TORIC DEGENERATIONS OF BÉZIER PATCHES: EXTENDED ABSTRACT

LUIS DAVID GARCÍA-PUENTE, FRANK SOTTILE, AND CHUNGANG ZHU

1. Introduction

In geometric modeling of curves and surfaces, the overall shape of an individual patch is intuitively governed by the placement of control points, and a patch may be finely tuned by altering the weights of the basis functions—large weights pull the patch towards the corresponding control points. The control points also have a global meaning as the patch lies within the convex hull of the control points, for any choice of weights.

This convex hull is often indicated by drawing some edges between the control points. The rational bicubic tensor product patches in Figure 1 have the same weights but different control points, and the same 3×3 quadrilateral grid of edges drawn between the control points. Unlike the control points or their convex hulls, there is no canonical choice of these

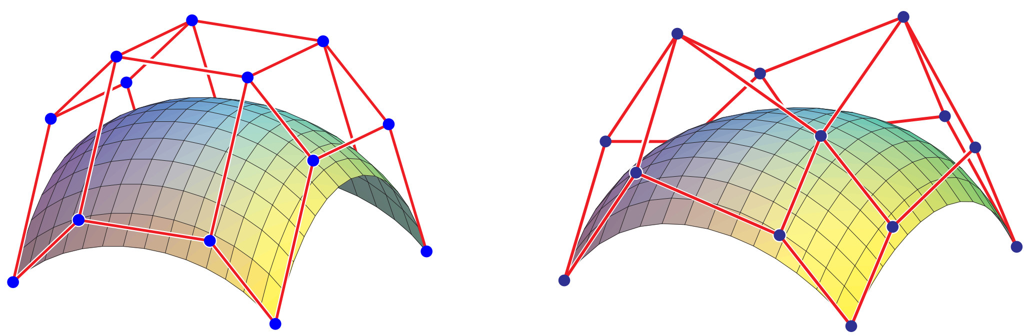

Figure 1. Two rational bicubic patches.

edges. We paraphrase a question posed to us by Carl de Boor and Ron Goldman: What is the significance for modeling of such control structures (control points plus edges)?

We provide an answer to this question. These control structures, the triangles, quadrilaterals, and other shapes implied by these edges, encode limiting positions of the patch when the weights assume extreme values. By Theorems 5.1 and 5.2, the only possible limiting positions of a patch are the control structures arising from *regular decompositions* (see Section 4) of the points indexing its basis functions and control points, and any such regular control structure is the limiting position of some sequence of patches. Our results rely

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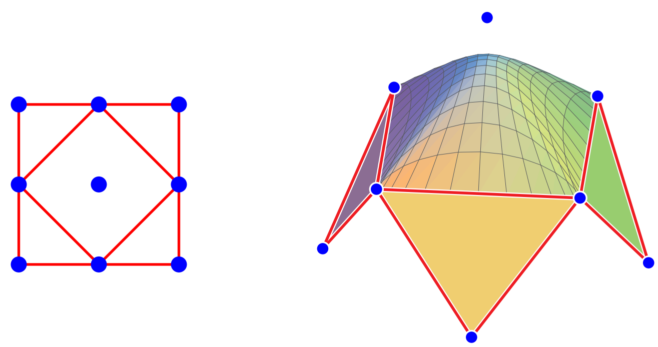
upon a construction in computational algebraic geometry called a toric degeneration [7, Ch. 8.3.1].

While our primary interest is to explain the meaning of control nets for the classical rational tensor product patches and rational Bézier triangles, we work in the generality of Krasauskas' toric Bézier patches [11, 12]. The reason for this is simple—any polygon may arise in a regular decomposition of the points underlying a classical patch. On the left below is a regular decomposition of the points in the 2×2 grid underlying a biquadratic patch and on the right is a degenerate patch, which consists of four triangles and Krasauskas's double pillow. The pillow corresponds to the central quadrilateral in the 2×2 grid, with the 'free' internal control point corresponding to the center point of the grid.

(1)



While our primary interest is in surface patches, our definitions and arguments make sense in any dimension. Our discussion will concern surface patches but the proofs are given for patches of any dimension.

2. Bézier patches and control nets

We define rational Bézier curves and surfaces and tensor product patches in a form that is convenient for our discussion, and then describe their control nets. Our presentation differs from the standard formulation [5] in that the degree is encoded by the domain. This linear reparametrization does not affect the resulting curve or surface. Write $\mathbb{R}_{\geq}$ for the nonnegative real numbers and $\mathbb{R}_{>}$ for the positive real numbers.

