

LINEAR PRECISION FOR PARAMETRIC PATCHES

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ABSTRACT. We give a precise mathematical formulation for the notions of a parametric patch and linear precision, and establish their elementary properties. We relate linear precision to the geometry of a particular linear projection, giving necessary (and quite restrictive) conditions for a patch to possess linear precision. A main focus is on linear precision for Krasauskas' toric patches, which we show is equivalent to a certain rational map on \mathbb{CP}^d being a birational isomorphism. Lastly, we establish the connection between linear precision for toric surface patches and maximum likelihood degree for discrete exponential families in algebraic statistics, and show how iterative proportional fitting may be used to compute toric patches.

INTRODUCTION

Bézier curves and surfaces are the fundamental units in geometric modeling. There are two basic shapes for surfaces—triangular Bézier patches and rectangular tensor product patches. Multi-sided patches are needed for some applications, and there are several control point schemes for C^∞ multi-sided patches. These include the S -patches of Loop and DeRose [17], Warren's hexagon [22], Karčiauskas's M -patches [12], and the multi-sided toric Bézier patches of Krasauskas [14]. (Relationships between these and other patches are discussed in [13].) Parametric patches are general control point schemes for C^∞ patches whose shape is a polygon or polytope. They include the patch schemes just mentioned, as well as barycentric coordinates for polygons and polytopes [9, 21, 23].

The success and widespread adoption of Bézier and tensor-product patches is due in part to their possessing many useful mathematical properties. Some, such as affine invariance and the convex hull property, are built into their definitions and also hold for the more general parametric patches. Other properties, such as de Casteljau's algorithm for computing Bézier patches, come from the specific form of their Bernstein polynomial blending functions. Linear precision is the ability of a parametric patch to replicate linear functions. When the blending functions of a parametric patch correspond to the vertices of a polytope, these blending functions give barycentric coordinates precisely when the patch has linear precision. In this way, blending functions for a parametric patch having linear precision are barycentric coordinates for general control point schemes.

2000 *Mathematics Subject Classification.* 65D17, 14M25.

Key words and phrases. tensor product Bézier surfaces; triangular Bézier surface patches; Barycentric coordinates; Iterative proportional fitting.

Work of Sottile supported by NSF CAREER grant DMS-0538734, by the Institute for Mathematics and its Applications with funds provided by the National Science Foundation, and by Peter Gritzmann of the Technische Universität München.

For us, a (parametric) *patch* is a collection of non-negative blending functions indexed by a finite set \mathcal{A} of points in \mathbb{R}^d , where the common domain of the blending functions is the convex hull Δ of \mathcal{A} . A collection of control points in \mathbb{R}^ℓ indexed by \mathcal{A} is used to define a map from Δ to \mathbb{R}^ℓ . The blending functions determine the internal structure of this map and the basic shape, Δ , of its image, while the control points determine how the image lies in \mathbb{R}^ℓ . Choosing the control points to be the points of \mathcal{A} gives the tautological map, and the patch has linear precision when this tautological map is the identity on Δ . We show that every patch has a unique reparametrization (composing the blending functions with a homeomorphism of Δ) having linear precision (Theorem 1.11). This generalizes Theorem 8.5 of [19], which was for toric patches.

The blending functions of a patch may be arbitrary non-negative C^∞ functions. A patch is rational if it has a reparametrization having polynomial blending functions. For a rational patch, a map given by control points corresponds to a linear projection of a projective algebraic variety associated to its blending functions. Its unique reparametrization having linear precision has rational blending functions if and only if a certain canonical map defined on this variety is a birational isomorphism (Theorem 2.9), which implies that this variety has a maximally degenerate position with respect to a canonical linear subspace given by the set \mathcal{A} (Theorem 2.10).

We apply this analysis to Krasauskas's toric patches, which are rational. The shape of a toric patch is a pair (\mathcal{A}, w) where \mathcal{A} is a set of integer points in \mathbb{Z}^d and w is a collection of positive real numbers indexed by \mathcal{A} . We show that iterative proportional fitting [5], a simple numerical algorithm from statistics, computes the blending functions that have linear precision. This algorithm was suggested to us by Bernd Sturmfels. It may form a basis for algorithms to manipulate these patches.

Krasauskas [15, Problem 3] asked whether any toric patches (besides the classical Bézier simploids [6]) admit a rational reparametrization having linear precision. A toric patch of shape (\mathcal{A}, w) corresponds to a homogeneous polynomial $f = f_{\mathcal{A}, w}$ whose dehomogenization is the sum of monomials with exponents \mathcal{A} and coefficients w . The toric patch admits a rational reparametrization having linear precision if and only if the toric differential

$$D_{\text{toric}} f := \left[x_0 \frac{\partial}{\partial x_0} f : x_1 \frac{\partial}{\partial x_1} f : \cdots : x_d \frac{\partial}{\partial x_d} f \right] : \mathbb{C}\mathbb{P}^d \dashrightarrow \mathbb{C}\mathbb{P}^d$$

defines a birational isomorphism (Corollary 3.13). This analysis of linear precision is used in [11] to classify which toric surface patches can have linear precision.

In Krasauskas's question about linear precision, he allowed the points \mathcal{A} indexing the blending functions of toric patches to move within their (fixed) convex hull, keeping the same blending functions (See Example 3.16). Our analysis of linear precision for general parametric patches will help to address that version of his question.

In Section 1, we define parametric patches and show that every patch has a unique reparametrization that has linear precision. In Section 2 we show that if a rational patch has linear precision, then an algebraic variety we obtain from the blending functions has an exceptional position with respect to a certain linear space. We define toric patches in Section 3, and in Section 4 we explain how iterative proportional fitting computes the blending functions for toric patches that have linear precision.

1. LINEAR PRECISION FOR PARAMETRIC PATCHES


We interpret the standard definition of a mapping via control points and blending functions (see for example [13, §2]) in a general form convenient for our discussion. All functions here are smooth (C^∞) where defined and real-valued unless otherwise stated. Let $\mathbb{R}_>$ be the set of strictly positive real numbers and \mathbb{R}_\geq the set of non-negative real numbers.

Let \mathcal{A} be a finite set of points in \mathbb{R}^d , which we shall use as geometrically meaningful indices. A control point scheme for parametric patches, or (*parametric*) *patch* for short, is a collection $\beta = \{\beta_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\}$ of non-negative functions, called *blending functions*. The common domain of the blending functions is the convex hull Δ of \mathcal{A} , which we call the *domain polytope*. We will always assume that Δ is full dimensional in that it has dimension d . We assume that the blending functions do not vanish simultaneously at any point of Δ . That is, the blending functions have no base points in their domain.

A set $\{\mathbf{b}_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\} \subset \mathbb{R}^\ell$ of *control points* gives a map $F: \Delta \rightarrow \mathbb{R}^\ell$ defined by

$$(1.1) \quad F(x) := \frac{\sum_{\mathbf{a} \in \mathcal{A}} \beta_{\mathbf{a}}(x) \mathbf{b}_{\mathbf{a}}}{\sum_{\mathbf{a} \in \mathcal{A}} \beta_{\mathbf{a}}(x)}.$$

The denominator in (1.1) is positive on Δ and so the map F is well-defined.

Remark 1.2. Positive weights, or shape parameters, $\{w_{\mathbf{a}} \in \mathbb{R}_> \mid \mathbf{a} \in \mathcal{A}\}$ scaling the blending functions are often included in the definition of the map F . We have instead absorbed them into our notion of blending functions. See Remarks 1.7 and 3.3. 

The blending functions $\{\beta_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\}$ are *normalized* if they form a partition of unity,

$$\sum_{\mathbf{a} \in \mathcal{A}} \beta_{\mathbf{a}}(x) = 1.$$

As the denominator in (1.1) is (strictly) positive on Δ , we may divide each blending function by this denominator to obtain normalized blending functions.

$$\text{Redefine } \beta_{\mathbf{a}} := \frac{\beta_{\mathbf{a}}}{\sum_{\mathbf{a} \in \mathcal{A}} \beta_{\mathbf{a}}}. \quad \text{Then } \beta_{\mathbf{a}}(x) \geq 0 \quad \text{and} \quad \sum_{\mathbf{a} \in \mathcal{A}} \beta_{\mathbf{a}}(x) = 1,$$

for $x \in \Delta$. For normalized blending functions, the formula (1.1) becomes

$$(1.3) \quad F(x) = \sum_{\mathbf{a} \in \mathcal{A}} \beta_{\mathbf{a}}(x) \mathbf{b}_{\mathbf{a}}.$$

We deduce two fundamental properties of parametric patches.

- **Convex hull property.** For $x \in \Delta$, $F(x)$ is a convex combination of the control points. Thus $F(\Delta)$ lies in the convex hull of the control points.
- **Affine invariance.** If Λ is an affine function on \mathbb{R}^ℓ , then $\Lambda(F(x))$ is the map with the same blending functions, but with control points $\{\Lambda(\mathbf{b}_{\mathbf{a}}) \mid \mathbf{a} \in \mathcal{A}\}$.

