$ as before are nonzero, have the same degree, and are irreducible. The degree of C_L is $|L \cap [j_r]|$, so only the terms in the sum in the second expression contribute to $\mathrm{top}(g)$. The rest of the argument proceeds as before. $\qquad\square$

We deduce three corollaries from this proof. In all, $g = \det M(i_\bullet; j_\bullet)$ is assumed to be nonzero and irreducible. Suppose that $\Upsilon(i_\bullet; j_\bullet) = \{s_1 < \cdots < s_m\} \neq \emptyset$. Set $s_0 := 0$ and $s_{m+1} := r$. For $t = 0, \ldots, m$, let

$$(6.4) \qquad M(i_\bullet; j_\bullet)_t \;:=\; \begin{pmatrix} * & \cdots & * & A^{1+s_t} \\ * & \cdots & A^{2+s_t} & I_{i_{2+s_t}, j_{1+s_t}} \\ \vdots & \ddots & \ddots & 0 \\ A^{s_{t+1}} & I_{i_{s_{t+1}}, j_{s_{t+1}-1}} & 0 & 0 \end{pmatrix},$$

which is a square submatrix of $M(i_\bullet; j_\bullet)$.

Corollary 6.3. *If $\Upsilon(i_\bullet; j_\bullet) \neq \emptyset$, then the irreducible factorization of $\mathrm{top}(g)$ is $f_0 \cdots f_m$, where $f_t = \mathrm{top}(\det(M(i_\bullet; j_\bullet)_t))$.*

Proof. That $\mathrm{top}(g) = f_0 \cdots f_m$ is a consequence of the proof of sufficiency in Lemma 6.2. The irreducibility of each f_t is a consequence of the proof of necessity (using mathematical induction and arguing as for the irreducibility of $\hat{C}_L$ therein). $\qquad\square$

Remark 6.4. When $s \in \Upsilon(i_\bullet; j_\bullet)$, let $m := i_1 + \cdots + i_s$. Then the matrix $M(i_\bullet; j_\bullet)$ has a block structure

$$(6.5) \qquad \begin{pmatrix} * & M \\ M' & P \end{pmatrix},$$

where $*$ is a $m \times (N-m)$ matrix of indeterminates, M and M' are structured matrices (6.1) with parameters

$$M \;=\; M(i_1, \ldots, i_s; j_1, \ldots, j_s) \qquad M' \;=\; M(i_{s+1}, \ldots, i_r; j_{s+1}, \ldots, j_r)\,,$$

and P is a $(N{-}m)\times m$ matrix with block structure $\left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$, where $I = I_{i_{s+1},j_s}$. In particular, the 2×2 submatrix on the anti-diagonal in rows $m, m{+}1$ (and columns $n{-}m, n{-}m{-}1$) is $\left(\begin{smallmatrix} * & * \\ * & 1 \end{smallmatrix}\right)$, where $*$ indicates an indeterminate. In particular,

$$\mathrm{top}(\det M(i_\bullet; j_\bullet)) \;=\; \mathrm{top}(\det M) \cdot \mathrm{top}(\det M') . \qquad \diamond$$

Corollary 6.5. *Every indeterminate in each matrix A^k for $k = 1, \ldots, r$ appears in $\mathrm{top}(g)$.*

Proof. This can be proven by induction on k, using the same arguments as in the proof of necessity in Lemma 6.2. $\qquad\square$

Corollary 6.6. *If $r = 1$, then $g = \mathrm{top}(g)$ is homogeneous and if $r > 1$, then $\mathrm{snd}(g) \neq 0$.*

Proof. If $r = 1$, then $i_1 = j_1 = N$, and $g = \det A^1$ is a homogeneous polynomial.