Let d be a positive integer. For each $i = 0, \ldots, d$ define the *Bernstein polynomial* $\beta_{i;d}(x)$,

$$\beta_{i;d}(x) \ := \ x^i (d-x)^{d-i} \, .$$

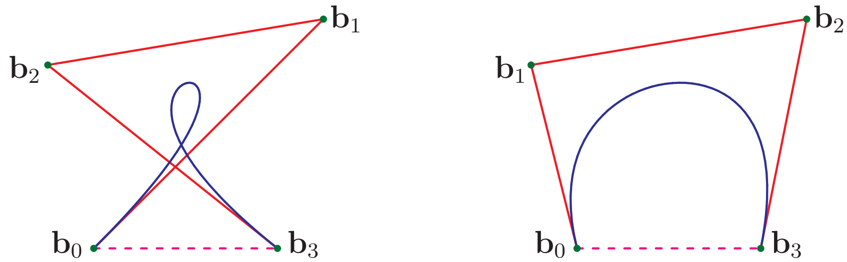
(Substituting $x = dy$ and multiplying by $\binom{d}{i} d^{-d}$ for normalization, this becomes the usual Bernstein polynomial. This nonstandard presentation, omitting the binomial coefficients, stresses the separate role of functions and weights.) Given weights $w_0, \ldots, w_d \in \mathbb{R}_{>}$ and control points $\mathbf{b}_0, \ldots, \mathbf{b}_d \in \mathbb{R}^n$ ($n = 2, 3$), we have the parameterized *rational Bézier curve*

$$F(x) \ := \ \frac{\sum_{i=0}^{d} w_i \mathbf{b}_i \beta_{i;d}(x)}{\sum_{i=0}^{d} w_i \beta_{i;d}(x)} \ : \ [0, d] \ \longrightarrow \ \mathbb{R}^n \, .$$

Our domain is $[0, d]$ rather than $[0, 1]$, for this is the natural convention for toric patches.

The *control polygon* of the curve is the union of segments $\overline{\mathbf{b}_0, \mathbf{b}_1}, \ldots, \overline{\mathbf{b}_{d-1}, \mathbf{b}_d}$. Here are two rational cubic Bézier planar curves with their control polygons.

(2)



There are two standard ways to extend this to surfaces. The most straightforward gives rational tensor product patches. Let c, d be positive integers and for each $i = 0, \ldots, c$ and $j = 0, \ldots, d$ let $w_{(i,j)} \in \mathbb{R}_>$ and $\mathbf{b}_{(i,j)} \in \mathbb{R}^3$ be a weight and a control point. The associated rational tensor product patch of bidegree (c, d) is the image of the map $[0, c] \times [0, d] \to \mathbb{R}^3$,

$$F(x, y) \;:=\; \frac{\sum_{i=0}^{c} \sum_{j=0}^{d} w_{(i,j)} \mathbf{b}_{(i,j)} \beta_{i;c}(x) \beta_{j;d}(y)}{\sum_{i=0}^{c} \sum_{j=0}^{d} w_{(i,j)} \beta_{i;c}(x) \beta_{j;d}(y)} \;.$$

Triangular Bézier patches are another extension. Set

$$d\triangle \;:=\; \{(x, y) \in \mathbb{R}^2 \mid 0 \le x, y \text{ and } x + y \le d\}$$

and set $\mathcal{A} := d\triangle \cap \mathbb{Z}^2$, the points with integer coordinates (lattice points) in the triangle $d\triangle$. For $(i, j) \in \mathcal{A}$, we have the bivariate Bernstein polynomial

$$\beta_{(i,j);d}(x, y) \;:=\; x^i y^j (d - x - y)^{d-i-j} \;.$$

Given weights $w = \{w_{(i,j)} \mid (i, j) \in \mathcal{A}\}$ and control points $\mathcal{B} = \{\mathbf{b}_{(i,j)} \mid (i, j) \in \mathcal{A}\}$, the associated triangular rational Bézier patch is the image of the map $d\triangle \to \mathbb{R}^3$,

$$F(x, y) \;:=:\; \frac{\sum_{(i,j) \in \mathcal{A}} w_{(i,j)} \mathbf{b}_{(i,j)} \beta_{(i,j);d}(x, y)}{\sum_{(i,j) \in \mathcal{A}} w_{(i,j)} \beta_{(i,j);d}(x, y)} \;.$$

The control points of a Bézier curve are naturally connected in sequence to give the control polygon, which is a piecewise linear caricature of the curve. For a surface patch there are however many ways to interpolate the control points by edges to form a control net. There also may not be a way to fill in these edges with polygons to form a polytope. Even when this is possible, the significance of this structure for the shape of the patch is not evident, except in special cases. For example, if the control points are the graph of a convex function over the lattice points, then the patch is convex [3, 4]. For such control points, the obvious net consists of the upward-pointing facets of the convex hull of the graph. This is the net of the bicubic patches in Figure 1.

3. Toric patches and toric varieties

Krasauskas's toric patches [11] are a natural extension of rational Bézier triangles and rational tensor product patches to arbitrary polygons whose vertices have integer coordinates, called *lattice polygons*. They are based on toric varieties [1, 6] from algebraic geometry which get their name as they are natural compactifications of algebraic tori $(\mathbb{C}^*)^n$, where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. They are naturally associated to lattice polygons (and in higher dimensions, lattice polytopes), and the positive real part [6, Ch. 4] [14] of a toric variety is canonically identified with the corresponding polygon/polytope.