A patch is *non-degenerate* if its blending functions are linearly independent. This implies in particular that $F(\Delta)$ cannot collapse into a point unless all the control points are equal. We make our key definition.

Definition 1.4. A patch $\{\beta_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\}$ has *linear precision* if for every affine function $\Lambda(x)$ defined on \mathbb{R}^d ,

$$\Lambda(x) = \sum_{\mathbf{a} \in \mathcal{A}} \beta_{\mathbf{a}}(x) \Lambda(\mathbf{a}).$$

That is, if the patch can replicate affine functions. This notion depends strongly on the positions of the points in \mathcal{A} . 

The *tautological map* τ of a given patch is the map (1.3) when the control points are taken to be the corresponding points of \mathcal{A} ,

$$(1.5) \quad \tau(x) := \sum_{\mathbf{a} \in \mathcal{A}} \beta_{\mathbf{a}}(x) \mathbf{a}.$$


By the convex hull property, $\tau(\Delta) \subset \Delta$. By affine invariance, the patch has linear precision if and only if τ is the identity function, $\tau(x) = x$. We record this fact.

Proposition 1.6. *A patch has linear precision if and only if its tautological map is the identity map on the domain polytope Δ .*

Remark 1.7. This is a more restrictive notion of linear precision than is typically considered. It is more common to define a patch to be a collection $\{\beta_i \mid i = 1, \dots, n\}$ of blending functions, and this has linear precision if we have

$$x = \frac{\sum_{i=1}^n w_i \beta_i(x) \mathbf{a}_i}{\sum_{i=1}^n w_i \beta_i(x)},$$

for some non-negative *weights* $\{w_i \mid i = 1, \dots, n\}$ and some points $\mathcal{A} = \{\mathbf{a}_i \mid i = 1, \dots, n\}$ whose convex hull is the domain polytope Δ of the blending functions.

Our more restrictive definition of patches and linear precision, where we incorporate the weights and the points \mathcal{A} into our definition of blending functions, allows us to derive precise criteria with which to study linear precision. Our intention is to employ these criteria to study patches in the generality in which other authors had worked. 

Example 1.8. Fix an integer $n > 0$ and let $\mathcal{A} := \{\frac{i}{n} \mid 0 \leq i \leq n\}$ be the set of $n+1$ equally distributed points in the unit interval. The i th *Bernstein polynomial* is $\beta_i := \binom{n}{i} x^i (1-x)^{n-i}$, which we associate to the point $\frac{i}{n} \in \mathcal{A}$. These form a partition of unity,

$$\sum_{i=0}^n \beta_i(x) = \sum_{i=0}^n \binom{n}{i} x^i (1-x)^{n-i} = (x + (1-x))^n = 1,$$

and are therefore normalized blending functions. Given control points $\mathbf{b}_i \in \mathbb{R}^3$, formula (1.1) becomes

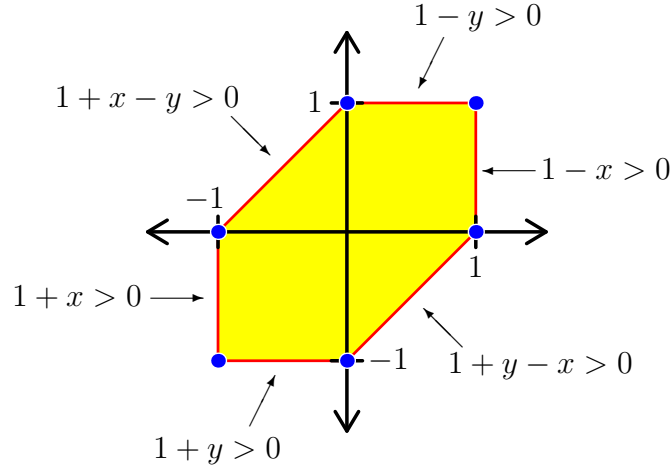
$$F(x) = \sum_{i=0}^n \binom{n}{i} x^i (1-x)^{n-i} \mathbf{b}_i,$$

which is the classical formula for a Bézier curve of degree n in \mathbb{R}^3 .

This patch has linear precision. First, note that $\frac{i}{n} \cdot \binom{n}{i} = \binom{n-1}{i-1}$. Then $\tau(x)$ is

$$\begin{aligned} \sum_{i=0}^n \binom{n}{i} x^i (1-x)^{n-i} \cdot \frac{i}{n} &= \sum_{i=1}^n \binom{n-1}{i-1} x^i (1-x)^{n-i} \\ &= x \cdot \sum_{j=0}^{n-1} \binom{n-1}{j} x^j (1-x)^{n-1-j} = x. \end{aligned} \quad \color{blue}\square$$

Example 1.9. Let \mathcal{A} be the vertices of the hexagon, given below by its defining inequalities.



Let the blending function associated to a vertex be the product of the linear forms defining edges that do not contain that vertex.

$$\begin{aligned} \beta_{\begin{bmatrix} -1 \\ -1 \end{bmatrix}} &:= (1+x-y)(1-y)(1-x)(1+y-x) \\ \beta_{\begin{bmatrix} -1 \\ 0 \end{bmatrix}} &:= (1-y)(1-x)(1+y-x)(1+y) \\ \beta_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} &:= (1-x)(1+y-x)(1+y)(1+x) \\ \beta_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} &:= (1+y-x)(1+y)(1+x)(1+x-y) \\ \beta_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} &:= (1+y)(1+x)(1+x-y)(1-y) \\ \beta_{\begin{bmatrix} 0 \\ -1 \end{bmatrix}} &:= (1+x)(1+x-y)(1-y)(1-x) \end{aligned} \quad (1.10)$$

The normalized blending functions have denominator $\varphi(x, y) := 2(3 + xy - x^2 - y^2)$, which is strictly positive on Δ . These normalized blending functions have linear precision.

$$\sum_{\mathbf{a} \in \mathcal{A}} \beta_{\mathbf{a}}(x, y) \mathbf{a} = \varphi(x, y) \begin{bmatrix} x \\ y \end{bmatrix}.$$

In fact, these normalized blending functions are Wachspress's barycentric coordinates [21] for this hexagon. $\color{blue}\square$

When \mathcal{A} is the set of vertices of Δ (so that \mathcal{A} is in convex position), then normalized blending functions of a patch with linear precision are *barycentric coordinates* for Δ . By

this we mean in the sense of [9, 21, 23]: nonnegative functions indexed by the extreme points of Δ that have linear precision in the sense of Definition 1.4.

A *reparametrization* of a patch $\{\beta_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\}$ by a homeomorphism $\psi: \Delta \rightarrow \Delta$ is a new patch with blending functions $\beta_{\mathbf{a}} \circ \psi$ for $\mathbf{a} \in \mathcal{A}$. A patch is *proper* if its tautological map (1.5) is a homeomorphism. This condition is necessary for the patch to have linear precision. If a patch is proper, then reparametrizing it by the inverse τ^{-1} of τ gives a new patch whose tautological map is $\tau \circ \tau^{-1}$, the identity function, and τ^{-1} is the unique homeomorphism of Δ with this property. We record this straightforward, yet fundamental result about linear precision.

Theorem 1.11. *A proper patch has a unique reparametrization that has linear precision.*

Theorem 1.11 suggests that it will be fruitful to discuss patches up to reparametrization. In Section 2, we show that a natural geometric object X_{β} associated to a patch is invariant under reparametrizations and thus represents the patch up to reparametrization.

A patch is *rational* if it has a reparametrization whose normalized blending functions are rational functions (quotients of polynomials). A patch has *rational linear precision* if it is proper and its reparametrization having linear precision has blending functions that are rational functions. Such a patch is necessarily rational. We seek criteria that determine when a rational patch has rational linear precision.

Theorem 1.11 and our preceding discussion concerns reparametrizations of a patch having linear precision. We may also alter a patch by multiplying its blending functions by positive real numbers, typically called weights, or by moving the points of \mathcal{A} that are not vertices of the domain polytope Δ . This second change does not alter any map (1.1) given by control points, but it will change the tautological map τ (1.5). Our analysis in the next section may be helpful in addressing whether it is possible to tune a given patch (using weights or moving the points in \mathcal{A}) to obtain one that has rational linear precision. Example 3.16 (due to Krasauskas and Karčiauskas) shows how one toric patch may be tuned to achieve linear precision.

2. THE GEOMETRY OF LINEAR PRECISION

We introduce an algebraic-geometric formulation of patches to clarify the discussion in Section 1 and to provide tools with which to understand and apply Theorem 1.11. This leads us to discuss the relevance for geometric modeling of the subtle difference between rational varieties and unirational varieties. Lastly, we use this geometric formulation to give a geometric characterization of when a patch has rational linear precision. We recommend the text [3] for additional background on algebraic geometry. We first review linear projections, which are the geometric counterpart of control points.