Assume that $r > 1$. Expand g along the last i_r rows of $M(i_\bullet; j_\bullet)$ as in the proof of necessity in Lemma 6.2,

$$g \;=\; \sum_{L \in \binom{[j_r+b]}{i_r}} C_L \hat{C}_L .$$

Recall that C_L is homogeneous of degree $|L \cap [j_r]|$ and that $\hat{C}_L$ is the determinant of a matrix with format (6.3), and thus these all have the same degree. As the maximum value for $|L \cap [j_r]|$ is $\min\{i_r, j_r\}$, we have

$$\mathrm{snd}(g) \;=\; \sum_{|L \cap [j_r]| = \min\{i_r, j_r\}} C_L \cdot \mathrm{snd}(\hat{C}_L) \;+\; \sum_{|L \cap [j_r]| = \min\{i_r, j_r\}-1} C_L \cdot \mathrm{top}(\hat{C}_L) .$$

The same arguments as before show that there is no cancellation in these sums. In particular, the second sum is nonempty and nonzero, which implies that $\mathrm{snd}(g) \neq 0$. $\quad\square$

Lemma 6.7. *Let g be the determinant of $M(i_\bullet; j_\bullet)$ and assume that g is nonzero, irreducible, and inhomogeneous. Then $\mathrm{snd}(g) \neq 0$ and $\mathrm{top}(g)$ is coprime to $\mathrm{snd}(g)$.*

Proof. Since g is inhomogeneous, $r > 1$ and $\mathrm{snd}(g) \neq 0$, by Corollary 6.6. If $\Upsilon(i_\bullet; j_\bullet) = \emptyset$, then $\mathrm{top}(g)$ is irreducible by Lemma 6.2 and thus is coprime to $\mathrm{snd}(g)$ as it has greater degree.

Now suppose that $\Upsilon(i_\bullet; j_\bullet) \neq \emptyset$, so that $\mathrm{top}(g)$ is reducible, and that one of its factors divides $\mathrm{snd}(g)$. We use the notation of Corollary 6.3. Suppose that for some $t \in \{0, \ldots, m\}$, we have $\mathrm{snd}(g) = h f_t$, for some polynomial h. Here, $f_t = \mathrm{top}(g_t)$, where g_t is the determinant of the submatrix $M_t := M(i_\bullet; j_\bullet)_t$ of $M(i_\bullet; j_\bullet)$ as defined in (6.4).

Suppose that M_t has columns indexed by the interval $[a, b]$ and rows by $[c, d]$, and $n := b - a + 1$ is its size. Let us consider the expansion of $g = \det M(i_\bullet; j_\bullet)$ along the rows $[c, d]$ of M_t,

$$(6.6) \qquad\qquad g \;=\; C_{[a,b]} \hat{C}_{[a,b]} \;+\; \sum_{L \in \binom{[N]}{n}, \; L \neq [a,b]} C_L \hat{C}_L .$$

Suppose that $\delta := \deg(g)$. By Corollary 6.3, only the first term in (6.6) has degree δ. As in Section 5, let $p_{\delta-1}$ be the homogeneous component of degree $\delta{-}1$ in the polynomial p. Then

$$(6.7) \qquad\qquad h f_t \;=\; \mathrm{snd}(g) \;=\; \mathrm{snd}(C_{[a,b]} \hat{C}_{[a,b]}) \;+\; \sum_{L \neq [a,b]} \left(C_L \hat{C}_L \right)_{\delta-1} .$$

If we specialize the indeterminates not appearing in $C_{[a,b]}\hat{C}_{[a,b]}$ to zero, we obtain

$$\overline{h}f_t \ = \ \text{snd}(C_{[a,b]}\hat{C}_{[a,b]}) \ = \ \text{top}(C_{[a,b]})\text{snd}(\hat{C}_{[a,b]}) \ + \ \text{snd}(C_{[a,b]})\text{top}(\hat{C}_{[a,b]}) \ ,$$

where $\overline{h}$ is the specialization of h. Since $f_t = \text{top}(C_{[a,b]})$ is irreducible and $\text{top}(\hat{C}_{[a,b]}) \neq 0$, we conclude that $\text{snd}(C_{[a,b]}) = 0$. Since $C_{[a,b]} = \det M_t$, and it is irreducible (as $\text{top}(C_{[a,b]})$ is irreducible), Corollary 6.6 implies that $1+s_t = s_{t+1}$ so that $M_t = A^{s_{t+1}}$ is a square matrix of indeterminates. Thus $f_t = \det M_t$ and every term of f_t involves a variable from each column of M_t.