Toric patches begin with a finite set $\mathcal{A} \subset \mathbb{Z}^2$ of (integer) lattice points. The convex hull of $\mathcal{A}$ is the set of all convex combinations

$$\sum_{\mathbf{a}\in\mathcal{A}} p_{\mathbf{a}}\mathbf{a} \qquad \text{where} \qquad p_{\mathbf{a}} \geq 0 \qquad \text{and} \qquad 1 \;=\; \sum_{\mathbf{a}\in\mathcal{A}} p_{\mathbf{a}}$$

of points of $\mathcal{A}$, which is a lattice polygon and is written $\Delta_{\mathcal{A}}$. To each edge e of $\Delta_{\mathcal{A}}$, there is a valid inequality $h_e(x,y) \geq 0$ on $\Delta_{\mathcal{A}}$, where $h_e(x,y)$ is a linear polynomial with integer coefficients having no common integer factors that vanishes on the edge e. For example, if $\mathcal{A} = d\triangle \cap \mathbb{Z}^2$, then $\Delta_{\mathcal{A}} = d\triangle$ and the inequalities are

$$x \geq 0, \quad y \geq 0, \quad \text{and} \quad d{-}x{-}y \geq 0.$$

Let E be the set of edges of the polygon $\Delta_{\mathcal{A}}$. To each lattice point $\mathbf{a} \in \mathcal{A}$, define the *toric basis function* $\beta_{\mathbf{a},\mathcal{A}} \colon \Delta_{\mathcal{A}} \to \mathbb{R}$ to be

$$\beta_{\mathbf{a},\mathcal{A}}(x,y) \;:=\; \prod_{e\in E} h_e(x,y)^{h_e(\mathbf{a})}.$$

This is strictly positive in the interior of $\Delta_{\mathcal{A}}$. If $\mathbf{a}$ lies on an edge e of $\Delta_{\mathcal{A}}$, then $\beta_{\mathbf{a},\mathcal{A}}$ is strictly positive on the relative interior of e, and if $\mathbf{a}$ is a vertex, then $\beta_{\mathbf{a},\mathcal{A}}(\mathbf{a}) > 0$. In particular the toric basis functions have no common zeroes in $\Delta_{\mathcal{A}}$.

Observe that the toric basis functions for $\mathcal{A} = [0,c] \times [0,d] \cap \mathbb{Z}^2$ and $\mathcal{A} = d\triangle \cap \mathbb{Z}^2$ are equal to the Bernstein polynomials $\beta_{i;c}(x)\beta_{j;d}(y)$ and $\beta_{(i,j);d}(x,y)$ underlying the tensor product and triangular Bézier patches.

Toric patch also require *weights* and control points. Let $\#\mathcal{A}$ be the number of points in $\mathcal{A}$. Let $\mathbb{R}^{\mathcal{A}}_{>}$ be $\mathbb{R}^{\#\mathcal{A}}_{>}$ with coordinates $(z_{\mathbf{a}} \in \mathbb{R}_{>} \mid \mathbf{a} \in \mathcal{A})$ indexed by elements of $\mathcal{A}$. A toric Bézier patch of shape $\mathcal{A}$ is given by a collection of positive weights $w = (w_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}) \in \mathbb{R}^{\mathcal{A}}_{>}$ and control points $\mathcal{B} = \{\mathbf{b}_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\} \subset \mathbb{R}^3$. These define a map $\Delta_{\mathcal{A}} \to \mathbb{R}^3$,

$$F_{\mathcal{A},w,\mathcal{B}}(x,y) \;:=\; \frac{\sum_{\mathbf{a}\in\mathcal{A}} w_{\mathbf{a}}\mathbf{b}_{\mathbf{a}}\beta_{\mathbf{a},\mathcal{A}}(x,y)}{\sum_{\mathbf{a}\in\mathcal{A}} w_{\mathbf{a}}\beta_{\mathbf{a},\mathcal{A}}(x,y)}\;. \tag{3}$$

Since the toric basis functions are nonnegative on $\Delta_{\mathcal{A}}$ and have no common zeroes, this denominator is strictly positive on $\Delta_{\mathcal{A}}$. Write $Y_{\mathcal{A},w,\mathcal{B}}$ for the image of $\Delta_{\mathcal{A}}$ under the map F, which is a *toric Bézier patch* of shape $\mathcal{A}$.

The map $F \colon \Delta_{\mathcal{A}} \to \mathbb{R}^3$ admits a factorization

$$F \;:\; \Delta_{\mathcal{A}} \xrightarrow{\;\beta_{\mathcal{A}}\;} \triangle^{\mathcal{A}} \xrightarrow{\;w\cdot\;} \triangle^{\mathcal{A}} \xrightarrow{\;\pi_{\mathcal{B}}\;} \mathbb{R}^3\,, \tag{4}$$