2.1. Linear projections. We consider \mathbb{R}^{ℓ} as a subset of the ℓ -dimensional real projective space \mathbb{RP}^{ℓ} via the embedding

$$\mathbb{R}^{\ell} \ni z \longmapsto [1, z] \in \mathbb{RP}^{\ell}.$$

A point $[z_0, z_1, \dots, z_\ell] \in \mathbb{RP}^\ell$ lies in this copy of \mathbb{R}^ℓ if and only if $z_0 \neq 0$. In that case, the corresponding point of \mathbb{R}^ℓ is

$$(2.1) \quad \left(\frac{z_1}{z_0}, \frac{z_2}{z_0}, \dots, \frac{z_\ell}{z_0} \right).$$

Let $\mathcal{A} \subset \mathbb{R}^d$ be a finite set of points and write $\mathbb{RP}^{\mathcal{A}}$ for the real projective space whose coordinates are indexed by \mathcal{A} . Let $b = \{\mathbf{b}_a \in \mathbb{R}^\ell \mid \mathbf{a} \in \mathcal{A}\}$ be a collection of control points, which we regard as points in \mathbb{RP}^ℓ so that $b = \{[1, \mathbf{b}_a] \in \mathbb{RP}^\ell \mid \mathbf{a} \in \mathcal{A}\}$.

Given a point $y = [y_a \mid \mathbf{a} \in \mathcal{A}] \in \mathbb{RP}^{\mathcal{A}}$, if the sum

$$(2.2) \quad \sum_{\mathbf{a} \in \mathcal{A}} y_a \cdot (1, \mathbf{b}_a) \in \mathbb{R}^{\ell+1}$$

is non-zero then it represents a point $\pi_b(y)$ in \mathbb{RP}^ℓ . This map π_b is a *linear projection*

$$(2.3) \quad \pi_b : \mathbb{RP}^{\mathcal{A}} \dashrightarrow \mathbb{RP}^\ell.$$

We use a broken arrow \dashrightarrow to indicate that π_b is defined only on the complement of the set $E_b \subset \mathbb{RP}^{\mathcal{A}}$ where the sum (2.2) vanishes. This linear subspace is the *center of projection*.

When the control points affinely span \mathbb{R}^ℓ , the center of projection E_b has codimension $\ell + 1$ in $\mathbb{RP}^{\mathcal{A}}$. The inverse image $\pi_b^{-1}(x)$ of a point $x \in \mathbb{RP}^\ell$ is the set $H \setminus E_b$, where $H \subset \mathbb{RP}^{\mathcal{A}}$ is a linear subspace of codimension ℓ containing E_b . If L is any ℓ -dimensional linear subspace of $\mathbb{RP}^{\mathcal{A}}$ that does not meet E_b , then π_b maps L isomorphically to \mathbb{RP}^ℓ . Identifying L with \mathbb{RP}^ℓ gives an explicit description of the map π_b . If $y \in \mathbb{RP}^{\mathcal{A}} \setminus E_b$, then

$$\pi_b(y) = L \cap (\overline{y, E_b}),$$

the intersection with L of the linear span of y and the center E_b . If $y \in L$, then its inverse image under π_b is $(\overline{y, E_b}) \setminus E_b$.

For example, Figure 1 shows the effect of a linear projection π on a cubic space curve C . The center of projection is a line, E , which meets the curve in a point, B .

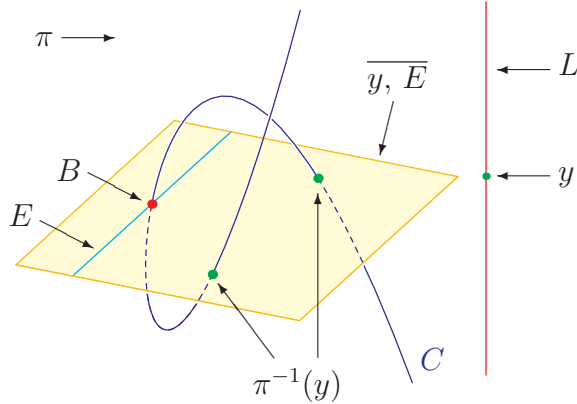


FIGURE 1. A linear projection π with center E .

2.2. Geometric formulation of a patch. Consider (1.1) from the point of view of projective geometry. Given a patch $\beta = \{\beta_{\mathbf{a}}(x) \mid \mathbf{a} \in \mathcal{A}\}$ with $\mathcal{A} \subset \mathbb{R}^d$, then

$$(2.4) \quad x \longmapsto [\beta_{\mathbf{a}}(x) \mid \mathbf{a} \in \mathcal{A}]$$

defines a map $\beta: \Delta \rightarrow \mathbb{RP}^{\mathcal{A}}$ whose \mathbf{a} th coordinate is the blending function $\beta_{\mathbf{a}}(x)$.

The map β is unchanged if we multiply all blending functions by the same positive function $\varphi(x)$ defined on Δ . Thus we obtain the same map if we use instead the normalized blending functions. The image X_β of Δ under β is a closed subset of $\mathbb{RP}^{\mathcal{A}}$ (but not Zariski closed!), as Δ is compact and the map β is continuous. It is non-degenerate (does not lie in a hyperplane) exactly when the patch is non-degenerate.

Suppose now that we have control points $b = \{\mathbf{b}_{\mathbf{a}} \in \mathbb{R}^\ell \mid \mathbf{a} \in \mathcal{A}\}$ and consider the composition of the map β and the linear projection π_b

$$\Delta \xrightarrow{\beta} X_\beta \subset \mathbb{RP}^{\mathcal{A}} \xrightarrow{\pi_b} \mathbb{RP}^\ell.$$

By our assumption on the positivity of the blending functions, formula (2.1) implies that the image lies in the standard copy of \mathbb{R}^ℓ , and there it is given by the formula (1.1). Thus this image, $F(\Delta)$, is the image of X_β under the projection π_b given by the control points.

A reparametrization of the patch β by a homeomorphism $\psi: \Delta \rightarrow \Delta$ gives a different map with the same image in $\mathbb{RP}^{\mathcal{A}}$

$$\tilde{\beta}: \Delta \xrightarrow{\psi} \Delta \xrightarrow{\beta} X_\beta.$$

Thus X_β is an invariant of the patch modulo reparametrization.

If we take \mathcal{A} to be our set of control points where the element $\mathbf{a} \in \mathcal{A}$ is the control point associated to the blending function $\beta_{\mathbf{a}}$, then the resulting linear projection $\pi_{\mathcal{A}}$ is the *tautological projection*,

$$(2.5) \quad \pi_{\mathcal{A}}: \mathbb{RP}^{\mathcal{A}} \dashrightarrow \mathbb{RP}^d.$$

Here, the coordinate points $(e_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A})$ are indexed by elements of \mathcal{A} and $\pi_{\mathcal{A}}(e_{\mathbf{a}}) = \mathbf{a} \in \mathbb{RP}^d$. The tautological map τ (1.5) is the composition


$$\Delta \xrightarrow{\beta} X_\beta \subset \mathbb{RP}^{\mathcal{A}} \xrightarrow{\pi_{\mathcal{A}}} \mathbb{RP}^d.$$

The patch is proper if this map is a homeomorphism onto its image $\Delta \subset \mathbb{RP}^d$, and the patch has linear precision if the composition is the identity map, by Proposition 1.6.

With these definitions, we have the following identification of the reparametrization having linear precision, which is a more precise version of Theorem 1.11.

Theorem 2.6. *Suppose that X_β is a proper patch and $\pi_{\mathcal{A}}: X_\beta \rightarrow \Delta$ is the tautological projection restricted to X_β . Then the blending functions for X_β that have linear precision are given by the coordinates of the inverse of $\pi_{\mathcal{A}}$.*

Remark 2.7. This geometric perspective, where a patch is first an embedding of Δ into $\mathbb{RP}^{\mathcal{A}}$ followed by a linear projection, may have been introduced into geometric modeling in [7]. It is fundamental for Krasauskas's toric Bézier patches [14], and was reworked in the tutorial [19]. There, an analog of Theorem 1.11 was formulated. We take this opportunity to correct an error in notation. The tautological projection is an algebraic version of the

moment map of the toric variety underlying a toric patch, and not the actual moment map from symplectic geometry. 

2.3. Rational varieties. We study the algebraic relaxation of our previous notions, replacing the real numbers by the complex numbers so that we may use notions from algebraic geometry. A rational patch has a parametrization by rational functions. Clearing their denominators gives a new collection $\{\beta_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\}$ of blending functions which are polynomials. These polynomials define complex-valued functions on \mathbb{C}^d , and so the blending functions give a map

$$\beta_{\mathbb{C}} : \mathbb{C}^d \setminus B \longrightarrow \mathbb{C}\mathbb{P}^{\mathcal{A}},$$

which is defined on the complement of the set B where all the blending functions vanish. This is called the *base locus* of the map $\beta_{\mathbb{C}}$. This map $\beta_{\mathbb{C}}$ extends the map $\beta : \Delta \rightarrow X_{\beta}$. We write β for $\beta_{\mathbb{C}}$, $\mathbb{P}^{\mathcal{A}}$ for $\mathbb{C}\mathbb{P}^{\mathcal{A}}$, and in general use the same notation for maps defined on complex algebraic varieties as for their restrictions to subsets of their real points.