The only variables from rows in $[c, d]$ in terms of g come from $C_{[a,b]}$ and the minors C_L in the sum in (6.6). Each C_L for $L \neq [a, b]$ is the determinant of a matrix with at least one column not from among $[a, b]$, consequently, there is no term of C_L and hence of $C_L\hat{C}_L$ that involves a variable from each column of M_t. This implies that f_t cannot divide the sum of (6.7), and thus no term in the sum of (6.6) has degree $\delta-1$. We will show that the sum of (6.6) has degree $\delta-1$, which is a contradiction. This will imply that $\text{top}(g)$ is coprime to $\text{snd}(g)$ and complete the proof.

Observe that the matrix $M(i_\bullet; j_\bullet)$ has the following block form

$$\begin{pmatrix} * & * & M(i'_\bullet; j'_\bullet) \\ * & M_t & P \\ M(i''_\bullet; j''_\bullet) & Q & 0 \end{pmatrix} ,$$

where $i'_\bullet = i_1, \ldots, i_{s_t}$ and $i''_\bullet = i_{1+s_{t+1}}, \ldots, i_r$, and the same for $j'_\bullet$ and $j''_\bullet$. Both P and Q have block structure $\left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$, where $I = I_{i_{1+s_t}, i_{s_t}}$ for P and $I = I_{i_{1+s_{t+1}}, i_{s_{t+1}}}$ for Q. If $t = 0$, then $M(i'_\bullet; j'_\bullet)$ and its rows and columns are omitted, while if $t = m$, then $M(i''_\bullet; j''_\bullet)$ and its rows and columns are omitted, but at most one of these occurs, as $m \geq 1$. Note that $\hat{C}_{[a,b]} = \det M(i'_\bullet; j'_\bullet) \cdot \det M(i''_\bullet; j''_\bullet)$.

If $t \neq m$, then $a > 1$ and let $L := \{a-1\}, (a, b]$. Then C_L is the determinant of the matrix obtained from M_t by replacing its first column of variables with another column of variables, so $\deg C_L = \deg C_{[a,b]}$. Similarly, $\hat{C}_L$ is the product $\det M(i'_\bullet; j'_\bullet) \cdot \det M$, where M is obtained from $M(i''_\bullet; j''_\bullet)$ by replacing its last column with the first column of Q. This amounts to setting all variables in the last column of $M(i''_\bullet; j''_\bullet)$ to zero, except for the first, which is set to 1. This variable was in the block $A^{1+s_{t+1}}$, and by Corollary 6.5 it appears in $\text{top}(\det M(i''_\bullet; j''_\bullet))$. This implies that the degree of $C_L\hat{C}_L$ is $\delta-1$.

If $t \neq 0$, then $b < N$ and we let $L = [a, b), \{b+1\}$. We have $\deg \hat{C}_L = \deg \hat{C}_{[a,b]}$, as they are determinants of matrices of the same format. However, C_L is obtained from $C_{[a,b]}$ by setting all variables in the last column of M_t to zero, except the for the first, which is set to 1. This again implies that the degree of $C_L\hat{C}_L$ is $\delta-1$, which shows that the sum of (6.6) has degree $\delta-1$, and completes the proof. $\square$

Proof of Lemma 5.2. Recall that we are considering $\mathbb{F}\ell(n_\bullet)$. Set $a_1 := n_1$, $a_t := n_t - n_{t-1}$ for $t = 2, \ldots r$, and $a_{r+1} := n - n_r$. Let us augment the coordinates (6.1) to a square

matrix by appending $(I_{a_{r+1}} \quad 0)$ in the remaining rows as follows:

$$(6.8) \qquad \begin{pmatrix} * & * & \cdots & * & * & I_{a_1} \\ * & * & \cdots & * & I_{a_2} & 0 \\ \vdots & \vdots & \cdot\cdot\cdot & \cdot\cdot\cdot & 0 & 0 \\ \vdots & * & \cdot\cdot\cdot & \cdot\cdot\cdot & 0 & 0 \\ * & I_{a_r} & 0 & \cdot\cdot\cdot & 0 & 0 \\ I_{a_{r+1}} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{n \times n} .$$

For $a \in [n]$, the divisor D_a is given by the $a \times a$ principal minor $f^{(a)}$ of this matrix. Each minor $f^{(a)}$ is nonzero and irreducible as D_a is irreducible. For $a \le \min\{n_r, n-n_1\}$, the $a \times a$ principal minor is the determinant of the first a rows and a columns of (6.8), and thus has the form (6.1). If $\min\{n_r, n-n_1\} < a \le n$, then the matrix formed by the first a rows and a columns of (6.8) does not have this form. When $n-n_1 < a$, its last $a+n_1-n$ columns have an identity matrix in the first $a + n_1 - n$ rows and 0s elsewhere, and when $n_r < a$, its last $a - n_r$ rows have an identity matrix in the first $a - n_r$ columns and 0s elsewhere.