where $\triangle^{\mathcal{A}} \subset \mathbb{R}^{\mathcal{A}}$ is the standard simplex of dimension $\#\mathcal{A} - 1$, the map $\beta_{\mathcal{A}}$ is given by the toric basis functions $\beta_{\mathbf{a},\mathcal{A}}$, the map $w\cdot$ is (essentially) coordinatewise multiplication by the weights w, and the map $\pi_{\mathcal{B}}$ is a projection given by the control points $\mathcal{B}$. The purpose of this factorization is to clarify the role of the weights in a toric patch by isolating their effect. The image $\beta_{\mathcal{A}}(\Delta_{\mathcal{A}}) \subset \triangle^{\mathcal{A}}$ is a standard toric variety $X_{\mathcal{A}}$. Acting on this by the map $w\cdot$ gives a translated toric variety $X_{\mathcal{A},w}$, which we call a *lift* of the patch $Y_{\mathcal{A},w,\mathcal{B}}$ as its image under the projection $\pi_{\mathcal{B}}$ is $Y_{\mathcal{A},w,\mathcal{B}}$. We use results on the limiting position of the translates $X_{\mathcal{A},w}$ as the weights are allowed to vary, which are called toric degenerations.

We make this precise. Let $\mathbb{R}^{\mathcal{A}}_{\geq}$ be $\mathbb{R}^{\#\mathcal{A}}_{\geq}$ with coordinates $(z_{\mathbf{a}} \in \mathbb{R}_{\geq} \mid \mathbf{a} \in \mathcal{A})$ indexed by elements of $\mathcal{A}$. The standard simplex

$$\triangle^{\mathcal{A}} \ := \ \{z \in \mathbb{R}^{\mathcal{A}}_{\geq} \mid \textstyle\sum_{\mathbf{a}\in\mathcal{A}} z_{\mathbf{a}} = 1\}$$

is the convex hull of the standard basis in $\mathbb{R}^{\mathcal{A}}$. It has homogeneous coordinates,

$$[z_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}] \ := \ \frac{1}{\sum_{\mathbf{a}\in\mathcal{A}} z_{\mathbf{a}}}(z_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A})\,. \tag{5}$$

Geometrically, $[z_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}] \in \triangle^{\mathcal{A}}$ is the unique point where the ray $\mathbb{R}_{>} \cdot (z_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A})$ meets the simplex $\triangle^{\mathcal{A}}$.

Let $\beta_{\mathcal{A}} \colon \Delta_{\mathcal{A}} \to \triangle^{\mathcal{A}}$ be the map $\beta_{\mathcal{A}}(x,y) = [\beta_{\mathbf{a},\mathcal{A}}(x,y) \mid \mathbf{a} \in \mathcal{A}]$ Given weights $w \in \mathbb{R}^{\mathcal{A}}_{>}$, we have the map $w\cdot \colon \triangle^{\mathcal{A}} \to \triangle^{\mathcal{A}}$ defined by

$$w \cdot [z_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}] \ = \ [w_{\mathbf{a}} z_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}]\,.$$

Lastly, given control points $\mathcal{B}$, define the linear map $\pi_{\mathcal{B}} \colon \mathbb{R}^{\mathcal{A}} \to \mathbb{R}^3$ by

$$\pi_{\mathcal{B}}(z) \ := \ \sum_{\mathbf{a}\in\mathcal{A}} \mathbf{b}_{\mathbf{a}} z_{\mathbf{a}}\,.$$

The image of the simplex $\triangle^{\mathcal{A}}$ under $\pi_{\mathcal{B}}$ is the convex hull of the control points $\mathcal{B}$, and by these definitons, the map F in (3) defining the toric Bézier patch is the composition (4).

We call $Y_{\mathcal{A},w,\mathcal{B}}$ a toric patch because the image $\beta_{\mathcal{A}}(\Delta_{\mathcal{A}})$ is a toric variety, which we now explain. Elements $\mathbf{a}$ of $\mathbb{Z}^2$ are exponents of monomials,

$$\mathbf{a} \ = \ (a,b) \ \longleftrightarrow \ x^a y^b\,,$$

which we will write as $x^{\mathbf{a}}$. The points of $\mathcal{A}$ define a map $\varphi_{\mathcal{A}} \colon \mathbb{R}^2_{>} \to \triangle^{\mathcal{A}}$ by

$$\varphi_{\mathcal{A}}(x,y) \ := \ [x^{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}]\,.$$

The closure in $\triangle^{\mathcal{A}}$ of the image of $\varphi_{\mathcal{A}}$ is the *toric variety $X_{\mathcal{A}}$*. We have the following result of Krasauskas [11].

Proposition 3.1. *The image of $\Delta_{\mathcal{A}}$ under the map $\beta_{\mathcal{A}}$ is the toric variety $X_{\mathcal{A}}$.*

Toric patches share with rational Bézier patches the following recursive structure. If $\mathbf{a}$ is a vertex of $\Delta_{\mathcal{A}}$, then $\mathbf{b}_{\mathbf{a}} = F_{\mathcal{A},w,\mathcal{B}}(\mathbf{a})$ is a point in the patch. If e is the edge between two vertices of $\Delta_{\mathcal{A}}$, then the restriction $F_{\mathcal{A},w,\mathcal{B}}|_e$ of $F_{\mathcal{A},w,\mathcal{B}}$ to e is the 1-dimensional toric patch given by the points of $\mathcal{A}$ lying on e and the corresponding weights, which is a rational Bézier curve. For example, the edges of the rational bicubic patches in Figure 1 are all rational cubic Bézier curves.