As with linear projections, we write $\beta : \mathbb{C}^d \dashrightarrow \mathbb{P}^{\mathcal{A}}$ to indicate that β is only defined on $\mathbb{C}^d \setminus B$. Such a map between algebraic varieties that is given by polynomials and defined on the complement of an algebraic subset is called a *rational map*.

Let Y_{β} be the Zariski closure of the image of β . One important consequence of the patch being rational is that X_{β} is a full-dimensional subset of the real points of the algebraic variety Y_{β} . In particular, X_{β} is defined locally in $\mathbb{R}\mathbb{P}^{\mathcal{A}}$ by the vanishing of some polynomials, and this remains true for any image $F(\Delta)$ of a map F (1.1) given by our original blending functions and any choice of control points. This is important in modeling, for these implicit equations are used to compute intersections of patches.

Suppose that X_{β} has rational linear precision and that $\beta = \{\beta_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\}$ are rational blending functions with linear precision. By Proposition 1.6, the composition

$$\Delta \xrightarrow{\beta} X_{\beta} \xrightarrow{-\pi_{\mathcal{A}}} \Delta$$

is the identity map. Since Δ has full dimension in \mathbb{R}^d , it is Zariski dense in \mathbb{C}^d , and so the composition of rational functions

$$(2.8) \quad \mathbb{C}^d \xrightarrow{-\beta} Y_{\beta} \xrightarrow{-\pi_{\mathcal{A}}} \mathbb{C}^d$$

is also the identity map (where it is defined).

We introduce some terminology to describe this situation. A rational map $\varphi : Y \dashrightarrow Z$ between complex algebraic varieties is a *birational isomorphism* if there is another map $\psi : Z \dashrightarrow Y$ such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are the identity maps where they are defined. In particular, Y has a Zariski open subset U and Z a Zariski open subset V such that $\varphi|_U$ is one-to-one between U and V . The projection $\pi : C \dashrightarrow L$ of Figure 1 is not a birational isomorphism. Indeed, if $x \in C$ is a point in $\pi^{-1}(p)$, then $\overline{p, E} = \overline{x, E}$ and this plane meets the complement $C \setminus B$ of the base locus in x and one other point.

Thus if the composition (2.8) is the identity map, then the map $\pi_{\mathcal{A}}$ is a birational isomorphism from $Y_{\beta} \dashrightarrow \mathbb{C}\mathbb{P}^{\mathcal{A}}$. We deduce the algebraic-geometric version of Proposition 1.6.

Theorem 2.9. *A patch $\beta = \{\beta_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\}$ has rational linear precision if and only if the complexified tautological projection*

$$\pi_{\mathcal{A}} : Y_{\beta} \dashrightarrow \mathbb{C}\mathbb{P}^{\mathcal{A}}$$

is a birational isomorphism.

Using well-known properties of birational projections, we use this characterization of linear precision to deduce necessary conditions for a patch to have linear precision. A *rational variety* of dimension d is one that is birational to \mathbb{C}^d .

Theorem 2.10. *If a patch $\beta = \{\beta_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\}$ has rational linear precision, then*


- (1) Y_β is a rational variety,
- (2) almost all codimension d planes L containing the center $E_{\mathcal{A}}$ of the tautological projection meet Y_β in at most one point outside of $E_{\mathcal{A}}$.

By condition (2), Y_β has an exceptionally singular position with respect to $E_{\mathcal{A}}$. Typically, Y_β does not meet a given codimension $d+1$ plane and its intersection with a given codimension d plane consists of $\deg(Y_\beta)$ points, counted with multiplicity. By Condition (2), not only does Y_β meet $E_{\mathcal{A}}$, but if L has codimension d and contains $E_{\mathcal{A}}$, then most of the $\deg(Y_\beta)$ points in $Y_\beta \cap L$ lie in $E_{\mathcal{A}}$. We will see this in Examples 2.11 and 3.16.

Proof. Statement (1) is immediate from Theorem 2.9.

Since the birational isomorphism $\pi_{\mathcal{A}}: Y_\beta \dashrightarrow \mathbb{C}\mathbb{P}^d$ is the restriction of a linear projection with center $E_{\mathcal{A}}$, then for almost all points $y \in Y_\beta \setminus E_{\mathcal{A}}$, we have

$$(\overline{y, E_{\mathcal{A}}}) \cap Y_\beta = \{y\} \cup (E_{\mathcal{A}} \cap Y_\beta).$$

Since all codimension d planes L that contain $E_{\mathcal{A}}$ have the form $(\overline{y, E_{\mathcal{A}}})$ for some $y \notin E_{\mathcal{A}}$ and almost all meet Y_β , statement (2) follows. 

Consider the geometric situation of Figure 1. Suppose now that E is the line tangent to C at the point B . Then $E \cap C$ is the point B , but counted with multiplicity 2. A linear projection with center E restricts to a birational isomorphism of C with \mathbb{P}^1 .

2.4. Unirational varieties. The necessary condition of Theorem 2.10(1) gives an important but subtle geometric restriction on patches that have rational linear precision.

The map β provides a parametrization of an open subset of Y_β by an open subset of \mathbb{C}^d . Such parametrized algebraic varieties are called *unirational*. Unirational curves and surfaces are also rational, but these two notions differ for varieties of dimension three and higher. Clemens and Griffiths [2] showed that a smooth hypersurface of degree 3 in \mathbb{P}^4 is not rational (these were classically known to be unirational, e.g. by Max Noether).

Thus parametric patches of dimension at least 3 will in general be unirational and not rational. This does not occur for Bézier simploids or toric patches.

Example 2.11. Consider again the Wachspress coordinates (1.10) of Example 1.9, which have linear precision. Let $Y_\beta \subset \mathbb{P}^5$ be the image of the blending functions (1.10). We study the base locus of the tautological projection $\pi_{\mathcal{A}}$ of this patch. We first find equations for Y_β as an algebraic subvariety of \mathbb{P}^5 .

If we divide $\beta_{\begin{smallmatrix} -1 \\ -1 \end{smallmatrix}} \beta_{\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}} + 2\beta_{\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}} \beta_{\begin{smallmatrix} -1 \\ -1 \end{smallmatrix}} + \beta_{\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}} \beta_{\begin{smallmatrix} -1 \\ -1 \end{smallmatrix}}$ by the product of the linear forms defining the hexagon we get

$$(1-x)(1+y-x) + 2(1+x)(1-x) + (1+x)(1+x-y) = 4 - 2xy,$$

which is symmetric in x and y , and so

$$(2.12) \quad \beta_{[-1]} \beta_{[0]} + 2\beta_{[0]} \beta_{[-1]} + \beta_{[1]} \beta_{[-1]} = \beta_{[-1]} \beta_{[0]} + 2\beta_{[0]} \beta_{[-1]} + \beta_{[1]} \beta_{[-1]}.$$

If we let $(y_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A})$ be natural coordinates for $\mathbb{P}^{\mathcal{A}}$, then (2.12) gives the quadratic polynomial which vanishes on Y_{β} ,

$$y_{[-1]} y_{[0]} + 2y_{[0]} y_{[-1]} + y_{[1]} y_{[-1]} - (y_{[-1]} y_{[0]} + 2y_{[0]} y_{[-1]} + y_{[1]} y_{[-1]}).$$

Cyclically permuting the vertices of the hexagon gives two other quadratics that vanish on Y_{β} , but these three sum to 0. There is another quadratic polynomial vanishing on Y_{β} ,

$$y_{[-1]} y_{[0]} + y_{[0]} y_{[1]} + y_{[1]} y_{[-1]} - (y_{[1]} y_{[-1]} + y_{[-1]} y_{[0]} + y_{[0]} y_{[1]}).$$

There is an additional cubic relation among the blending functions, which gives a cubic polynomial that vanishes on Y_{β}

$$(2.13) \quad y_{[-1]} y_{[0]} y_{[1]} - y_{[1]} y_{[-1]} y_{[0]}.$$

These relations, three independent quadratics and one cubic, define Y_{β} as an algebraic subset of $\mathbb{P}^{\mathcal{A}} = \mathbb{P}^5$. The quadratic equations define Y_{β} , together with the 2-dimensional linear space cut out by the three linear equations.

$$(2.14) \quad \begin{aligned} 0 &= y_{[-1]} + y_{[0]} + y_{[1]} + y_{[-1]} + y_{[-1]} + y_{[-1]} \\ 0 &= y_{[1]} - y_{[-1]} + y_{[0]} - y_{[-1]} \\ 0 &= y_{[1]} - y_{[-1]} + y_{[0]} - y_{[-1]} \end{aligned}$$


Thus Y_{β} has degree $7 = 2^3 - 1$. A general codimension 2 plane L containing the center $E_{\mathcal{A}}$ of the tautological projection $\pi_{\mathcal{A}}$ will meet Y_{β} in at most 1 point outside of $E_{\mathcal{A}}$, by Theorem 2.10(2). Since $L \cap Y_{\beta}$ has degree at least 7, the other 6 points must lie in the base locus $E_{\mathcal{A}} \cap Y_{\beta}$, which is in fact a reducible cubic plane curve.