In the first case, removing the first $a + n_1 - n$ rows and last $a + n_1 - n$ columns does not change the determinant, and in the second case, removing the first $a - n_r$ columns and $a - n_r$ columns does not change the determinant. After these removals, we are left with a matrix having the form (6.1). Hence, by Lemma 6.1 and Corollary 6.3, every polynomial $\text{top}(f^{(a)})$ is either irreducible or a product of distinct irreducible polynomials, and hence is square-free. By Lemma 6.7, $\text{top}(f^{(a)})$ and $\text{snd}(f^{(a)})$ are coprime whenever $f^{(a)}$ is not homogeneous (in which case $\text{snd}(f^{(a)}) \ne 0$). This proves statement (1) of Lemma 5.2.

For statement (2), let us first consider the irreducible factorization of $\text{top}(f^{(a)})$ for $a \in [n-1]$. Let $M^{(a)}$ be the principal $a \times a$ submatrix of (6.8). By Corollary 6.3 and Remark 6.4, the factorization of $\text{top}(f^{(a)})$ is determined by decompositions of $M^{(a)}$ as in (6.5). That is, by the rows of $M^{(a)}$ whose 2×2 block along the anti-diagonal is $\left(\begin{smallmatrix} * & * \\ * & 1 \end{smallmatrix}\right)$. From the form of (6.8) this occurs when the northwest 1 of some I_{a_s} is in the indicated position. In this case, $a + a_s = n$ and it occurs in row $n_s + 1$ and column $a - n_s + 1$.

Thus each row n_s giving the block structure of (6.8) will contribute to the factorization of a unique $\text{top}(f^{(a)})$, namely when $a = n - a_s$. Suppose that

$$\text{top}(f^{(a)}) \;=\; f_0^{(a)} \cdot f_1^{(a)} \cdots f_{m_a}^{(a)}$$

is the irreducible factorization of $\text{top}(f^{(a)})$. Here, m_a is the number of indices s such that $a + a_s = n$ and $f_i^{(a)} := \text{top}(\det(M_i^{(a)}))$, where $M_i^{(a)}$ is the corresponding submatrix of $M^{(a)}$. These matrices $M_0^{(a)}, \ldots, M_{m_a}^{(a)}$ lie along the anti-diagonal of $M^{(a)}$ between adjacent rows $n_s, n_{s'}$ such that $a_s = a_{a'} = n - a$ (or row 1 when $i = 0$ or row a when $i = m_a$).

Statement (2) follows from the claim that if $a \ne b$, then for all i, j, $f_i^{(a)} \ne f_j^{(b)}$, as these are irreducible. To prove the claim, let M' be the matrix $M_i^{(a)}$, after removing rows and columns coming from I_{a_1} if $a + a_1 > n$ and $i = 0$ and after removing rows and columns corresponding to $I_{a_{r+1}}$ if $a > n_r$ and $i = m_a$. Then M' has structure as in (6.1) and by Corollary 6.5 each variable of each anti-diagonal block A^t of M' appears in $f_i^{(a)}$. The claim now follows, as this set of variables is different for distinct a and i. $\qquad\square$

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School of Mathematics, Sun Yat-sen University, Guangzhou 510275, P.R. China
E-mail address: lichangzh@mail.sysu.edu.cn
URL: http://math.sysu.edu.cn/gagp/czli

Department of Mathematics. Texas A&M University College Station, TX 77843, USA
E-mail address: sottile@math.tamu.edu
URL: http://www.math.tamu.edu/~sottile

School of Mathematics, Sun Yat-sen University, Guangzhou 510275, P.R. China
E-mail address: zhangch223@mail2.sysu.edu.cn; chizhmath@gmail.com