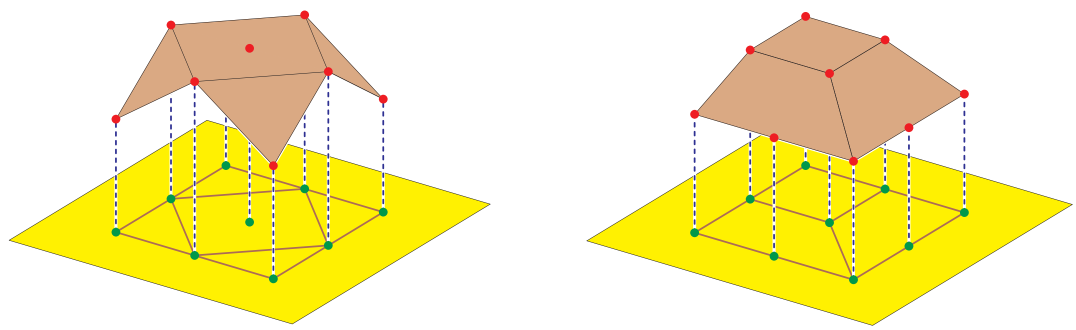
4. Regular polyhedral decompositions

We recall the definitions of regular (or coherent) polyhedral subdivisions from geometric combinatorics, which were introduced in [7, § 7.2]. Because subdivision has a different meaning in modeling, we instead use the term *decomposition*. Let $\mathcal{A} \subset \mathbb{R}^2$ be a finite set and suppose that $\lambda \colon \mathcal{A} \to \mathbb{R}$ is a function. We use λ to lift the points of $\mathcal{A}$ into $\mathbb{R}^3$. Let P_λ be the convex hull of the lifted points,

$$P_\lambda = \operatorname{conv}\{(\mathbf{a}, \lambda(\mathbf{a})) \mid \mathbf{a} \in \mathcal{A}\} \subset \mathbb{R}^3.$$

Each face of P_λ has an outward pointing normal vector, and its *upper facets* are those whose normal has positive last coordinate. If we project these upper facets back to $\mathbb{R}^2$, they cover the polygon $\Delta_{\mathcal{A}}$ and are the facets of the *regular polyhedral decomposition* $\mathcal{T}_\lambda$ of $\Delta_{\mathcal{A}}$ induced by λ. (Lower facets also induce a regular polyhedral subdivision, which equals $\mathcal{T}_{-\lambda}$, and so it is no loss of generality to work with upper facets.)

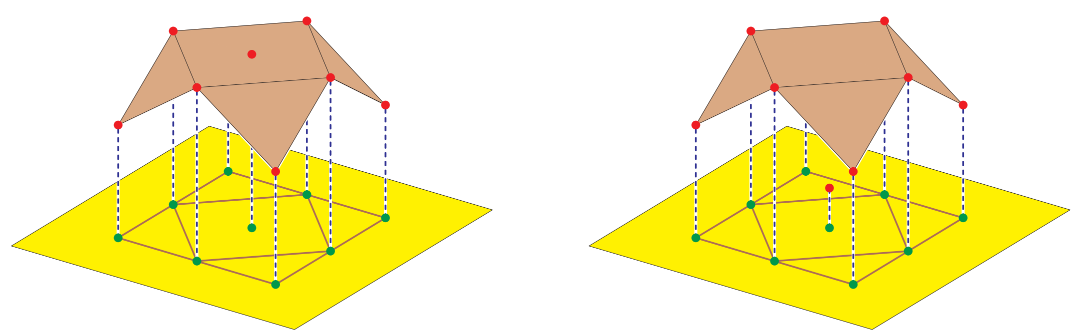
The edges and vertices of $\mathcal{T}_\lambda$ are the images of the edges and vertices lying on upper facets. Here are the upper facets and regular polyhedral decompositions given by two different lifting functions for the points $\mathcal{A}$ underlying a biquadratic tensor product patch.



More generally, a *polyhedral decomposition* of $\Delta_{\mathcal{A}}$ is a collection $\mathcal{T}$ of polygons, line segments, and points of $\mathcal{A}$, whose union is $\Delta_{\mathcal{A}}$, where any edge, vertex, or endpoint of a segment also lies in $\mathcal{T}$, and any two elements of $\mathcal{T}$ are either disjoint or their intersection is an element of $\mathcal{T}$. A decomposition $\mathcal{T}$ is *regular* if it is induced from a lifting function.