The three linear forms (2.14) also define the tautological projection, $\pi_{\mathcal{A}}$, so this linear subspace is the center of projection $E_{\mathcal{A}}$, and the base locus B is defined in $E_{\mathcal{A}}$ by the cubic (2.13) defining Y_{β} . Let us parametrize the center $E_{\mathcal{A}}$ as follows. For $[a, b, c] \in \mathbb{P}^2$, set

$$\begin{aligned} y_{[-1]} &:= a + b & y_{[1]} &:= -a + b \\ y_{[0]} &:= a - c & y_{[-1]} &:= -a - c \\ y_{[1]} &:= a - b + c & y_{[-1]} &:= -a - b + c \end{aligned}$$

Then we have

$$y_{[-1]} y_{[0]} y_{[1]} - y_{[1]} y_{[-1]} y_{[-1]} = 2a(a^2 - b^2 + bc - c^2),$$

the product of a linear and a quadratic form. 

3. TORIC PATCHES

Toric patches, which were introduced by Krasauskas [14], are a class of patches based on certain special algebraic varieties, called toric varieties. For a basic reference on toric varieties as parametrized varieties, see the book [20], particularly Chapters 4 and 13, and the articles [4, 19], which are introductions aimed at people in geometric modeling. We first define toric patches, give some examples, reinterpret Theorem 2.9 for toric patches, and then state the main open problem about linear precision for toric patches.

3.1. Toric Patches. We regard elements $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{Z}^d$ as exponent vectors for monomials in the variables $x = (x_1, \dots, x_d)$

$$x^{\mathbf{a}} := x_1^{a_1} x_2^{a_2} \cdots x_d^{a_d}.$$

Let $\mathcal{A} \subset \mathbb{Z}^d$ be a finite set of integer vectors whose convex hull Δ is a polytope of dimension d in \mathbb{R}^d . This implies that \mathcal{A} affinely spans \mathbb{R}^d . Suppose that $w = \{w_{\mathbf{a}} \in \mathbb{R}_{>} \mid \mathbf{a} \in \mathcal{A}\}$ is a set of positive weights indexed by \mathcal{A} . These data (\mathcal{A}, w) define a map $\varphi_{\mathcal{A}, w}: (\mathbb{C}^\times)^d \rightarrow \mathbb{P}^{\mathcal{A}}$,

$$(3.1) \quad \varphi_{\mathcal{A}, w} : x \longmapsto [w_{\mathbf{a}} x^{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}].$$


Let $Y_{\mathcal{A}, w}$ be the closure of the image of $(\mathbb{C}^\times)^d$ under the map $\varphi_{\mathcal{A}, w}$. When the weights $w_{\mathbf{a}}$ are equal, this is the toric variety parametrized by the monomials in \mathcal{A} . In general, it is the translate of that toric variety by the element w of the positive real torus $\mathbb{R}_{>}^{\mathcal{A}}$, which acts on $\mathbb{P}^{\mathcal{A}}$ by scaling the coordinates.


Let $X_{\mathcal{A}, w} \subset \mathbb{R}\mathbb{P}^{\mathcal{A}}$ be the closure $\varphi_{\mathcal{A}, w}(\mathbb{R}_{>}^d)$. This is the non-negative part of the translated toric variety $Y_{\mathcal{A}, w}$. That is, if $\mathbb{R}\mathbb{P}_{\geq}^{\mathcal{A}} \subset \mathbb{P}^{\mathcal{A}}$ is the set of points having non-negative (real) homogeneous coordinates, then $X_{\mathcal{A}, w} = Y_{\mathcal{A}, w} \cap \mathbb{R}\mathbb{P}_{\geq}^{\mathcal{A}}$. It is homeomorphic to the convex hull Δ of the vectors in \mathcal{A} , as a manifold with corners [10, § 4].

The set \mathcal{A} is *primitive* if it affinely spans \mathbb{Z}^d . We may assume that \mathcal{A} is primitive, for there is always some primitive set $\mathcal{A}' \subset \mathbb{Z}^d$ with $Y_{\mathcal{A}, w} = Y_{\mathcal{A}', w}$. First note that translating every vector in \mathcal{A} by a fixed vector \mathbf{a}' multiplies each coordinate of the map $\varphi_{\mathcal{A}, w}$ by the monomial $x^{\mathbf{a}'}$, which does not change $\varphi_{\mathcal{A}, w}$ as a map to projective space. Thus we may assume that \mathcal{A} contains the zero vector. Since \mathcal{A} affinely spans \mathbb{R}^d , it generates a subgroup $\mathbb{Z}\mathcal{A}$ of \mathbb{Z}^d which is isomorphic to \mathbb{Z}^d . Let $\mathbf{b}_1, \dots, \mathbf{b}_d \in \mathbb{Z}\mathcal{A}$ be the basis corresponding to the basis of \mathbb{Z}^d under an isomorphism $\mathbb{Z}\mathcal{A} \xrightarrow{\sim} \mathbb{Z}^d$. If we let $\mathcal{A}' \subset \mathbb{Z}^d$ be the image of \mathcal{A} under this isomorphism and $\mathcal{B} := \{\mathbf{b}_1, \dots, \mathbf{b}_d\}$, then the map $\varphi_{\mathcal{A}, w}$ factors

$$(\mathbb{C}^\times)^d \xrightarrow{\chi_{\mathcal{B}}} (\mathbb{C}^\times)^d \xrightarrow{\varphi_{\mathcal{A}', w}} \mathbb{P}^{\mathcal{A}},$$

where $\chi_{\mathcal{B}}$ is the surjective map that sends x to $(x^{\mathbf{b}_1}, \dots, x^{\mathbf{b}_d})$. Note that $\varphi_{\mathcal{A}, w}$ is injective but that $\chi_{\mathcal{B}}$ has fibers of size $|\mathbb{Z}^d/\mathbb{Z}\mathcal{A}|$. Thus $\varphi_{\mathcal{A}, w}$ is not proper unless \mathcal{A} is primitive.

Definition 3.2. A *toric patch* of *shape* (\mathcal{A}, w) is any patch $\{\beta_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\}$ such that the image, X_{β} , of Δ under the map (2.4) given by the blending functions is equal to the non-negative part $X_{\mathcal{A}, w}$ of the translated toric variety $Y_{\mathcal{A}, w}$. 


Remark 3.3. Our geometric perspective that the fundamental object is the image X_{β} requires us to absorb the weights w into our definition and distinguishes patches with different choices of weights. 

Example 3.4. Let $\mathcal{A} := \{0, 1, \dots, n\} \subset \mathbb{Z}$ and let $w \in \mathbb{R}_{>}^{n+1}$ be any set of weights. Then the map $\varphi_{\mathcal{A},w}$ is

$$x \longmapsto [w_0, w_1x, w_2x^2, \dots, w_nx^n].$$

The map $x \mapsto \frac{x}{n-x}$ sends $\Delta = [0, n]$ to $[0, \infty)$. Precomposing $\varphi_{\mathcal{A},w}$ with this map gives a toric patch of shape (\mathcal{A}, w) .

$$\begin{aligned} [0, n] \ni x &\longmapsto \left[w_0, w_1 \left(\frac{x}{n-x} \right), w_2 \left(\frac{x}{n-x} \right)^2, \dots, w_n \left(\frac{x}{n-x} \right)^n \right] \\ &= [w_0(n-x)^n, w_1x(n-x)^{n-1}, w_2x^2(n-x)^{n-2}, \dots, w_nx^n]. \end{aligned}$$

Observe that if we choose weights $w_i = \binom{n}{i}$, substitute ny for x , and remove the common factors of n^n , then we obtain the blending functions for the Bézier curve of Example 1.8. Note that replacing x by ny also replaces $i \in \mathcal{A}$ by $\frac{i}{n}$. 

Example 3.5. Krasauskas [14] generalized the classical Bézier parametrization to any polytope with integer vertices. A polytope Δ is defined by its facet inequalities

$$\Delta = \{x \in \mathbb{R}^d \mid h_i(x) \geq 0, i = 1, \dots, N\}.$$

Here, Δ has N facets and for each $i = 1, \dots, N$, $h_i(x) = \langle \mathbf{v}_i, x \rangle + c_i$ is the linear function defining the i th facet, where $\mathbf{v}_i \in \mathbb{Z}^d$ is the (inward oriented) primitive vector normal to the facet and $c_i \in \mathbb{Z}$. (Compare these to the functions in Example 1.9.)

Let $\mathcal{A} \subset \Delta \cap \mathbb{Z}^d$ be any subset of the integer points of Δ that includes its vertices. Let $w = \{w_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\} \subset \mathbb{R}_{>}$ be a collection of positive weights. For every $\mathbf{a} \in \mathcal{A}$, Krasauskas defined *toric Bézier functions*

$$(3.6) \quad \beta_{\mathbf{a}}(x) := w_{\mathbf{a}} h_1(x)^{h_1(\mathbf{a})} h_2(x)^{h_2(\mathbf{a})} \dots h_N(x)^{h_N(\mathbf{a})}.$$

For $x \in \Delta$, these parametrize a patch X_{β} .