A *decomposition* $\mathcal{S}$ of the configuration $\mathcal{A}$ of points is a collection $\mathcal{S}$ of subsets of $\mathcal{A}$ called *faces*. The convex hulls of these faces are required to be the polygons, line segments, and vertices of a polyhedral decomposition $\mathcal{T}(\mathcal{S})$ of $\Delta_{\mathcal{A}}$. In particular, the intersection of any face with the convex hull $\Delta_{\mathcal{F}}$ of another face $\mathcal{F}$ of $\mathcal{S}$ is either empty, a vertex of $\Delta_{\mathcal{F}}$, or the points of $\mathcal{F}$ lying in some edge of $\Delta_{\mathcal{F}}$. A face $\mathcal{F}$ is a *facet*, *edge*, or *vertex* of $\mathcal{S}$ as its convex hull $\Delta_{\mathcal{F}}$ is a polygon, line segment, or vertex. The decomposition $\mathcal{S}$ is *regular* if the polyhedral decomposition $\mathcal{T}(\mathcal{S})$ is regular.

Below are two different lifting functions that induce the same regular polyhedral decomposition of the 2×2 square underlying a biquadratic patch, but different regular decompositions of $\mathcal{A}$.



The center point of $\mathcal{A}$ does not lie in any face of the decomposition on the right as its lift does not lie on any upper facet.

5. Regular control surfaces

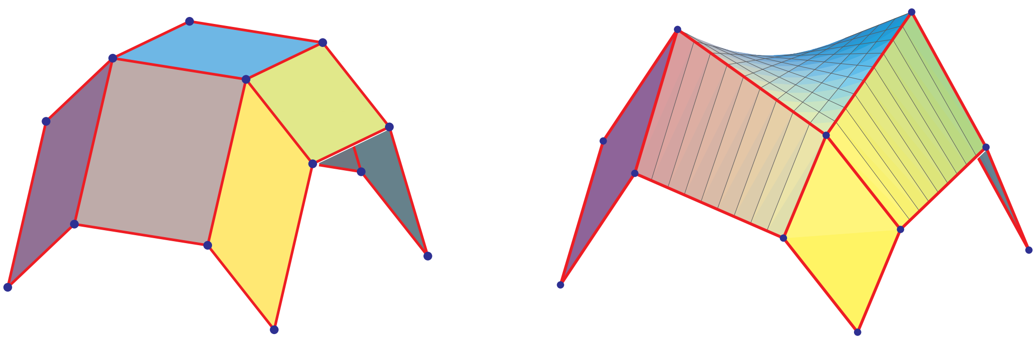
Regular control surfaces are possible limiting positions of patches. Let $\mathcal{A} \subset \mathbb{Z}^2$ be a finite set, $w \in \mathbb{R}^{\mathcal{A}}_{>}$ be weights and $\mathcal{B} = \{\mathbf{b_a} \mid \mathbf{a} \in \mathcal{A}\}$ be control points for a toric patch $Y_{\mathcal{A},w,\mathcal{B}}$ of shape $\mathcal{A}$.

Suppose that we have a decomposition $\mathcal{S}$ of $\mathcal{A}$. We may use the weights w and control points $\mathcal{B}$ indexed by elements of a facet $\mathcal{F}$ as weights and control points for a toric patch of shape $\mathcal{F}$, written $Y_{\mathcal{F},w|_{\mathcal{F}},\mathcal{B}|_{\mathcal{F}}}$. In fact, this can be done for any face of $\mathcal{S}$. The union

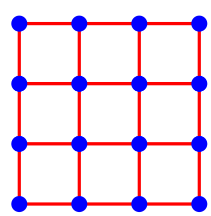
$$Y_{\mathcal{A},w,\mathcal{B}}(\mathcal{S}) \ := \ \bigcup_{\mathcal{F}\in\mathcal{S}} Y_{\mathcal{F},w|_{\mathcal{F}},\mathcal{B}|_{\mathcal{F}}}\,,$$

of these patches is the *control surface* induced by the decomposition $\mathcal{S}$. As the domain of a patch of shape $\mathcal{F}$ is the convex hull $\Delta_{\mathcal{F}}$ of $\mathcal{F}$ and faces of toric patches are again toric patches, the control surface of a patch induced by a decomposition is naturally a C^0 spline surface. A control surface $Y_{\mathcal{A},w,\mathcal{B}}(\mathcal{S})$ is *regular* if the decomposition $\mathcal{S}$ is regular.

Here are the control surfaces of the bicubic patches from Figure 1.

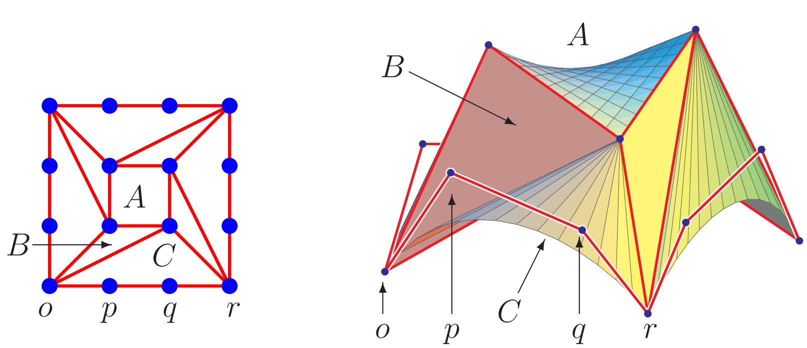


These control surfaces are regular as they are induced by the 3×3 grid below, which is a regular decomposition.



Here is an irregular decomposition of the 3×3 grid and the corresponding irregular control surface.

(6)



We show that regular control surfaces are exactly the possible limits of toric patches when the control points $\mathcal{B}$ are fixed but the weights w are allowed to vary. In particular, the irregular control surface (6) cannot be the limit of toric Bézier patches.

Let $\lambda\colon \mathcal{A} \to \mathbb{R}$ be a lifting function. We use this and a given set of weights $w = \{w_{\mathbf{a}} \in \mathbb{R}_{>} \mid \mathbf{a} \in \mathcal{A}\}$ to get a set of weights which depends upon a parameter, $w_\lambda(t) := \{t^{\lambda(\mathbf{a})} w_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\}$. These weights are used to define a *toric degeneration* of the patch,

$$F_{\mathcal{A},w,\mathcal{B},\lambda}(x;t) \ := \ \frac{\sum_{\mathbf{a}\in\mathcal{A}} t^{\lambda(\mathbf{a})} w_{\mathbf{a}} \mathbf{b}_{\mathbf{a}} \beta_{\mathbf{a}}(x)}{\sum_{\mathbf{a}\in\mathcal{A}} t^{\lambda(\mathbf{a})} w_{\mathbf{a}} \beta_{\mathbf{a}}(x)}\,.$$

Let $\mathcal{S}_\lambda$ be the regular decomposition of $\mathcal{A}$ induced by λ. Our first main result is that the regular control surface $Y_{\mathcal{A},w,\mathcal{B}}(\mathcal{S}_\lambda)$ induced by $\mathcal{S}_\lambda$ is the limit of the patches $Y_{\mathcal{A},w,\mathcal{B},\lambda}(t)$ parameterized by $F_{\mathcal{A},w,\mathcal{B},\lambda}(x;t)$ as $t \to \infty$.

This limit is with respect to the Hausdorff distance between two subsets of $\mathbb{R}^3$. Two subsets X and Y of $\mathbb{R}^3$ are *within Hausdorff distance ϵ* if for every point x of X there is some point y of Y within a distance ϵ of x, and vice-versa. With this notion of distance, we have the following result.

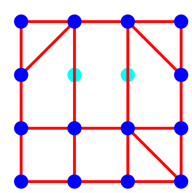
Theorem 5.1. $\lim\limits_{t\to\infty} Y_{\mathcal{A},w,\mathcal{B},\lambda}(t) = Y_{\mathcal{A},w,\mathcal{B}}(\mathcal{S}_\lambda)$.

That is, for every $\epsilon > 0$ there is a number M such that if $t \geq M$, then the patch $Y_{\mathcal{A},w,\mathcal{B},\lambda}(t)$ and the regular control surface $Y_{\mathcal{A},w,\mathcal{B}}(\mathcal{S}_\lambda)$ are within Hausdorff distance ϵ.

We illustrate Theorem 5.1 on a bicubic patch. On the left below are the weights of a bicubic patch, in the center are the values of a lifting function, and the corresponding regular decomposition is on the right.

1	3	3	1
3	9	9	3
3	9	9	3
1	3	3	1

0	2	2	0
1	1	1	1
1	2	2	1
0	1	1	0.5



The two lighter points, $(1,2)$ and $(2,2)$, lie in no face of the decomposition. Figure 2 shows the toric degeneration of this bicubic patch at values $t = 1$ and $t = 6$, and the regular control surface, all with the indicated control points.

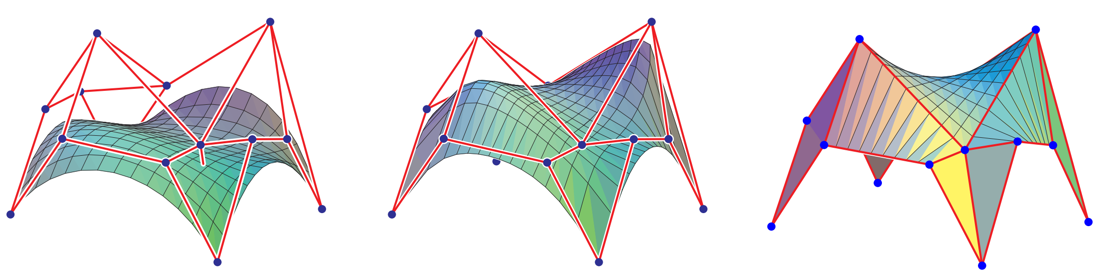


FIGURE 2. Toric degeneration of a rational tensor product patch of bidegree $(3,3)$.

We outline the prove of Theorem 5.1 below. The key idea is the factorization (4) of the map $F_{\mathcal{A},w,\mathcal{B},\lambda}(x;t)$ through the simplex $\triangle^{\mathcal{A}}$. This factorization allows us to study the limit in Theorem 5.1 by considering the effect of the family of weights $w_\lambda(t)$ on the toric variety $X_{\mathcal{A}}$ in $\triangle^{\mathcal{A}}$. Using equations for $X_{\mathcal{A}}$, we can show the limit as $t \to \infty$ of the translated toric variety $X_{\mathcal{A},w_\lambda(t)}$ is a regular control surface in $\mathbb{R}^{\mathcal{A}}$ whose projection to $\mathbb{R}^3$ is the regular control surface $Y_{\mathcal{A},w,\mathcal{B}}(\mathcal{S}_\lambda)$.