Krasauskas defined a toric patch to be any reparametrization of such a patch. Our definition of a toric patch (Definition 3.2) agrees with Krasauskas's, when $\mathcal{A} = \Delta \cap \mathbb{Z}^d$ consists of all the integer points in a polytope: Observe that the map β in (3.6) is the composition of a map $H : \Delta \rightarrow \mathbb{R}^N$ given by $x \mapsto (h_1(x), \dots, h_N(x))$ with a rational map $\varphi : \mathbb{C}^N \dashrightarrow \mathbb{RP}^{\mathcal{A}}$ given by


$$\varphi : (u_1, \dots, u_N) \longmapsto [w_{\mathbf{a}} u_1^{h_1(\mathbf{a})} \dots u_N^{h_N(\mathbf{a})} \mid \mathbf{a} \in \mathcal{A}].$$

This map φ factors through the map $\varphi_{\mathcal{A},w}$. Indeed, define a map

$$\begin{aligned} f_{\Delta} : (\mathbb{C}^{\times})^N &\longrightarrow (\mathbb{C}^{\times})^n \\ u = (u_1, \dots, u_N) &\longmapsto t = (t_1, \dots, t_n) \quad \text{where} \quad t_j := \prod_{i=1}^N u_i^{\langle \mathbf{v}_i, \mathbf{e}_j \rangle} \end{aligned}$$

Then

$$\prod_{i=1}^N u_i^{h_i(\mathbf{a})} = \prod_{i=1}^N u_i^{c_i} \cdot \prod_{i=1}^N u_i^{\langle \mathbf{v}_i, \mathbf{a} \rangle} = u^c \cdot t^{\mathbf{a}}.$$


And so $\varphi(u) = u^c \cdot \varphi_{\mathcal{A},w}(f_{\Delta}(u))$. This shows that $X_{\beta} \subset X_{\mathcal{A},w}$, and Krasauskas [15, §4] shows that this inclusion is an equality. 

Example 3.7. A *standard d -simplex of degree n* is the simplex in \mathbb{R}^d ,

$$n\Delta_d := \{x \in \mathbb{R}^d \mid x_1 + \cdots + x_d \leq n \text{ and } x_i \geq 0\}.$$

For a point $\mathbf{a} = (a_1, \dots, a_d)$ in $n\Delta_d \cap \mathbb{Z}^d$ set $|\mathbf{a}| := a_1 + \cdots + a_d$ and let $w_{\mathbf{a}} := \binom{n}{\mathbf{a}} = \frac{n!}{a_1! \cdots a_d! (n-|\mathbf{a}|)!}$ be the multinomial coefficient. This gives a system of weights for $\mathcal{A} = n\Delta \cap \mathbb{Z}^d$. Then Krasauskas's toric Bézier patch for $n\Delta_d$ has blending functions

$$\beta_{\mathbf{a}} := \binom{n}{\mathbf{a}} (n - \sum_i x_i)^{n - \sum_i a_i} \prod_{i=1}^d x_i^{a_i}.$$

If we substitute ny_i for x_i and then remove the common factors of n^n , we recover the classical Bernstein polynomials [8, §4.1]. 

Example 3.8. The *Bézier simploids* [6] are toric patches based on products of Bézier simplices. Suppose that d_1, \dots, d_m and n_1, \dots, n_m are positive integers. Set $d := d_1 + \cdots + d_m$. Write \mathbb{R}^d as a direct sum $\mathbb{R}^{d_1} \oplus \cdots \oplus \mathbb{R}^{d_m}$ and suppose that for each $i = 1, \dots, m$, the scaled simplex $n_i\Delta_{d_i}$ lies in the i th summand \mathbb{R}^{d_i} . Consider the Minkowski sum

$$\Delta := n_1\Delta_{d_1} + n_2\Delta_{d_2} + \cdots + n_m\Delta_{d_m} \subset \mathbb{R}^d,$$


which is just the product of the simplices $n_i\Delta_{d_i}$.

If $\mathbf{x}_i := (x_{i,1}, \dots, x_{i,d_i})$ are coordinates on \mathbb{R}^{d_i} , then

$$\Delta = \{\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_m) \mid x_{i,j} \geq 0 \text{ and } x_{i,1} + \cdots + x_{i,d_i} \leq n_i \text{ } i = 1, \dots, m\}.$$

Let $\mathcal{A} := \Delta \cap \mathbb{Z}^d$ be the set of integer points of Δ . Given an integer point $\mathbf{a} = (\mathbf{a}_1, \dots, \mathbf{a}_m) \in \mathcal{A}$, set $w_{\mathbf{a}} := \prod_{i=1}^m \binom{n_i}{\mathbf{a}_i}$, the product of multinomial coefficients. This gives a system of weights, and let $Y_{\mathcal{A},w}$ be the resulting toric patch. Krasauskas's toric patch of shape Δ has blending functions

$$\beta_{\mathbf{a}}(\mathbf{x}) := \prod_{i=1}^m \beta_{\mathbf{a}_i}(\mathbf{x}_i),$$

where $\beta_{\mathbf{a}_i}(\mathbf{x}_i)$ for $\mathbf{x}_i \in \mathbb{R}^{d_i}$ are the Bernstein polynomials. The resulting patch is called a *Bézier simploid* in [6]. 

We state our main result about linear precision for toric patches, which is a useful reformulation of linear precision for toric patches.

Theorem 3.9. *Let $\mathcal{A} \subset \mathbb{Z}^d$ be a primitive collection of exponent vectors, $w \in \mathbb{R}_{>}^{\mathcal{A}}$ be a system of weights, and define the Laurent polynomial*

$$f = f_{\mathcal{A},w} := \sum_{a \in \mathcal{A}} w_a x^a.$$

A toric patch of shape (\mathcal{A}, w) has rational linear precision if and only if the rational function $\psi_{\mathcal{A},w}: \mathbb{C}^d \dashrightarrow \mathbb{C}^d$ defined by

$$(3.10) \quad \frac{1}{f} \left(x_1 \frac{\partial}{\partial x_1} f, x_2 \frac{\partial}{\partial x_2} f, \dots, x_d \frac{\partial}{\partial x_d} f \right)$$

is a birational isomorphism.

Remark 3.11. The map (3.10) has an interesting reformulation in terms of toric derivatives. The i th *toric derivative* of a Laurent polynomial f is $x_i \frac{\partial}{\partial x_i} f$. The toric differential $D_{\text{torus}} f$ is the vector whose components are the toric derivatives of f . Thus the map $\psi_{\mathcal{A},w}$ is the logarithmic toric differential $D_{\text{torus}} \log f$.

In the proof below, we show that the map $\psi_{\mathcal{A},w}$ comes from a map $\mathbb{C}^d \dashrightarrow \mathbb{C}\mathbb{P}^d$,

$$(3.12) \quad x \longmapsto \left[f : x_1 \frac{\partial}{\partial x_1} f : x_2 \frac{\partial}{\partial x_2} f : \cdots : x_d \frac{\partial}{\partial x_d} f \right]$$

Let $F_{\mathcal{A},w}$ be the homogenization of f with respect to a new variable x_0 . Then

$$\deg(F_{\mathcal{A},w}) F_{\mathcal{A},w} = \sum_{i=0}^d x_i \frac{\partial}{\partial x_i} F_{\mathcal{A},w}.$$

Thus, after homogenizing and a linear change of coordinates, the map (3.12) is defined by the formula

$$D_{\text{torus}} F_{\mathcal{A},w} = \left[x_0 \frac{\partial}{\partial x_0} F_{\mathcal{A},w} : x_1 \frac{\partial}{\partial x_1} F_{\mathcal{A},w} : \cdots : x_d \frac{\partial}{\partial x_d} F_{\mathcal{A},w} \right] \quad \color{blue}{\square}$$

This last form of the map (3.12) gives an appealing reformulation of Theorem 3.9 which we feel is the most useful for further analysis of linear precision for toric patches.

Corollary 3.13. *The toric patch of shape (\mathcal{A}, w) has linear precision if and only if the map $D_{\text{torus}} F_{\mathcal{A},w} : \mathbb{C}\mathbb{P}^d \dashrightarrow \mathbb{C}\mathbb{P}^d$ is a birational isomorphism.*

Proof of Theorem 3.9. By Theorem 2.9, a toric patch of shape (\mathcal{A}, w) has linear precision if and only if the complexified tautological projection

$$\pi_{\mathcal{A}} : Y_{\mathcal{A},w} \dashrightarrow \mathbb{C}\mathbb{P}^d$$

is a birational isomorphism. Precomposing this with the defining parametrization of $Y_{\mathcal{A},w}$, $\varphi_{\mathcal{A},w} : (\mathbb{C}^\times)^d \hookrightarrow Y_{\mathcal{A},w}$ (which is a rational map $\mathbb{C}^d \dashrightarrow Y_{\mathcal{A},w}$), gives a map

$$(3.14) \quad \begin{array}{ccccc} \mathbb{C}^d & \xrightarrow{\varphi_{\mathcal{A},w}} & Y_{\mathcal{A},w} & \xrightarrow{-\pi_{\mathcal{A}}} & \mathbb{C}\mathbb{P}^d \\ x & \longmapsto & [w_{\mathbf{a}} x^{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}] & \longmapsto & \sum_{\mathbf{a} \in \mathcal{A}} w_{\mathbf{a}} x^{\mathbf{a}}(1, \mathbf{a}). \end{array}$$

The initial (0th) coordinate of this map is

$$\sum_{\mathbf{a} \in \mathcal{A}} w_{\mathbf{a}} x^{\mathbf{a}}$$

which is the polynomial $f_{\mathcal{A},w}$. For $i > 0$ the i th coordinate of this composition is


$$\sum_{\mathbf{a} \in \mathcal{A}} a_i \cdot w_{\mathbf{a}} x^{\mathbf{a}} = x_i \frac{\partial}{\partial x_i} f_{\mathcal{A},w}.$$

If we divide the composite map (3.14) by its 0th coordinate, $f_{\mathcal{A},w}$, we obtain the map $\psi_{\mathcal{A},w}$. This map is birational if and only if $\pi_{\mathcal{A}}$ is birational, as $\varphi_{\mathcal{A},w}$ is birational. 

Example 3.15. Let $\mathcal{A} := \{0, 1, \dots, n\} \subset \mathbb{Z}$ as in Example 3.4. Let $w \in \mathbb{R}_{>}^{n+1}$ be any set of weights. Then the polynomial $f = f_{\mathcal{A},w}$ is

$$f(x) = w_0 + w_1x + \cdots + w_nx^n.$$

Suppose that this toric patch has rational linear precision. By Theorem 3.9, the logarithmic toric differential $xf'(x)/f(x)$ is a birational isomorphism. Thus the fraction $xf'(x)/f(x)$ reduces to a quotient of linear polynomials and so $f(x)$ is necessarily a pure power, say $a(x + \alpha)^n$ with $\alpha \neq 0$.

We conclude that the weights are $w_i = \alpha^{n-i} \binom{n}{i}$, which are not quite the weights in the Bézier curve 1.8. If we rescale x (which does not change $Y_{\mathcal{A},w}$), setting $y = x/\alpha$, then the weights become $|\alpha|^n \binom{n}{i}$. Removing the common factor of $|\alpha|^n$, reveals that this toric patch is the classical Bézier curve of Example 1.8. 


We close with two problems.

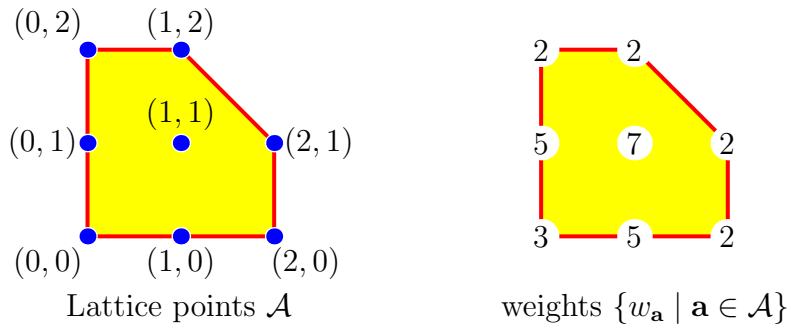
Problem 1. Classify the toric patches of dimension d that have linear precision.

The analysis of this section was recently used to classify toric surface patches ($d = 2$) having linear precision [11]. It remains an open problem to understand how to tune a toric patch (moving the points \mathcal{A}) to achieve linear precision.

Problem 2. Classify the toric patches of dimension d that may be tuned to have linear precision.

This is a strictly larger class of patches. For example, the classification of [11] shows if a toric surface patch has linear precision, then it is triangular or quadrilateral, but it is possible to tune a pentagonal patch to achieve linear precision.

Example 3.16. We discuss Krasauskas's version [15] of Karčiauskas's pentagonal M -patch [12], which has linear precision. Let \mathcal{A} be the integer points in the pentagon  shown below, and let $\{w_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\}$ be the indicated weights.



These define a toric patch $X_{\mathcal{A},w}$ and translated toric variety $Y_{\mathcal{A},w}$. This subvariety of \mathbb{P}^7 has degree 7 and is defined by 14 quadratic equations, all of the form

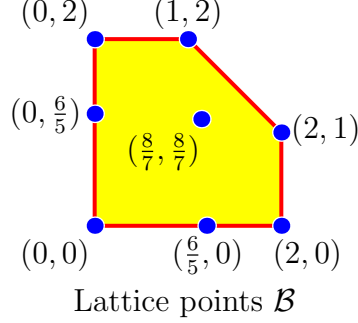
$$y_{\mathbf{a}}y_{\mathbf{b}} - y_{\mathbf{c}}y_{\mathbf{d}} \quad \text{where } \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathcal{A} \quad \text{with } \mathbf{a} + \mathbf{b} = \mathbf{c} + \mathbf{d}.$$

This has a monomial parametrization

$$\varphi_{\mathcal{A},w} : (s, t) \longmapsto [s^{a_1}t^{a_2} \mid \mathbf{a} = (a_1, a_2) \in \mathcal{A}].$$

Its composition with the tautological projection $\pi_{\mathcal{A}}$ gives a map $g: \mathbb{P}^2 \rightarrow \mathbb{P}^2$ of degree 4, with 3 base points at $(-1, -1)$, $(-1, -\frac{1}{2})$, and $(-\frac{1}{2}, -1)$

We tune this patch (or rather the projection $\pi_{\mathcal{A}}$) by moving the non-extreme points of \mathcal{A} within the pentagon. Let \mathcal{B} consist of the vertices of the pentagon, together with the three non-vertices



The composition of $\varphi_{\mathcal{A},w}$ with this new tautological projection $\pi_{\mathcal{B}}$ gives the rational map

$$(s, t) \mapsto \left(\frac{2s(2s + t + 3)}{(s + 1)(2s + 2t + 3)}, \frac{2t(s + 2t + 3)}{(t + 1)(2s + 2t + 3)} \right).$$

This is given by quadrics and has 3 base points $(-1, -1)$, $(0, -\frac{3}{2})$, and $(-\frac{3}{2}, 0)$, and so it has degree $1 = 2 \cdot 2 - 3$, and is therefore a birational isomorphism. By Theorem 2.9, the tuned toric patch (with the lattice points \mathcal{A} replaced by \mathcal{B}) has linear precision.

Thus there is a rational parametrization $\beta: \square \rightarrow X_{\mathcal{A},w}$ whose composition with the tautological projection $\pi_{\mathcal{B}}$ is the identity map. Indeed, Krasauskas [15] gives the following modification of the toric Bézier functions and shows that the map $\pi_{\mathcal{B}} \circ \beta$ is the identity.

$$(3.17) \quad \begin{aligned} \beta_{[0]} &:= 3(2-s)^2(3-s-t)^3(2-t)^2 \\ \beta_{[1]} &:= 5(2-s)(3-s-t)^2(2-t)^2s(3-s-\frac{t}{2}) \\ \beta_{[0]} &:= 5(3-\frac{s}{2}-t)t(2-s)^2(3-s-t)^2(2-t) \\ \beta_{[1]} &:= 7(3-\frac{s}{2}-t)t(2-t)(3-s-t)(2-s)s(3-s-\frac{t}{2}) \\ \beta_{[2]} &:= 2(3-s-t)(2-t)^2s^2(3-s-\frac{t}{2})^2 \\ \beta_{[0]} &:= 2(3-\frac{s}{2}-t)^2t^2(2-s)^2s(3-s-t) \\ \beta_{[2]} &:= 2(3-\frac{s}{2}-t)t(2-t)s^2(3-s-\frac{t}{2})^2 \\ \beta_{[1]} &:= 2(3-\frac{s}{2}-t)^2t^2(2-s)s(3-s-\frac{t}{2}) \end{aligned}$$

This is a modification of the toric Bézier functions for the pentagon. If, in the definition (3.6), we replace the form s by $s(3-s-\frac{t}{2})$ and the form t by $(3-\frac{s}{2}-t)t$, then we get these modified blending functions (3.17).