Figure 3 illustrates this lift, showing a toric degeneration of a rational cubic Bézier curve, together with the corresponding degeneration of the curve $X_{\mathcal{A},w}$ in the simplex $\triangle^{\mathcal{A}}$. Here, the weights are $w_\lambda(t) = (1, 3t^2, 3t^2, 1)$. That is, the control points $\mathbf{b}_0$ and $\mathbf{b}_3$ have weight 1, while the internal control points $\mathbf{b}_1$ and $\mathbf{b}_2$ have weights $3t^2$.

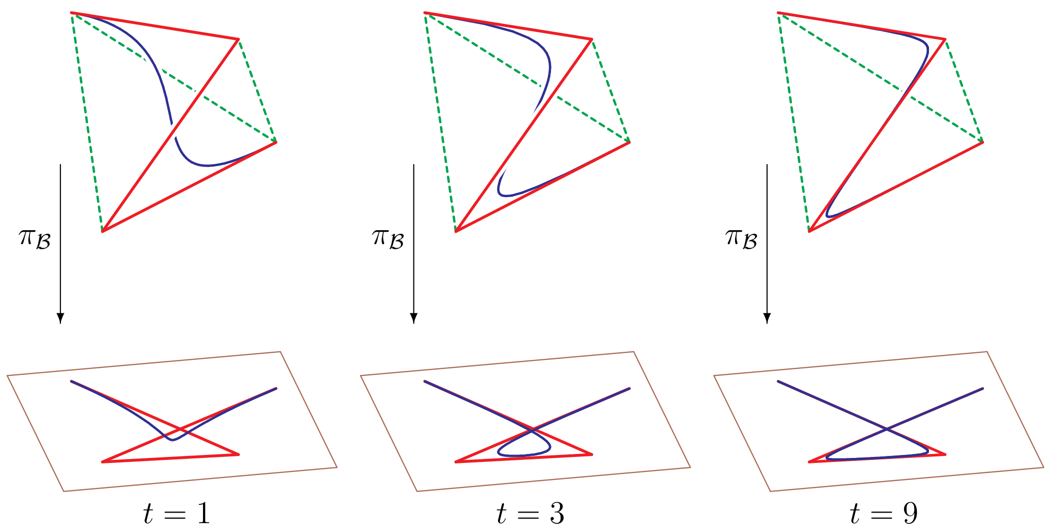


FIGURE 3. Toric degenerations of a rational cubic Bézier curve.

By Theorem 5.1, every regular control surface is the limit of the corresponding patch under a toric degeneration. Our second main result is a converse: If a space Y is the limit of patches of shape $\mathcal{A}$ with control points $\mathcal{B}$, but differing weights, then Y is a regular control surface of shape $\mathcal{A}$ and control points $\mathcal{B}$.

Theorem 5.2. *Let $\mathcal{A} \subset \mathbb{Z}^2$ be a finite set and $\mathcal{B} = \{\mathbf{b}_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\} \subset \mathbb{R}^3$ a set of control points. If $Y \subset \mathbb{R}^3$ is a set for which there is a sequence $w^1, w^2, \ldots$ of weights so that*

$$\lim_{i\to\infty} Y_{\mathcal{A},w^i,\mathcal{B}} \;=\; Y\,.$$

then there is a lifting function $\lambda\colon \mathcal{A} \to \mathbb{R}$ and a weight $w \in \mathbb{R}^{\mathcal{A}}_{>}$ such that $Y = Y_{\mathcal{A},w,\mathcal{B}}(\mathcal{S}_\lambda)$, a regular control surface.

To prove Theorem 5.2, we consider the sequence of translated toric varieties $X_{\mathcal{A},w^i} \subset \triangle^{\mathcal{A}}$. We show how work of Kapranov, Sturmfels, and Zelevinsky [9, 10] implies that the set of all translated toric varieties is naturally compactified by the set of all regular control surfaces in $\triangle^{\mathcal{A}}$. Thus some subsequence of $\{X_{\mathcal{A},w^i}\}$ converges to a regular control surface in $\triangle^{\mathcal{A}}$, whose image must coincide with Y, implying that Y is a regular control surface.

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Department of Mathematics and Statistics, Sam Houston State University, Huntsville, TX 77341, USA
E-mail address: `lgarcia@shsu.edu`
URL: `www.shsu.edu/~ldg005`

Department of Mathematics, Texas A&M University, College Station, Texas 77843, USA
E-mail address: `sottile@math.tamu.edu`
URL: `http://www.math.tamu.edu/~sottile`

School of Mathematical Sciences, Dalian University of Technology, Dalian 116024 China
E-mail address: `cgzhu@dlut.edu.cn`
URL: `http://gs1.dlut.edu.cn/Supervisor/zhuchungan.page`