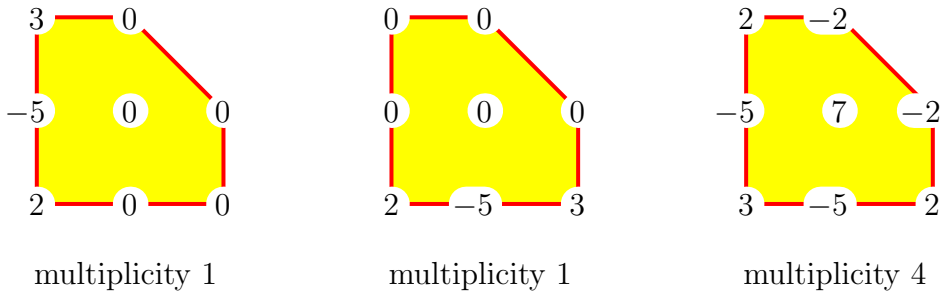
Let us consider the geometry of linear precision for this tuned patch in the spirit of Section 2. The projection $\pi_{\mathcal{B}}$ is defined by the three linear forms.

$$\begin{aligned} X &:= \frac{6}{5}y_{[0]} + 2y_{[2]} + \frac{8}{7}y_{[1]} + 2y_{[1]} + y_{[2]} \\ Y &:= \frac{6}{5}y_{[0]} + 2y_{[2]} + \frac{8}{7}y_{[1]} + 2y_{[1]} + y_{[1]} \\ Z &:= y_{[0]} + y_{[0]} + y_{[1]} + y_{[1]} + y_{[2]} + y_{[2]} + y_{[1]} + y_{[2]} \end{aligned}$$

Its center $E_{\mathcal{B}}$ is defined by the vanishing of these three forms. A general linear subspace L of codimension 2 containing $E_{\mathcal{B}}$ is defined by equations of the form

$$X = xZ \quad \text{and} \quad Y = yZ.$$

This subspace meets $Y_{\mathcal{A},w}$ in four points. One point lies outside of $E_{\mathcal{B}}$ while three points lie in the center $E_{\mathcal{B}}$. We give the three points.



We write the coordinate $y_{\mathbf{a}}$ of a point $y \in \mathbb{R}P^A$ at the position of the point \mathbf{a} . ▭

4. ITERATIVE PROPORTIONAL FITTING FOR TORIC PATCHES

The toric patch $X_{\mathcal{A},w}$ from geometric modeling appears naturally in algebraic statistics in the form of a toric model, which leads to a dictionary between the subjects. We show that a toric patch has rational linear precision if and only if it has maximum likelihood degree 1 as a statistical model. As a consequence, we present a new family of toric patches with rational linear precision. Finally, we propose the iterative proportional fitting algorithm from statistics as a tool to compute the unique reparametrization of a toric patch having linear precision.

In algebraic statistics, the image of $\mathbb{R}_{>}^d$ under the map $\varphi_{\mathcal{A},w}$ (3.1) is known as a *toric model* [18, §1.2]. They are more commonly called *log-linear models*, as the logarithms of the coordinates of $\varphi_{\mathcal{A},w}$ are linear functions in the logarithms of the coordinates of $\mathbb{R}_{>}^d$, or *discrete exponential families*, as the coordinates of $\varphi_{\mathcal{A},w}$ are exponentials of the coordinates of \mathbb{R}^d , which are themselves logarithms of the coordinates of $\mathbb{R}_{>}^d$.

We identify the non-negative orthant $\mathbb{R}P_{\geq}^A$ with the probability simplex

$$\Delta^A := \left\{ y \in \mathbb{R}_{\geq 0}^A \mid |y| := \sum_{a \in \mathcal{A}} y_a = 1 \right\},$$

and so we may also regard $X_{\mathcal{A},w}$ as a subvariety of Δ^A .

The tautological map $\pi_{\mathcal{A}}$ (2.5) appears in statistics. Given (normalized) data $q \in \Delta^{\mathcal{A}}$, the problem of *maximum likelihood estimation* is to find the toric model parameters (a point in $\mathbb{R}_{>}^d$) that best explain the data q . By Lemma 4 in [5], the maximum likelihood estimate for the log-linear model $X_{\mathcal{A},w}$ is the unique point $p \in X_{\mathcal{A},w}$ such that

$$\pi_{\mathcal{A}}(p) = \pi_{\mathcal{A}}(q) \quad \text{so that} \quad p = \pi_{\mathcal{A}}^{-1}(\pi_{\mathcal{A}}(q)).$$

Thus inverting the tautological projection is necessary for maximum likelihood estimation.

Catanese, *et. al.* [1] defined the *maximum likelihood degree* of the model $X_{\mathcal{A},w}$ to be the degree of $\pi_{\mathcal{A}}^{-1}$, as an algebraic function. Equivalently, this is the number of complex solutions to the critical equations of the likelihood function, which is the degree of the tautological map $\pi_{\mathcal{A}}$ from the Zariski closure $Y_{\mathcal{A},w}$ of $X_{\mathcal{A},w}$ to $\mathbb{C}\mathbb{P}^d$. By definition, a toric patch has rational linear precision if $\pi_{\mathcal{A}}^{-1}$ is a rational function (an algebraic function of degree 1).

Proposition 4.1. *A toric patch $X_{\mathcal{A},w}$ has rational linear precision if and only if $X_{\mathcal{A},w}$ has maximum likelihood degree 1.*

An important family of toric models called *decomposable graphical models* are known to have maximum likelihood degree 1 [16, p. 91]. Therefore, these models, which are not in general Bézier simplexes, have linear precision. They typically have large $d > 2$ dimension.

Darroch and Ratcliff [5] introduced the numerical algorithm of iterative proportional fitting, also known as generalized iterative scaling, for computing the inverse of the tautological projection $\pi_{\mathcal{A}}^{-1}$. Their algorithm requires that the data \mathcal{A} be in a normal, homogeneous form.

Observe first that the toric patch $X_{\mathcal{A},w}$ does not change if we translate all elements of \mathcal{A} by a fixed vector \mathbf{b} , ($\mathbf{a} \mapsto \mathbf{a} + \mathbf{b}$), so we may assume that \mathcal{A} lies in the positive orthant $\mathbb{R}_{>}^d$. Scaling the exponent vectors in \mathcal{A} by a fixed positive scalar $t \in \mathbb{R}_{>}$ also does not change $X_{\mathcal{A},w}$ as $x \mapsto x^t$ is a homeomorphism of $\mathbb{R}_{>}$ that extends to a homeomorphism of $\mathbb{R}_{>}^d$. Thus we may assume that \mathcal{A} lies in the standard simplex Δ_d in \mathbb{R}^d :

$$\Delta_d = \{x \in \mathbb{R}^d \mid x \geq 0 \quad \text{and} \quad |x| \leq 1\}.$$

Lastly, we lift this to the probability simplex $\Delta_d^+ \in \mathbb{R}_{\geq}^{d+1}$,

$$\Delta_d^+ := \{y \in \mathbb{R}^{1+d} \mid y_i \geq 0 \quad \text{and} \quad |y| = 1\},$$

by

$$\mathcal{A} \ni \mathbf{a} \mapsto \mathbf{a}^+ := (1 - |\mathbf{a}|, \mathbf{a}) \in \Delta_d^+.$$

Since for $t \in \mathbb{R}_{>}$ and $x \in \mathbb{R}_{>}^d$,

$$(t, tx)^{\mathbf{a}^+} = t^{1-|\mathbf{a}|} (tx)^{\mathbf{a}} = tx^{\mathbf{a}},$$

we see that replacing \mathcal{A} by its lifted version \mathcal{A}^+ also does not change $X_{\mathcal{A},w}$. We remark that this is just a way to homogenize the data for our problem.

Since these are affine transformations, the property of affine invariance for patches shows that the affine map that is the composition of these transformations intertwines the original tautological projection with the new one.

We describe the algorithm of *iterative proportional fitting*, which is Theorem 1 in [5].

Proposition 4.2. *Suppose that $\mathcal{A} \subset \Delta_d$ and $y \in \text{conv}(\mathcal{A})$. Then the sequence*

$$\{\mathbf{p}^{(n)} \mid n = 0, 1, 2, \dots\}$$

whose \mathbf{a} -coordinates are defined by $p_{\mathbf{a}}^{(0)} := w_{\mathbf{a}}$ and, for $n \geq 0$,

$$p_{\mathbf{a}}^{(n+1)} := p_{\mathbf{a}}^{(n)} \cdot \frac{y^{\mathbf{a}}}{(\pi_{\mathcal{A}}(\mathbf{p}^{(n)}))^{\mathbf{a}}},$$

converges to the unique point $\mathbf{p} \in X_{\mathcal{A},w}$ such that $\pi_{\mathcal{A}}(\mathbf{p}) = \pi_{\mathcal{A}}(y)$.

We remark that if \mathcal{A} is not homogenized then to compute $\pi_{\mathcal{A}}^{-1}(y)$ for $y \in \text{conv}(\mathcal{A})$, we first put \mathcal{A} into homogeneous form \mathcal{A}^+ using an affine map ψ , and then use iterative proportional fitting to compute $\pi_{\mathcal{A}^+}^{-1}(\psi(y)) = \pi_{\mathcal{A}}^{-1}(y)$. We also call this modification of the algorithm of Proposition 4.2 iterative proportional fitting. Thus iterative proportional fitting computes the inverse image of the tautological projection, which, by Theorem 2.6 gives the unique parametrization of $X_{\mathcal{A},w}$ having linear precision.

Corollary 4.3. *Iterative proportional fitting computes the unique parametrization of a toric patch having linear precision.*

Little is known about the convergence of iterative proportional fitting. We think that it is an interesting question to investigate this convergence, particularly how it may relate to maximum likelihood degree.

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